Black Holes I

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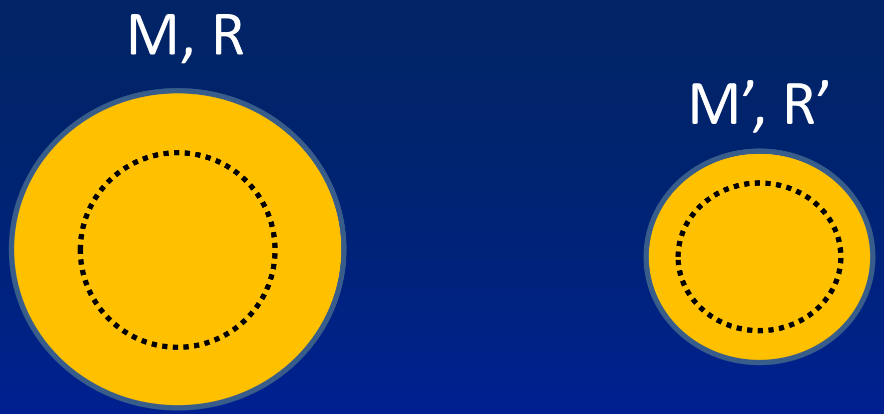
Stability of Stars

- For a star to be stable, γ, the ratio of specific heats, should be greater than 4/3.
- If $\gamma < 4/3$, the compressibility would be negative.
- Since the pressure varies over the star, the relevant parameter is the pressure-averaged value of γ over the star.

$$\gamma_{ad} = \left(\frac{\delta \ln P}{\delta \ln \rho} \right)_s$$

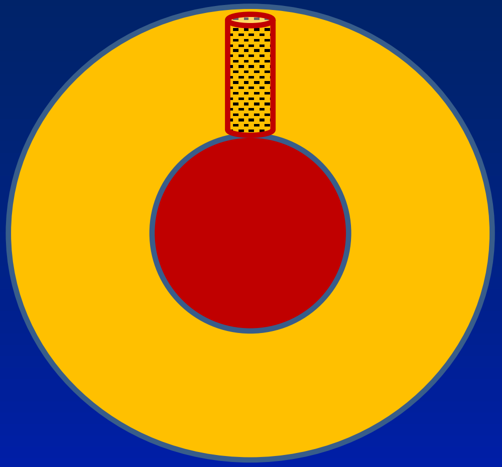
$$\bar{\gamma} = \frac{\int_0^R \gamma \, \mathrm{P} r^2 \, dr}{\int_0^R \mathrm{P} r^2 \, dr}$$



The two configurations are said to be “homologous” if the

$$\frac{r}{R} = \frac{r'}{R'} \qquad \frac{m}{M} = \frac{m'}{M'}$$

Homologous mass shells are located at homologous points.

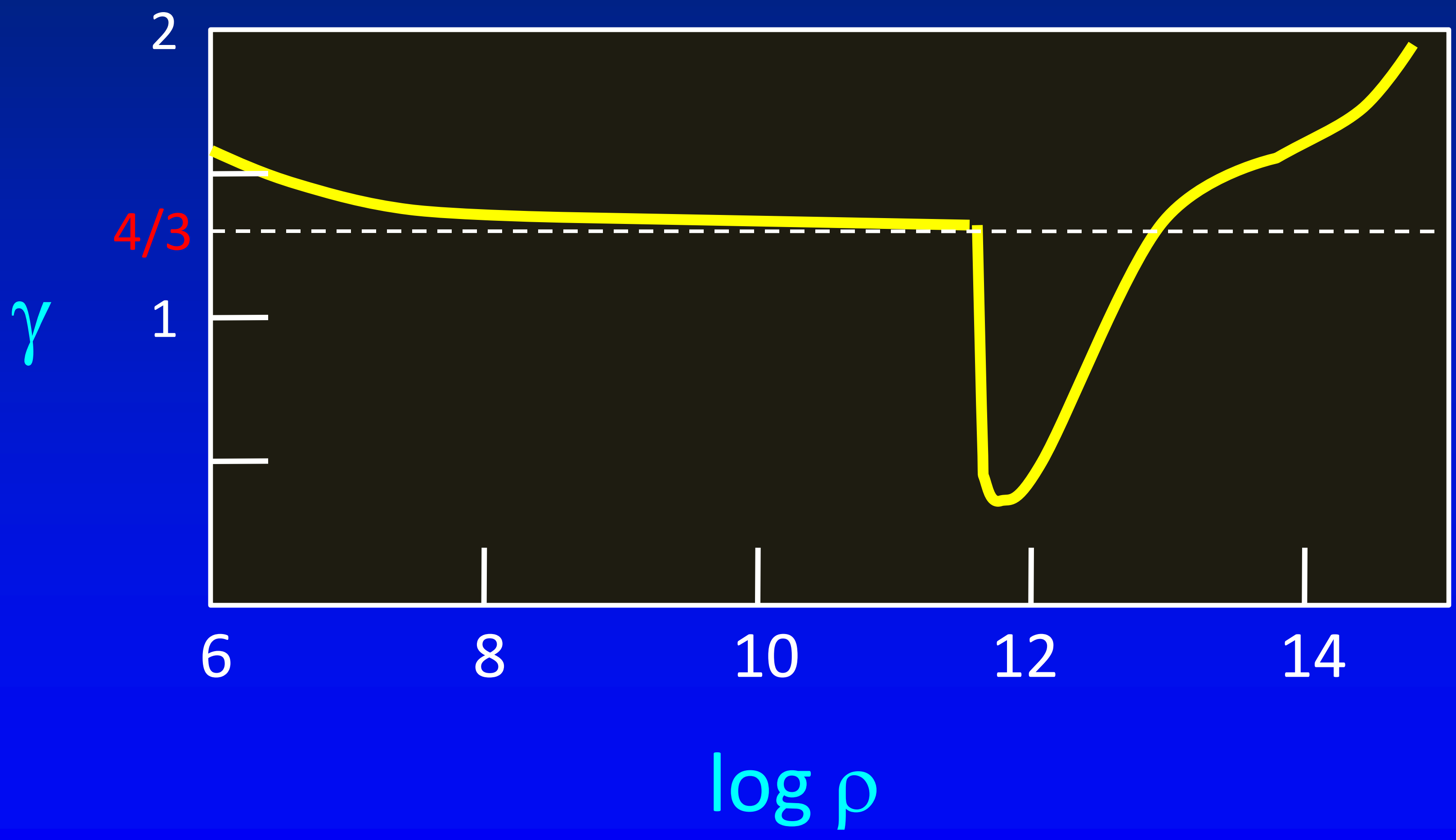


Pressure = weight of unit cylinder

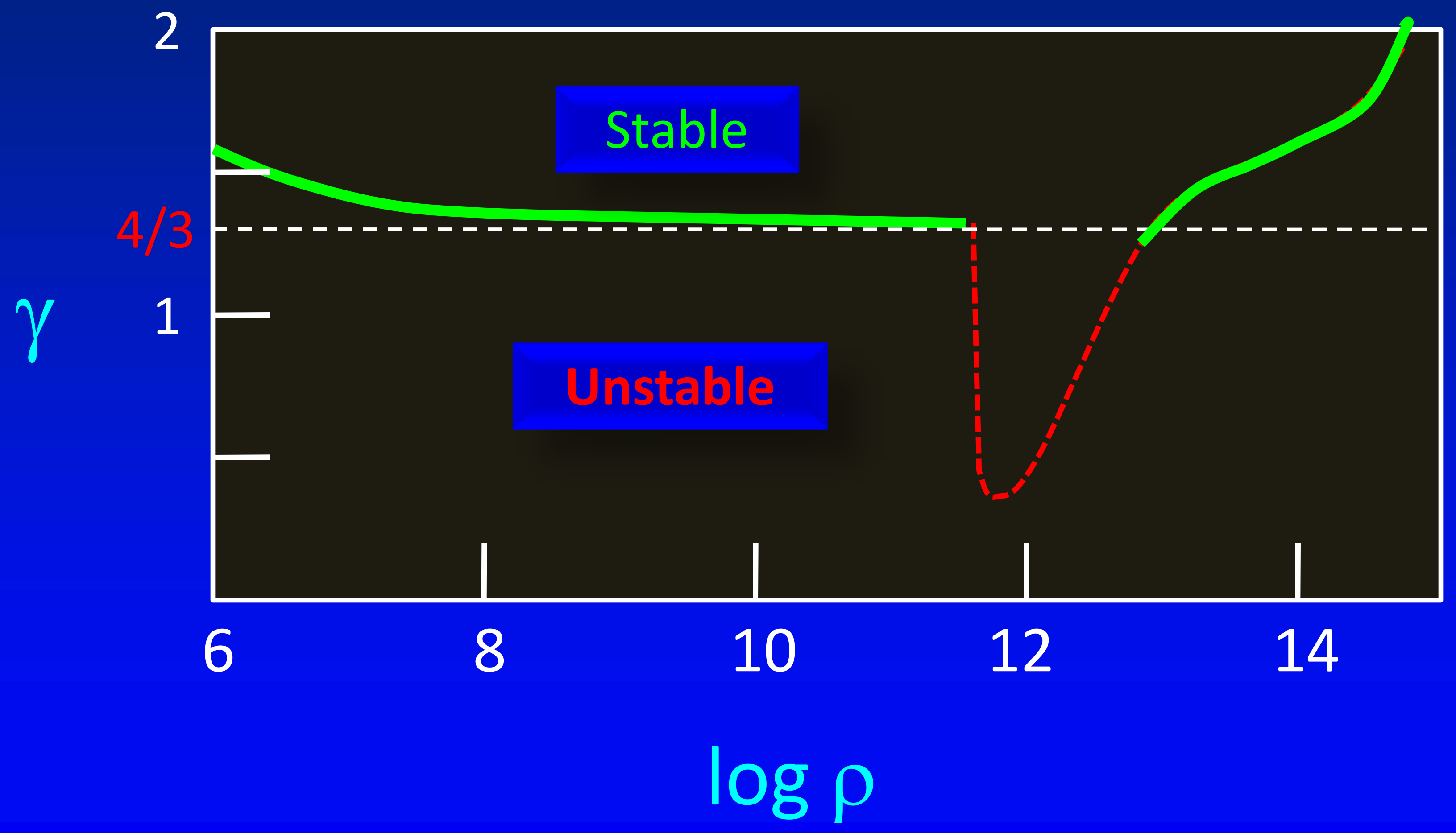
$$P = \int_m^M \frac{Gm'dm'}{4\pi r^2}$$

- Compress sphere adiabatically and homologously.
- Will the star still be in hydrostatic equilibrium?
- RHS varies as $\left(\frac{R'}{R}\right)^{-4}$
- LHS varies as $\left(\frac{\rho'}{\rho}\right)^{\gamma} = \left(\frac{R'}{R}\right)^{-3\gamma}$
- If $\gamma_{ad} > 4/3$, LHS increases STRONGER with compression then the weight on RHS. Star will BOUNCE BACK!

Stability of Matter



Stability of Matter



Stability of Stars in General Relativity

- In the 1960s, Chandrasekhar made a definitive study of stability of stars in GTR. As is to be expected, there is an important correction to Newtonian criterion for stability.

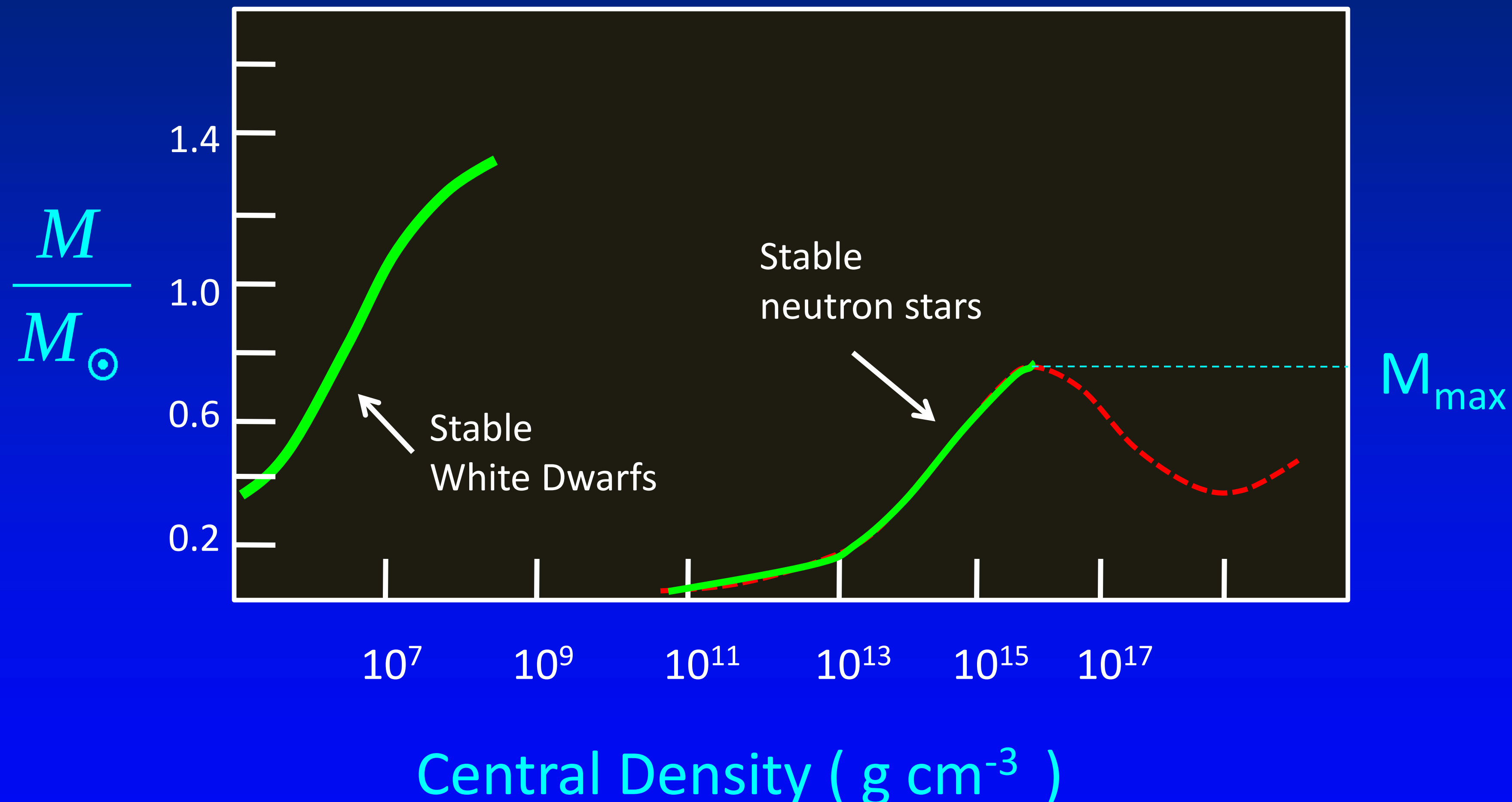
- In the post-Newtonian limit,

$$\gamma_{crit} = \frac{4}{3} + \kappa \frac{GM}{Rc^2}$$

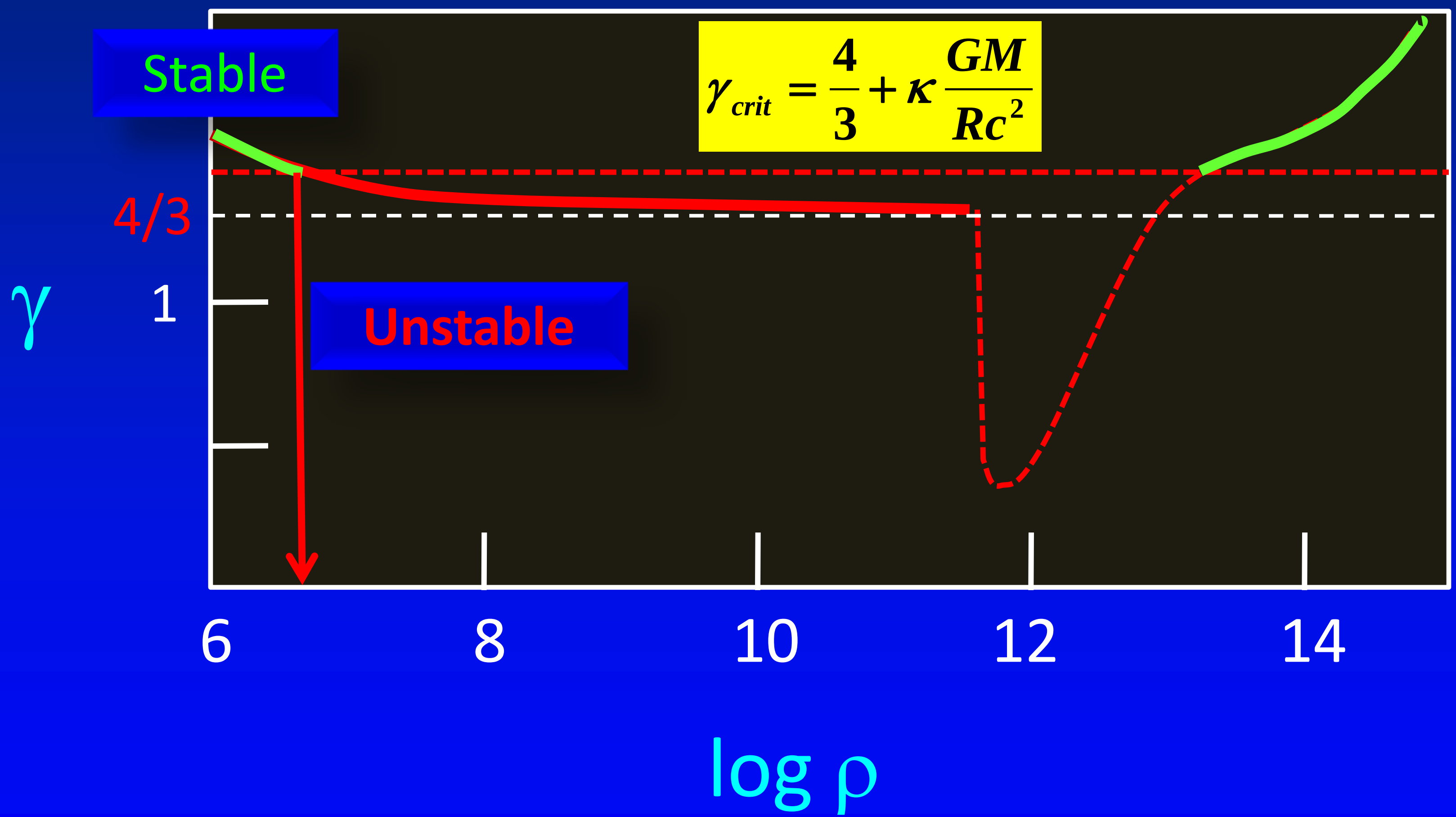
κ is of order unity

- **Gravity is stronger in GTR and can render a star UNSTABLE to radial pulsation when they are nearly Newtonian.**

OV: Oppenheimer – Volkoff Equation of State



Stability of Stars in GTR



$$\gamma_{crit} = \frac{4}{3} + \kappa \frac{GM}{Rc^2}$$

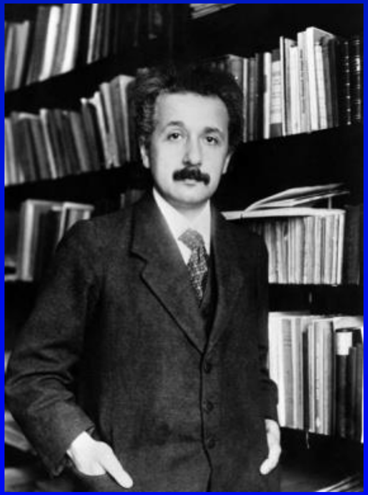
$$\boxed{R_{crit} = \kappa \frac{R_g}{\gamma - \frac{4}{3}}}$$

- For example, a white dwarf becomes unstable in GTR when its radius is ~ 5000 R_g . (Chandrasekhar and Tooper, 1964).
- The early models for Quasars invoked "Supermassive stars". These would be supported by radiation pressure for which $\gamma = 4/3$.
- Chandrasekhar pointed out (1964) that such stars would be unstable in GTR. (Fowler & Feynman).

General Relativistic Instability

- Oppenheimer and Volkoff (1938) derived the maximum mass of neutron stars using a “turning point criterion”.
- After Chandra’s paper of 1964, Misner & Zapolsky found that the neutron star was unstable to radial pulsations precisely at the extremum.
- Similarly, the upper mass limit of a white dwarf coincides with the point of instability to collapse.
- But this coincidence arises only because the EOS used for equilibrium analysis is the same one used to model radial pulsations.
- The above digression sets the stage for our discussion of black holes of general relativity.

$$R^{ij} - \frac{1}{2} g^{ij} R = -\left(\frac{8\pi G}{c^4}\right) T^{ij}$$



"Scarcely anyone who fully understands this theory can escape from its magic."

25 November 1915

Laws of gravitation

In formulating the laws of gravitation, one may ask three questions.

1) When can we say that there is no gravitational field?

2) What are the equations which determine the gravitational field in vacuum, outside material bodies?

3) What are the equations that obtain in regions of space where matter is present?

Newtonian Gravity

- Gravitational field is described by a scalar potential U.

(1) In the absence of gravitational field, $U = 0$

(2) In regions of space outside matter, $\nabla^2 U = 0$

(3) In regions where matter density is ρ, $\nabla^2 U = -4\pi G\rho$

Einstein’s generalizations

- In the absence of gravity, particles experience no acceleration.

$$\frac{d^2x^i}{ds^2} = 0.$$

- In a curvilinear coordinate system particles will experience “inertial acceleration” due to “fictitious forces”.

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

- If there is no gravity, we should be able to find a transformation such that the Christoffel symbols vanish identically.

Einstein's Zeroth Law of Gravity

- According to Riemannian geometry, the condition that the Christoffel symbols vanish identically is that the Riemann-Christoffel tensor vanishes.

$$\boxed{R^{i}_{jkl} = 0.} \quad \Leftarrow \quad U = 0$$

- This is a necessary and sufficient condition.

- This is Einstein's zeroth law of gravity.

In regions where no matter is present

- Einstein guessed that the generalization of Laplace's equation is the vanishing of the Riicci tensor.

$$\nabla^2 U = 0 \quad \Longrightarrow \quad \boxed{R^l_{jkl} = R_{jk} = 0.}$$

- The above equation is, like Laplace's eqn., a second order differential equation. There are 10 equations for the 10 coefficients of the metric tensor g_{ij}

- The above equation reduces to Laplace's equation in the limit

$$c \to \infty$$

In regions where matter is present

- In Newtonian theory one relates the gravitational potential to the material density ρ. $\nabla^2 U = -4\pi G\rho$
- In relativity, density alone will not suffice to specify the distribution of matter and its motions.
- In special relativity one introduces the energy-momentum tensor, T_{ij}, with ten components. The (0,0) component of this tensor is ρc^2.
- One would therefore be inclined to relate the Ricci tensor (the generalization of $\nabla^2 U$) to the energy-momentum tensor T_{ij} .

$$\nabla^2 U = -4\pi G\rho \quad \xrightarrow[?]{} \quad R_{ij} = \kappa T_{ij}$$

Conservation Laws

- In special relativity, conservation laws follow from the relation that the 'divergence of the energy-momentum tensor vanishes'.

$$\mathrm{div}(T_{ij}) = 0$$

- In a non-Euclidean geometry, this should be replaced by

$$\text{covariant divergence}\left(T_{ij}\right) = 0$$

- The "guess" $R_{ij} = \kappa T_{ij}$ will not work because

$$\text{Covariant divergence of } R_{ij} \neq 0$$

Einstein's field equations

- However, $\boxed{\text{Covariant divergence of } \left(R^{ij} - \frac{1}{2} g^{ij} R \right) = 0}$

- Therefore, the generalization of Poisson's equation is

$$\nabla^2 U = -4\pi G\rho \Longrightarrow \boxed{R^{ij} - \frac{1}{2} g^{ij} R = \kappa T^{ij}.}$$

- The coupling constant κ has to be determined.

Newtonian limit

- In the limit of 'weak gravity', the metric is given by,

$$ds^2 = c^2 dt^2 \left(1 - \frac{2U}{c^2}\right) - \left(dx^2 + dy^2 + dz^2\right)$$

- Using this metric, one can find the limiting form of Einstein's equations. In the limit $C \longrightarrow \infty$ Einstein's equations reduce to

$$\nabla^2 U = \frac{1}{2}\kappa\rho c^4 \Longleftarrow R^{ij} - \frac{1}{2}g^{ij}R = \kappa T^{ij}.$$

$$\Updownarrow$$

$$\nabla^2 U = -4\pi G\rho$$

$$\boxed{\kappa = -\frac{8\pi G}{c^4}}$$

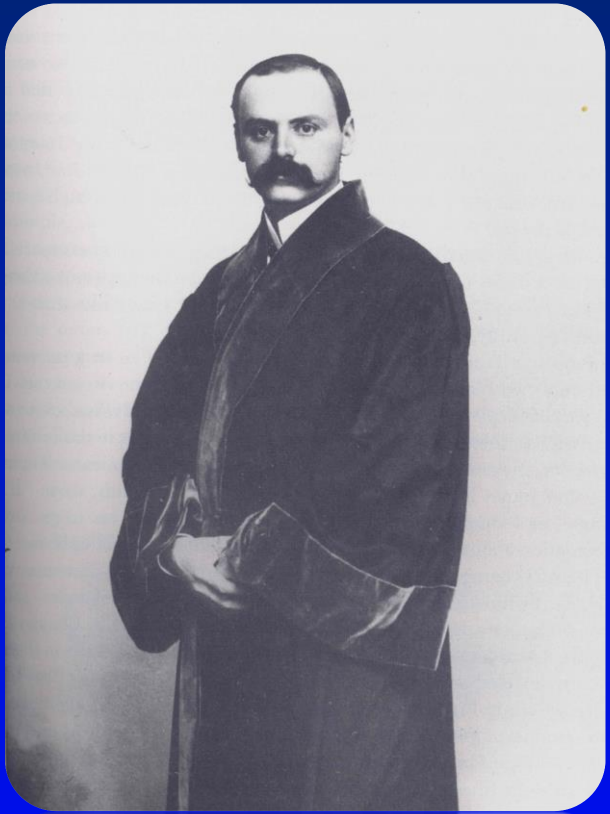
Einstein’s field equations

$$R^{ij} - \frac{1}{2} g^{ij} R = -\left(\frac{8\pi G}{c^4} \right) T^{ij}$$

- These equations determine both the geometry (LHS) and the states of motion (RHS).

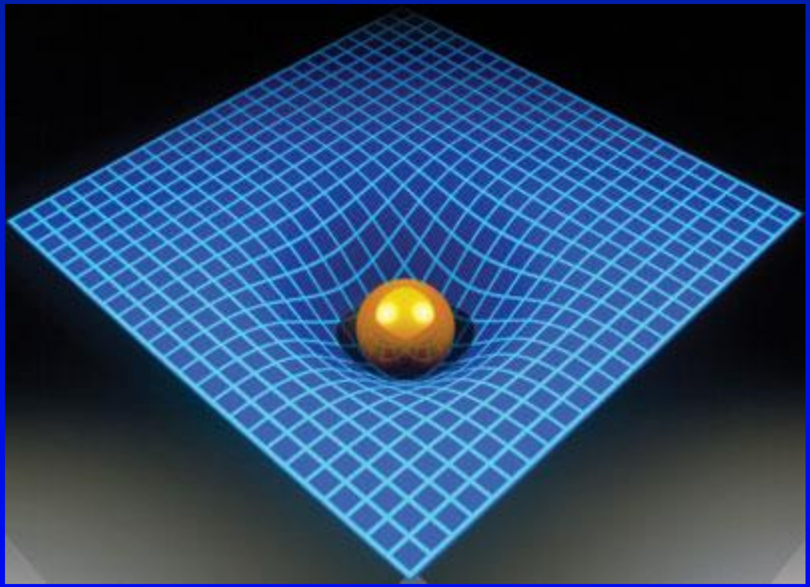
- The conservation laws are built into them.

Karl Schwarzschild



Within a month of the announcement of Einstein's discovery, Karl Schwarzschild found an EXACT solution to Einstein's equations. Einstein was astonished by this!

Schwarzschild's solution described the curvature of space and time near a spherically symmetric massive body.



$$ds^2 = g_{ij} dx^i dx^j$$

Schwarzschild metric

- Schwarzschild's solution of Einstein's equations for the gravitational field describes the curvature of space and time near a spherically symmetric massive body.

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} dr^2 - r^2\left(\sin^2\theta \, d\phi^2 + d\theta^2\right)$$

"warpage of time"

"warpage of space"

The Schwarzschild Geometry

- This describes not only the “curvature of space” but also the “warpage of time” near the star. This is just dilatation of time due to gravity.

- This leads to gravitational redshift of light emitted from the stars surface. Near the star, time flows more slowly than far away. Therefore, the frequency of light will be lowered by the same amount.

- Radiation from the star will be shifted to longer wavelengths.

The Schwarzschild Radius

- Exact result in General Relativity,

$$\boxed{\nu_2 = \nu_1 \left(1 - \frac{2GM / c^2}{R} \right)}$$

- At the critical radius $R_g = 2GM / c^2$, the time dilatation becomes infinite. Time does not flow at all!

- As seen by an outside observer, frequency goes to zero. The wavelength of radiation becomes infinite.

- The star will appear BLACK.

- The surface of the black hole is known as the EVENT HORIZON

- It is a semi permeable surface. Things can only go into the horizon. Nothing can come out.

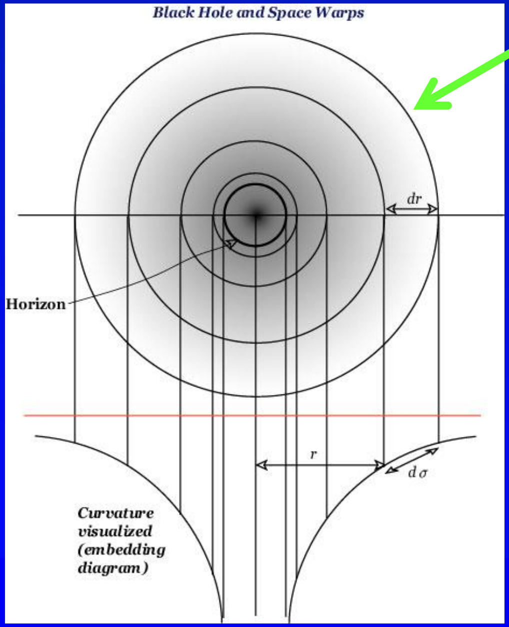
- Inside the event horizon, there are no allowed trajectories with a constant distance from the centre.

- The infalling matter cannot arrest its collapse.

- The ONLY allowed trajectories are those along which all matter will fall to the centre of the black hole.

Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 - \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} dr^2 - r^2\left(\sin^2\theta\, d\phi^2 + d\theta^2\right)$$



Circumference = 2πr.

Distance between r_1 and r_2 is $\int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{R_g}{r}}} > (r_2 - r_1)$

$\left(1 - \frac{R_g}{r}\right)$ is the curvature factor.

Approximate metric at large distances from the centre.

$$ds^2 = ds_0^2 \underbrace{- \frac{2GM}{rc^2}\left(dr^2 + c^2dt^2\right)}_{\text{Small correction}}$$

At large distance from the masses, every field appears centrally symmetric. Therefore, the above metric is the metric at large distances from ANY system of bodies.

The above metric can also be written as

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 + \frac{2GM}{rc^2}\right)\left(dx^2 + dy^2 + dz^2\right)$$

Effective potential in Newtonian theory

$$E = \frac{1}{2} m \mathrm{v}^2 - \frac{GMm}{r}; \quad L = mr^2 \frac{d\phi}{dt}$$

$$\boxed{\mathrm{v}^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{mr^2}}$$

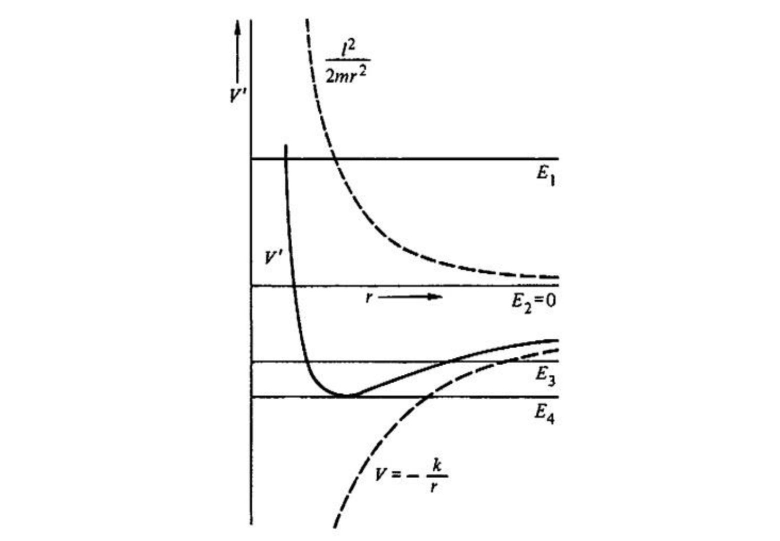


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left[-\frac{GM}{r} + \frac{L^2}{2mr^2}\right]$$

V(r)

Effective potential in Newtonian theory

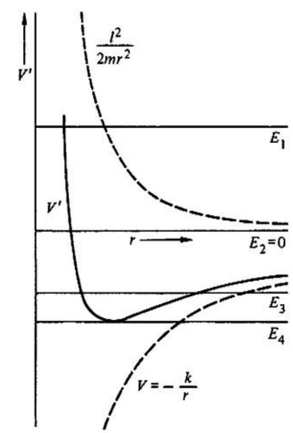
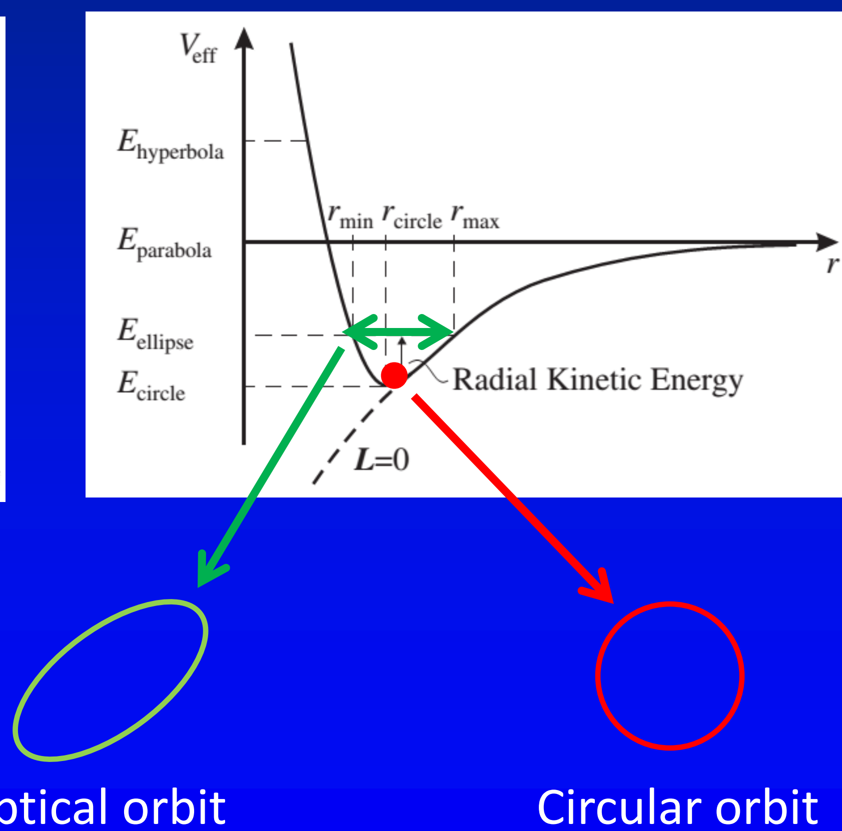


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.



Effective potential in Einstein's theory

Constants of motion: $r^2 \frac{d\phi}{d\tau} = \frac{L}{m} = \text{ const.} \qquad \frac{E}{m} = \left(1 - \frac{r_g}{r}\right)\frac{dt}{d\tau}$

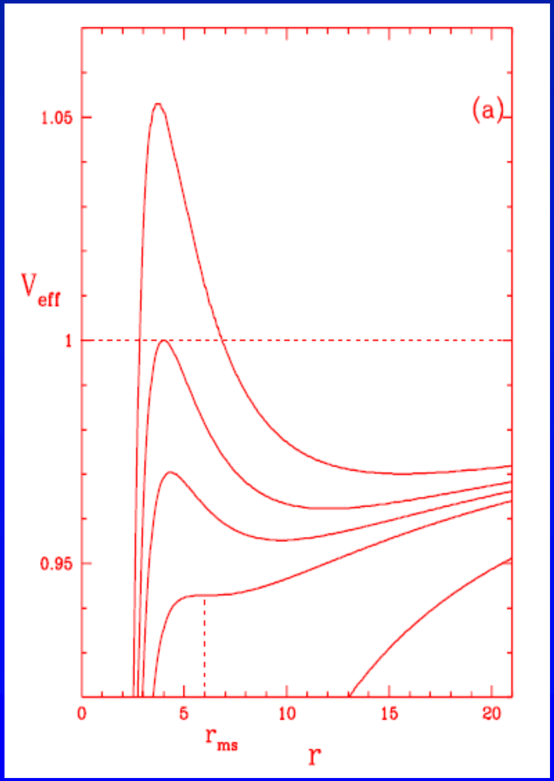
$$c^2 d\tau^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{r_g}{r}\right)} dr^2 - r^2 d\phi^2$$

Substitute for dt and dϕ from the constants of motion. Simplify.

$$dr = c\,d\tau \left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{r_g}{r}\right)\left\{1 + \frac{L^2}{m^2 r^2}\right\}\right]^{\frac{1}{2}} \qquad \frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 = \left[\left(\frac{E}{m}\right)^2 - \left(1 - \frac{r_g}{r}\right)\left\{1 + \frac{L^2}{m^2 r^2}\right\}\right]$$

Effective potential in Einstein's theory

$$\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 = \left[\left(\frac{E}{m}\right)^2 - \left(1-\frac{r_g}{r}\right)\left\{1+\frac{L^2}{m^2r^2}\right\}\right] = \left(\frac{E}{m}\right)^2 - \left(\frac{V}{m}\right)^2$$



$$\frac{V}{m} = \left[\left(1-\frac{r_g}{r}\right)\left\{1+\frac{L^2}{m^2r^2}\right\}\right]^{\frac{1}{2}}$$

$$\left(\frac{V}{m}\right)_{\mathrm{Newton}} = -\frac{GM}{r} + \frac{L^2}{2m^2r^2}$$

Effective potential in Einstein's theory

$$\frac{V}{m} = \left[\left(1-\frac{r_g}{r}\right)\left\{1+\frac{L^2}{m^2 r^2}\right\}\right]^{\frac{1}{2}}$$

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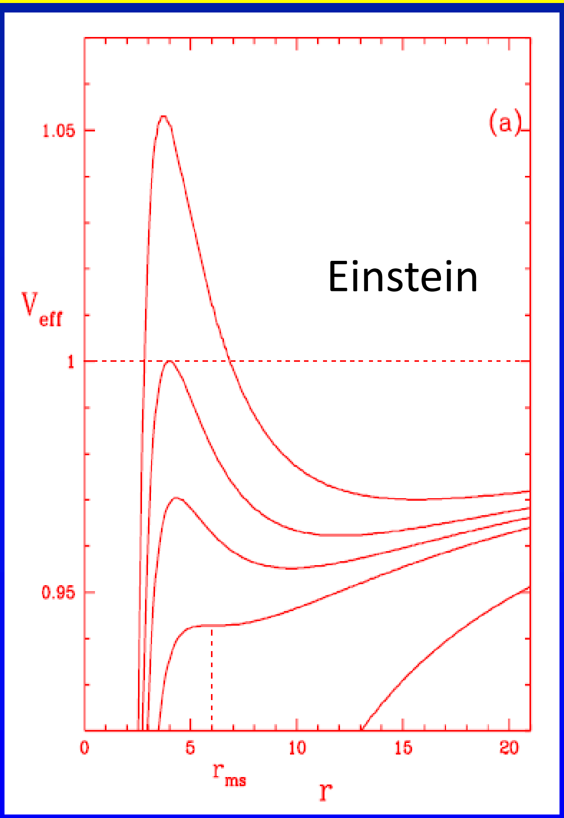
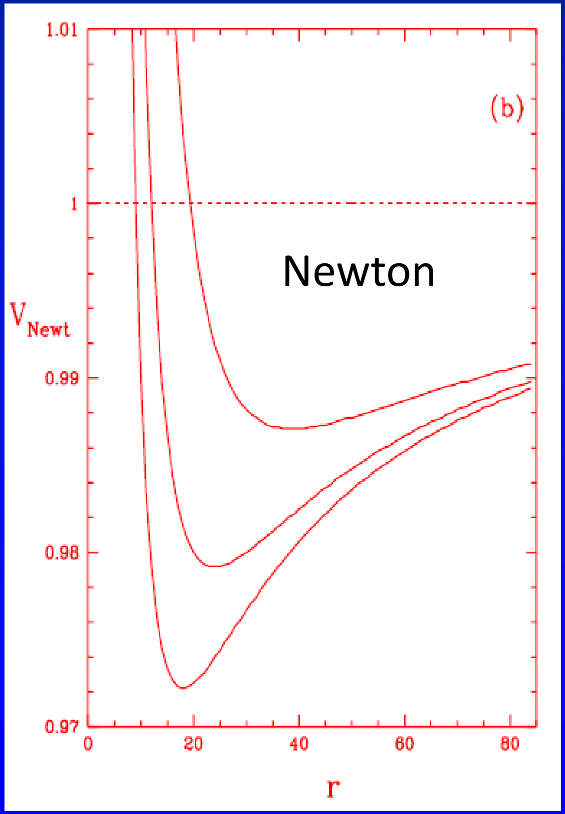
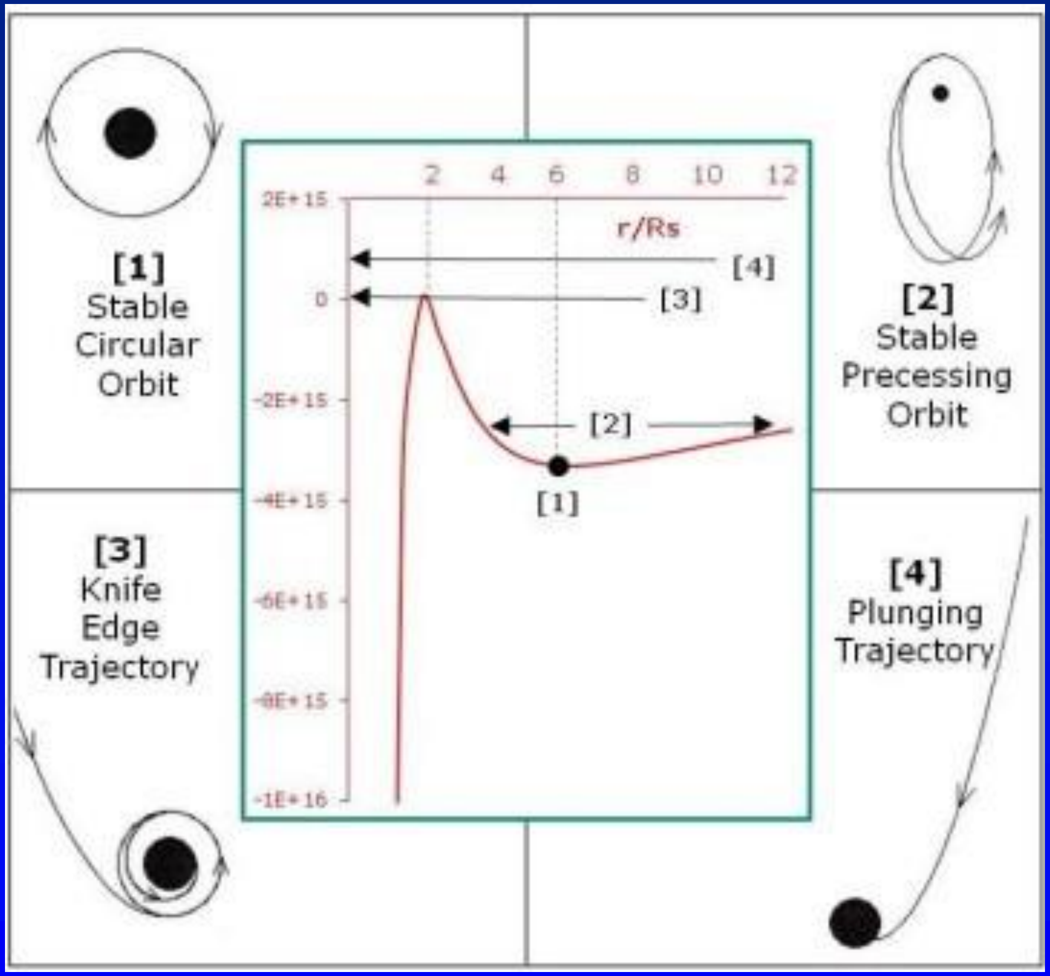
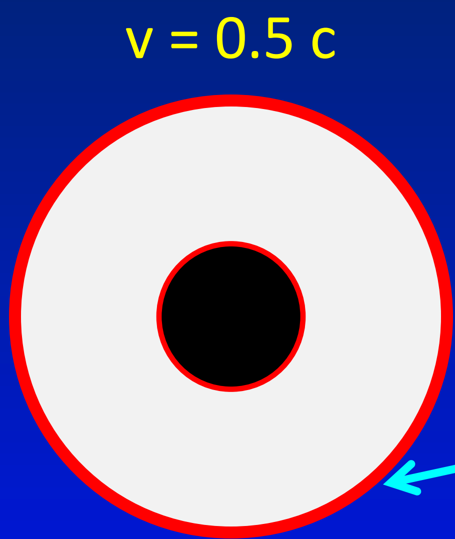
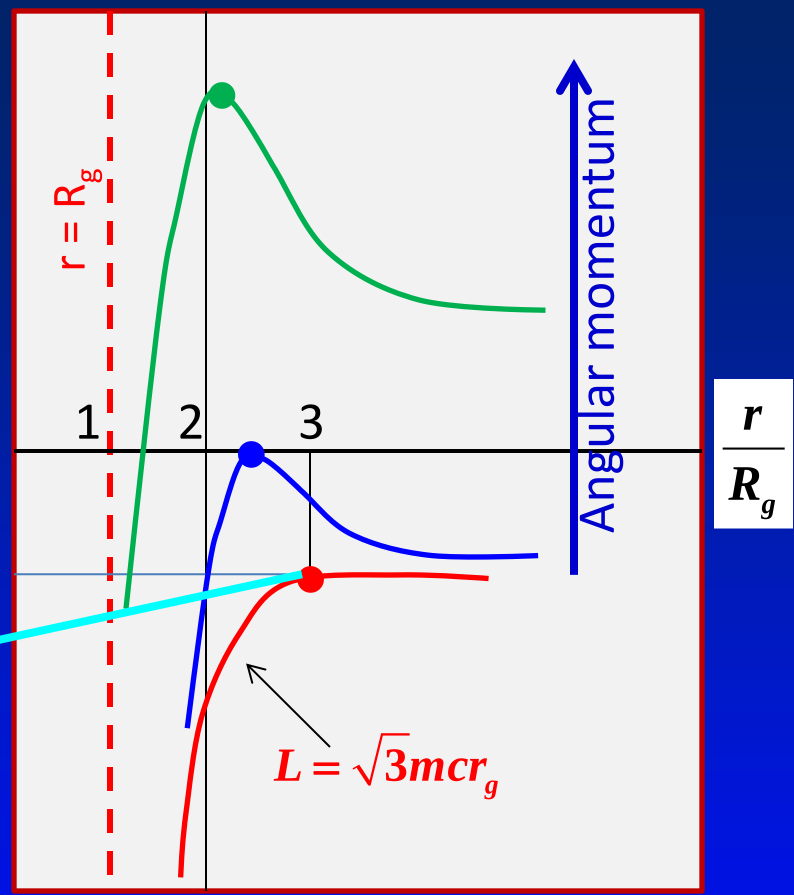




ORBITS in Einstein's theory



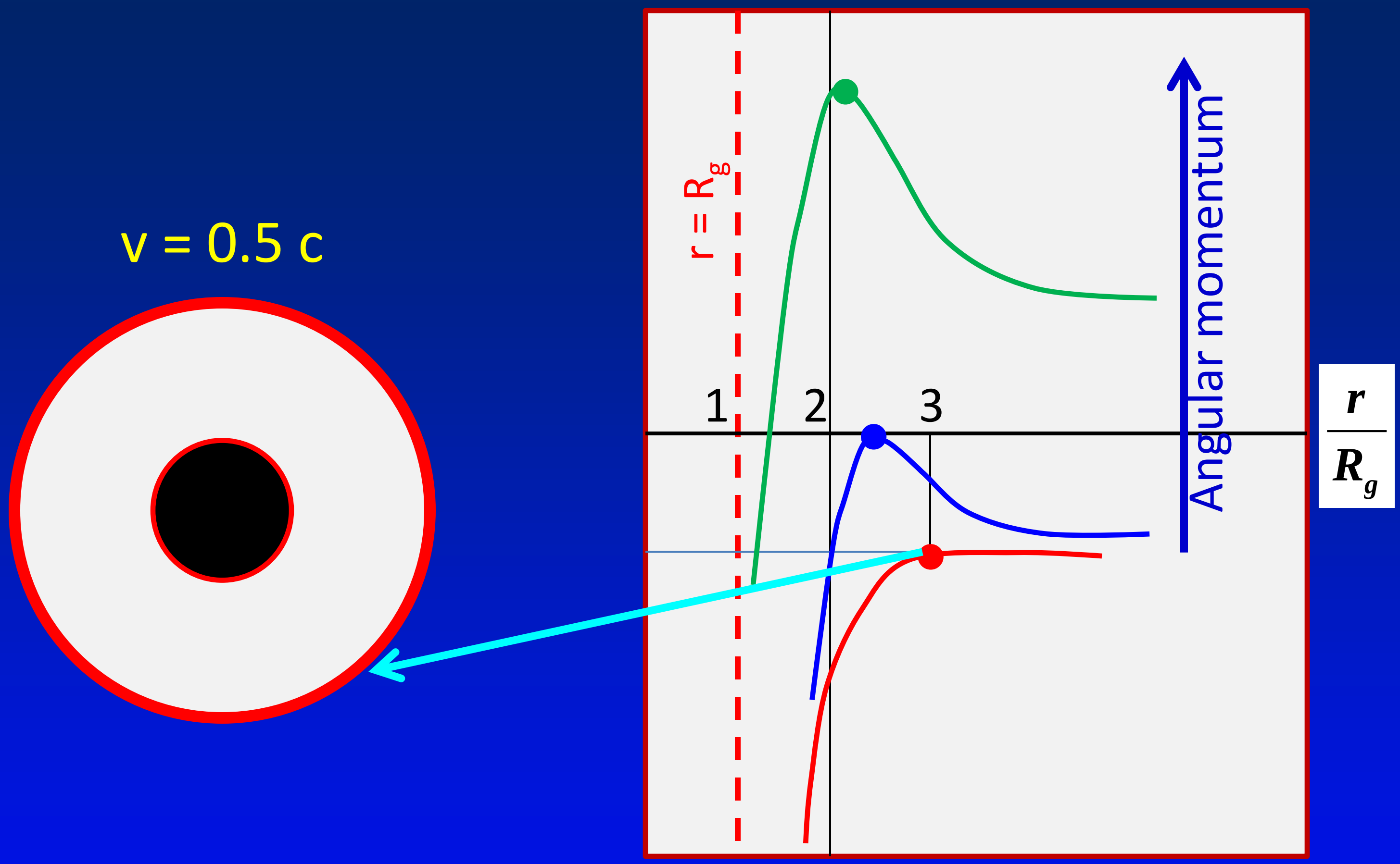




Last stable circular orbit

$$\boxed{r = 3R_g}$$

The last "unstable" circular orbit is at a radius of 1.5 R_g .
The velocity of the particle will be 'c'.



Last stable circular orbit

$$r = 3R_g$$

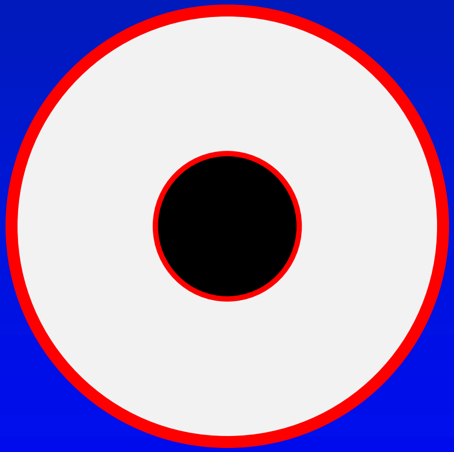
$$L = \sqrt{3}mcr_g$$

$$E = \sqrt{\frac{8}{9}}mc^2$$

Important new result

In GTR, there exists a minimum radius for stable circular orbits, and a corresponding minimum energy for circular motion.

$$r = 3R_g \qquad L = \sqrt{3}mcr_g \qquad E = \sqrt{\frac{8}{9}}mc^2$$

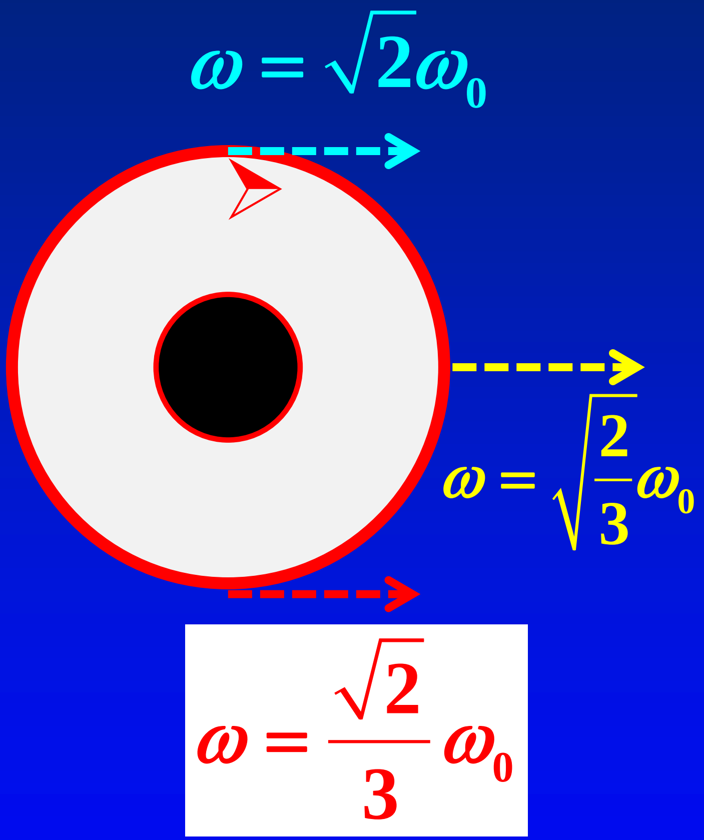


An observer at infinity will see the particle orbiting with a period T.

$$d\tau = \sqrt{1 - \frac{r_g}{r}}dt$$

$$T = \frac{1}{\sqrt{1 - \frac{r_g}{r}}}\left(\frac{2\pi(3r_g)}{\frac{1}{2}c}\right) = \frac{12\pi r_g}{\sqrt{2/3c}}$$

Important new result

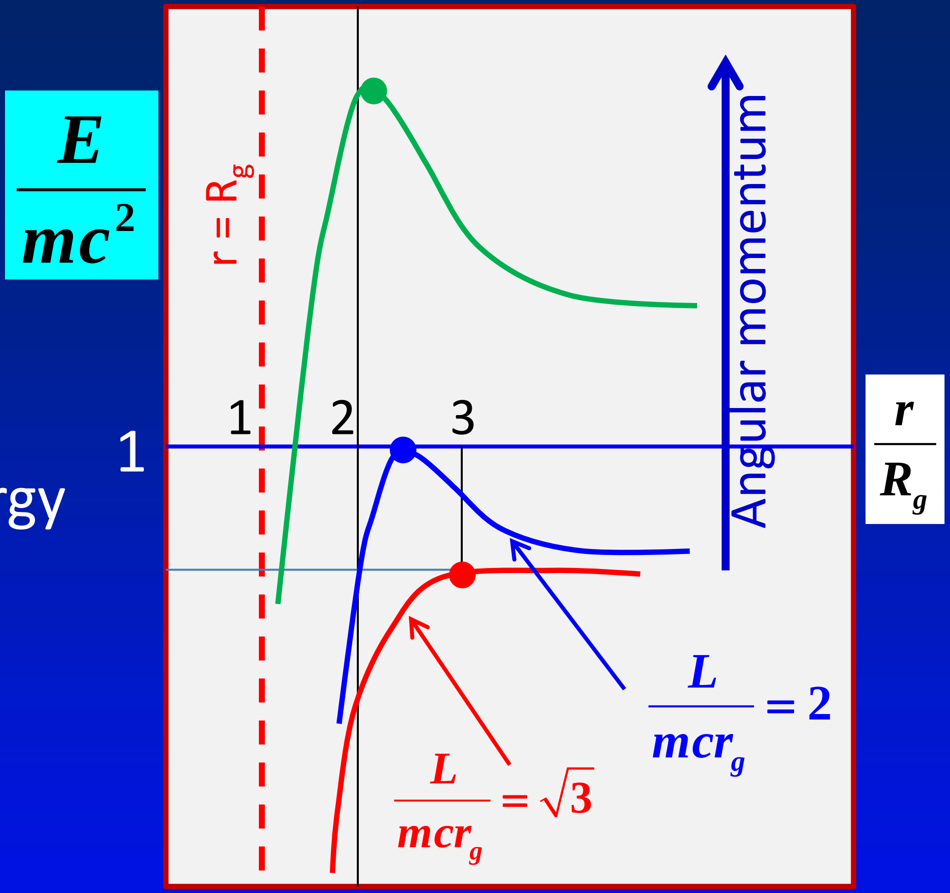


Orbiting particle emits frequency ω_0. The received frequency ω at infinity is given by:

$$\boxed{\omega = \omega_0 \left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}} \left[\frac{1 \pm v/c}{1 \mp v/c}\right]^{\frac{1}{2}}}$$

Gravitational Capture

The particle is at rest at infinity. E= rest mass energy



If L < $2mcr_g$, then E=1 does NOT intersect the potential curve V(r, L).

Therefore there is no turning point. The particle will be gravitationally captured by the body.

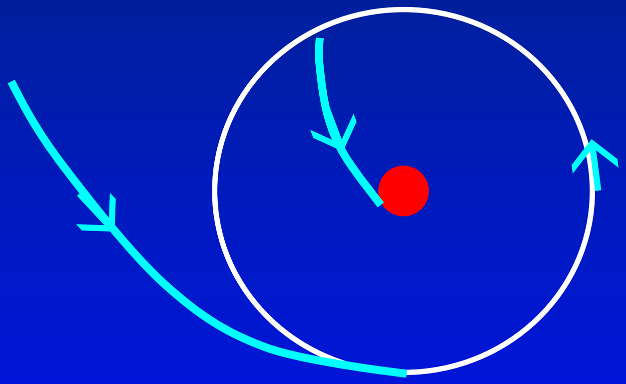
Cross Section for Gravitational capture

Angular momentum: $L = m\mathrm{v}_{\infty}b$.

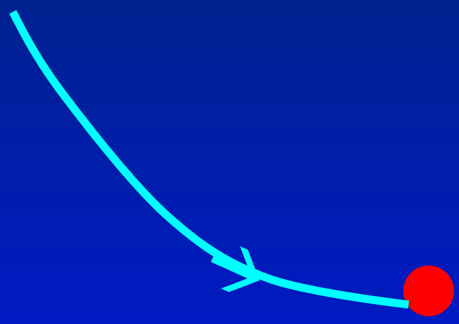
$L < 2mcr_g$ corresponds to $b < \dfrac{2cr_g}{\mathrm{v}_{\infty}}$. All particles with impact parameter smaller than this will be captured.

The capture cross section is $\boxed{\sigma = \pi b^2 = 4\pi r_g^2 \left(\dfrac{c}{\mathrm{v}_{\infty}}\right)^2}$

Gravitational Capture



$$L = 2mcr_g$$



$$L < 2mcr_g$$

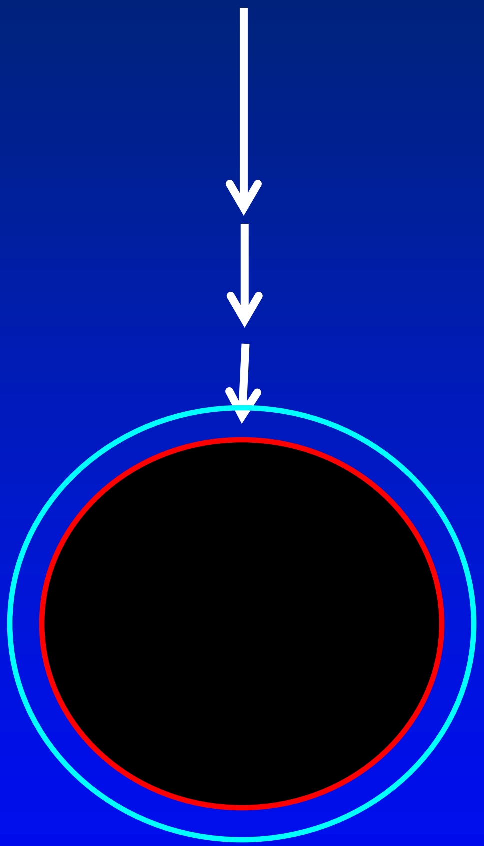
Gravitational capture is not possible in Newtonian mechanics.

Radial Motion of Particles

- Let the particle be at rest at infinity. $E=mc^2$ is a constant.
- General expression for the energy is $\frac{E}{m}=\left(1-\frac{r_g}{r}\right)\frac{dt}{d\tau}$
- Since energy is a constant, $d\tau^2=\left(1-\frac{r_g}{r}\right)^2 dt^2$
- Plug this into Schwarzschild metric (radial motion: $d\theta=d\phi=0$)

$$\frac{dr}{dt}=\pm\left(1-\frac{r_g}{r}\right)\sqrt{\frac{r_g}{r}}\,c$$

Falling towards a black hole



The “velocity” of a particle as seen by distant observer

$$\frac{dr}{dt} = -\left(1 - \frac{2GM}{rc^2}\right)\left(\frac{2GM}{rc^2}\right)^{\frac{1}{2}}$$

"Physical time" at a given point r:

$$d\tau = \sqrt{g_{00}}\,dt = \left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}} dt$$

$d\tau \rightarrow 0$, as $r \rightarrow r_g$.

"Proper radial distance" measured by observer at position r.

$$dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}}}$$

$$\left(\frac{dr}{dt}\right)_{\text{shell}} = \frac{1}{1 - \frac{r_g}{r}} \frac{dr}{dt} = \sqrt{\frac{r_g}{r}}\, c$$

"Physical time" at a given point r:

$$d\tau = \sqrt{g_{00}}dt = \left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}} dt$$

$d\tau \to 0$, as $r \to r_g$.

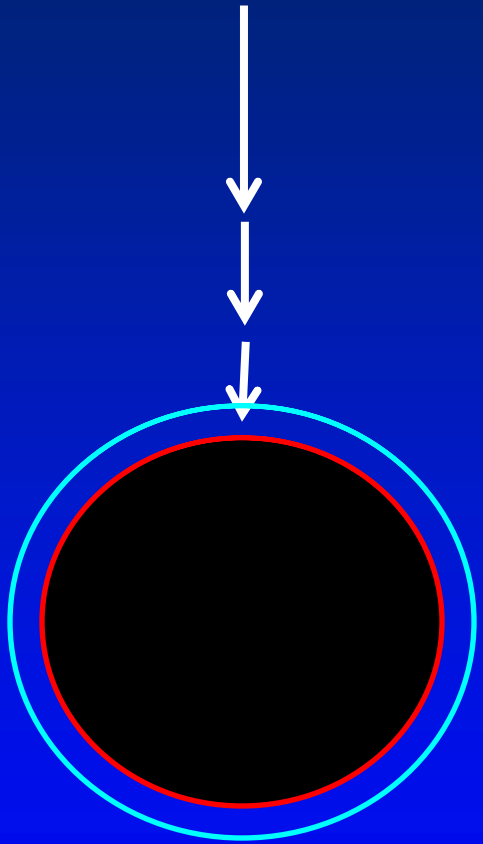
"Proper radial distance" measured by observer at position r.

$$dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{r_g}{r}\right)^{\frac{1}{2}}}$$

Acceleration:

$$a = \frac{1}{\sqrt{1 - \frac{r_g}{r}}}\left(\frac{GM}{r^2}\right) \to \infty, \quad \text{as } r \to r_g$$

Falling towards a black hole



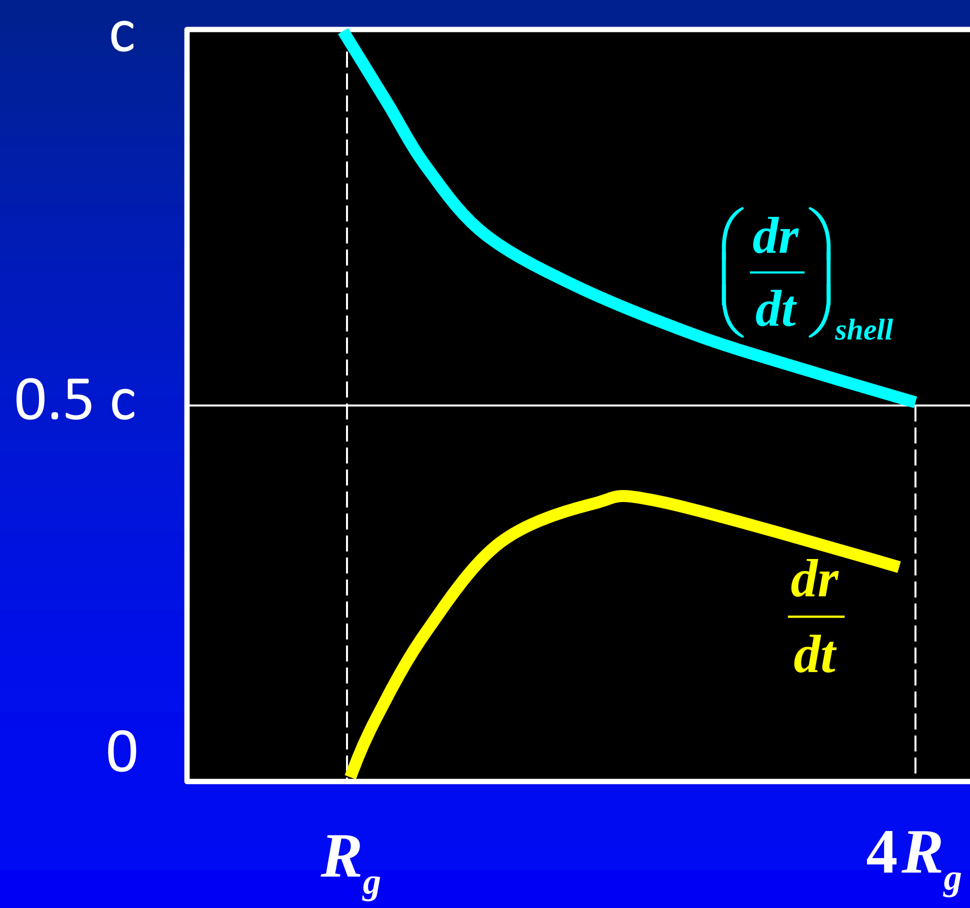
The “velocity” of a particle as seen by distant observer

$$\frac{dr}{dt} = -\left(1 - \frac{2GM}{rc^2}\right)\left(\frac{2GM}{rc^2}\right)^{\frac{1}{2}}$$

The velocity as seen by an observer on a stationary infinitely rigid shell

$$\left(\frac{dr}{dt}\right)_{\text{shell}} = -\left(\frac{2GM}{rc^2}\right)^{\frac{1}{2}}$$

Falling towards a black hole



Time to reach R_g from $r = r_1$

- The free fall time can be obtained by integrating dr/dt.

$$\frac{dr}{dt} = -\left(1 - \frac{2GM}{rc^2}\right)\left(\frac{2GM}{rc^2}\right)^{\frac{1}{2}}$$

- This will diverge. As seen by the distant observer, the free fall time to R_g is infinite.
- The "proper time interval" for free fall is given by:

$$\Delta\tau = \frac{2}{3}\left(\frac{r_g}{c}\right)\left[\left(\frac{r_1}{r_g}\right)^{\frac{3}{2}} - \left(\frac{r}{r_g}\right)^{\frac{3}{2}}\right]$$

Time to reach R_g from $r = r_1$

- The "proper time interval" for free fall is given by:

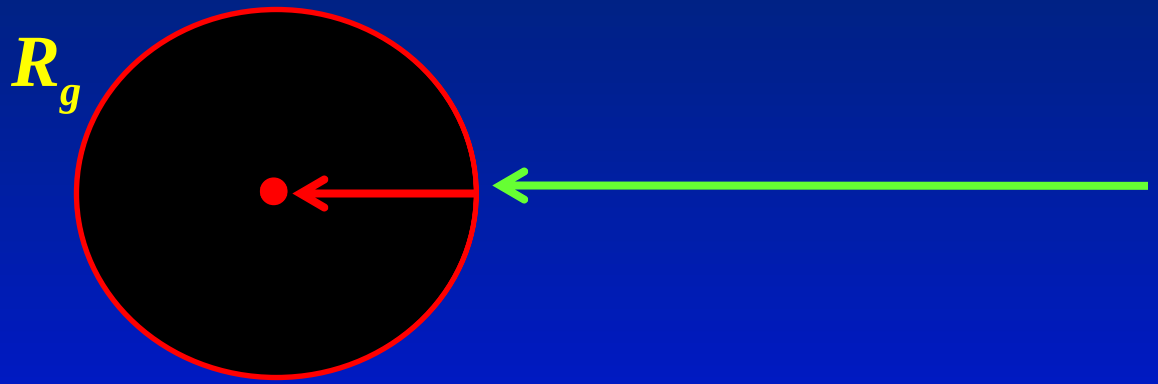
$$\boxed{\Delta\tau = \frac{2}{3}\left(\frac{r_g}{c}\right)\left[\left(\frac{r_1}{r_g}\right)^{\frac{3}{2}} - \left(\frac{r}{r_g}\right)^{\frac{3}{2}}\right]}$$

- This is finite due to two reasons.

- Time is "slowed down" in a gravitational field.

- The "physical velocity" of the free falling clock tends to c, as r tends to r_g . There is a consequent "Lorentz contraction".

Falling into a black hole



- Let a body, at rest at infinity, fall towards the BH. The time taken to fall from the event horizon (R_g) to the centre is

$$\Delta\tau = \frac{2}{3}\left(\frac{R_g}{c}\right)$$

<u>as measured by the infalling clock</u>.

Radial motion of photons

- For light, ds = 0. For radial motion, $d\theta = d\phi = 0$.
- Schwarzschild metric gives us:

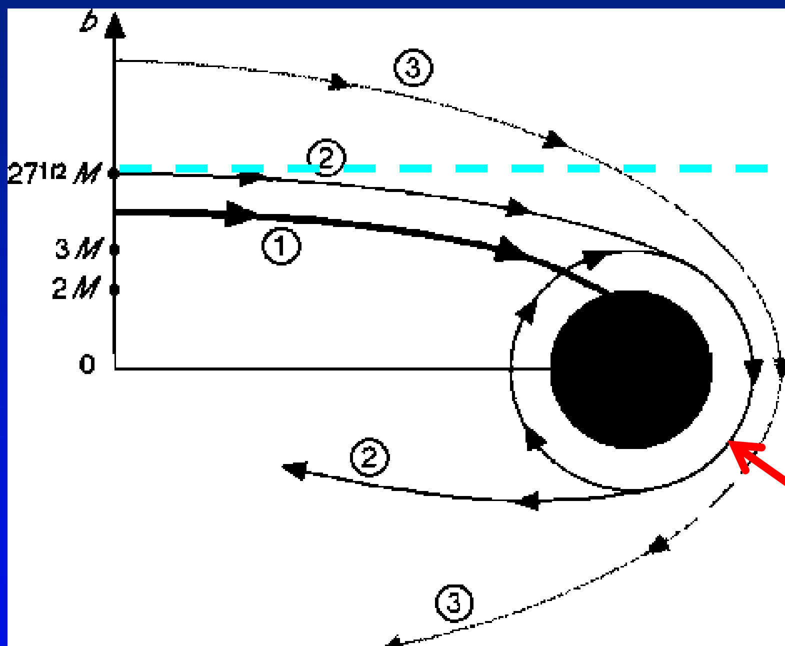
$$\frac{dr}{dt} = c\left(1 - \frac{r_g}{r}\right)$$

- The distant observer would find that light has slowed down!
- But the "physical velocity" is given by

$$\frac{dx}{d\tau} = -\frac{\sqrt{|g_{11}|}}{\sqrt{g_{00}}}\frac{dr}{dt} = c$$

- Locally, Special Relativity must always prevail!

Motion of light around a Schwarzschild BH



Critical "impact parameter"

$$\frac{3\sqrt{3}}{2} R_g$$

$$\frac{3}{2} R_g$$

Inside the horizon

- The Schw. Metric cannot be extended inside the horizon.

$$\left(1-\frac{2GM}{rc^2}\right)<0$$

- The dt^2 term becomes NEGATIVE, and dr^2 term becomes POSITIVE.
- t-coordinate becomes SPACELIKE. r-cord becomes TIMELIKE.
- r = constant line cannot serve as the 'radial coordinate'.
- **The only TIMELIKE trajectories are those with decreasing r.**

The Lemaitre coordinates (1933)

Choose co-ords { R, τ, θ, ϕ }

$$r = \left(\frac{3}{2}(R - c\tau)\right)^{\frac{2}{3}} r_g^{\frac{1}{3}}$$

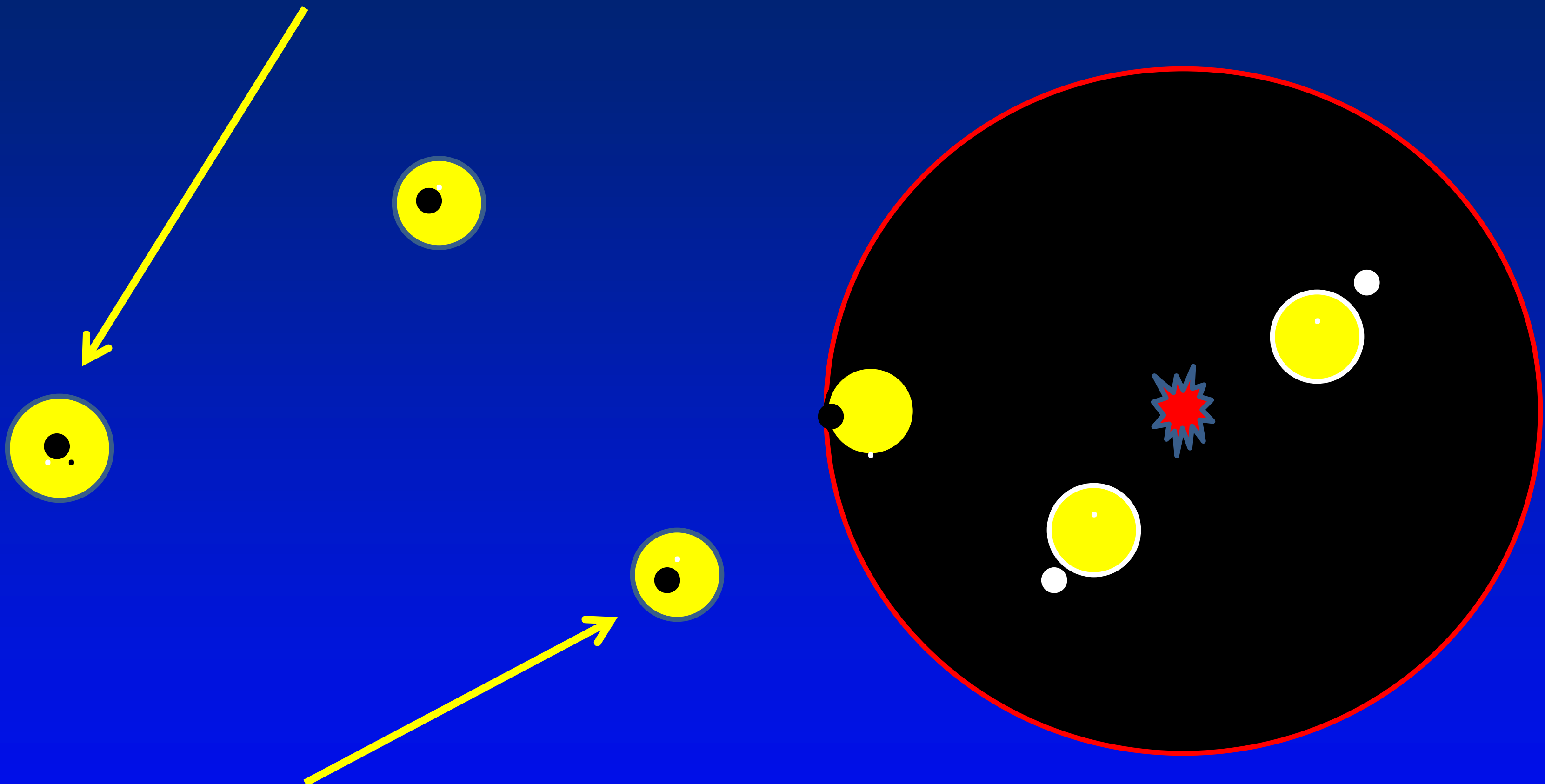
$$\boxed{ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{\frac{2}{3}}} - \left[\frac{3}{2}(R - c\tau)\right]^{\frac{4}{3}} r_g^{\frac{2}{3}} (d\theta^2 + \sin^2\theta\, d\phi^2)}$$

In these coords, there is no singularity at r_g :

$$r_g = \frac{3}{2}(R - c\tau)$$

τ is timelike and R is spacelike everywhere. The metric is not stationary; τ enters explicitly.

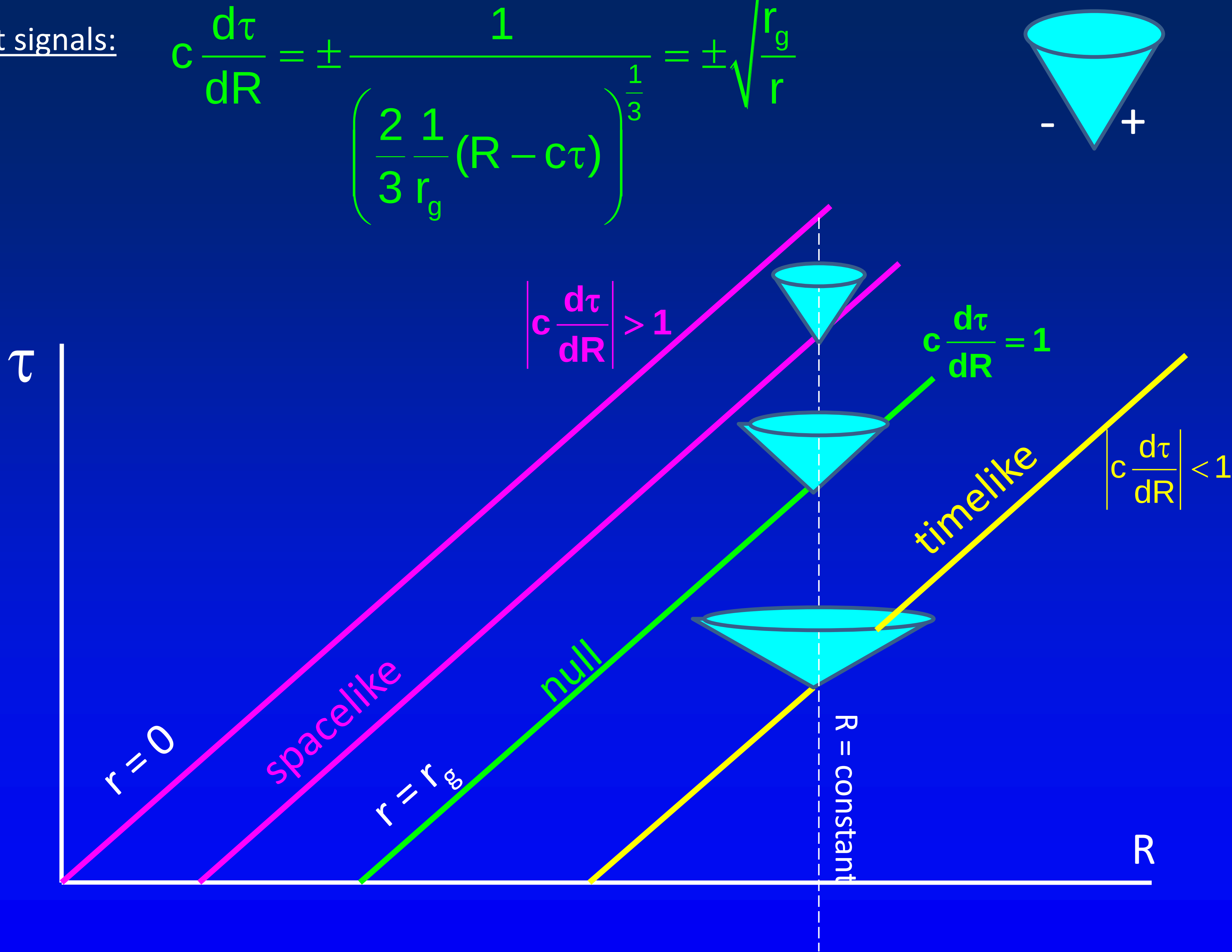
At large distances, the point of emission lies at the centre of the expanding spherical wavefront surface.

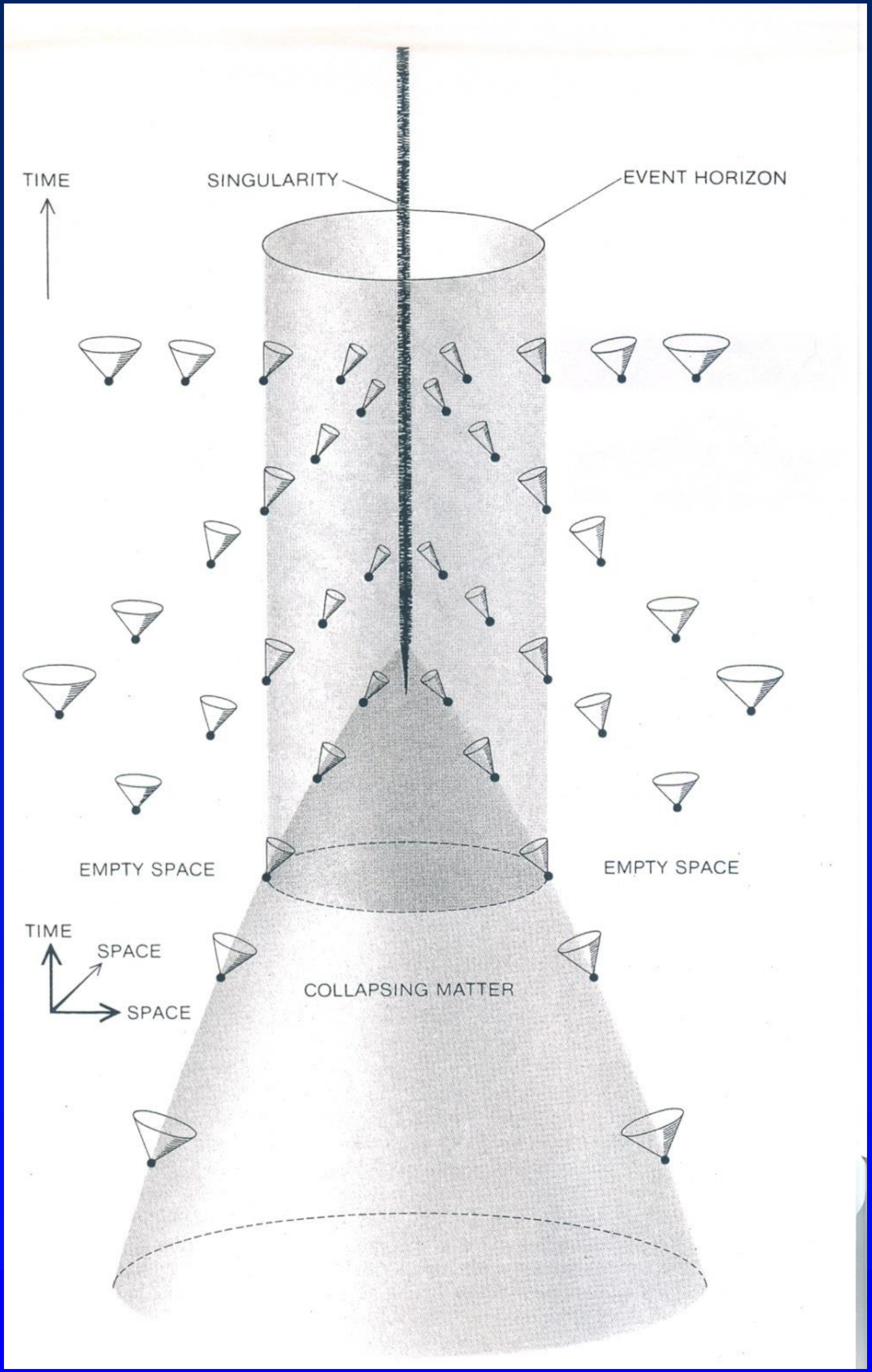


At short distances from the event horizon, the wavefront surface is displaced by the strong gravity. At precisely the horizon, the spherical wavefront surface touches the horizon INTERNALLY. Inside, the expanding wavefront detaches itself from the point of emission. Light emitted in any direction is pulled towards the central singularity.

Light signals:

$$c\frac{d\tau}{dR} = \pm\frac{1}{\left(\frac{2}{3}\frac{1}{r_g}(R - c\tau)\right)^{\frac{1}{3}}} = \pm\sqrt{\frac{r_g}{r}}$$





Roger Penrose