Wave Zone Physics: Radiative losses

Balance equations

$$\frac{dE}{dt} = -c \oint_\infty (-g) t^{0k}_{\rm LL}\, dS_k$$

$$\frac{dP^j}{dt} = - \oint_\infty (-g) t^{jk}_{\rm LL}\, dS_k$$

$$\frac{dJ^{jk}}{dt} = - \oint_\infty \left[x^j (-g) t^{kn}_{\rm LL} - x^k (-g) t^{jn}_{\rm LL} \right] dS_n$$

Tools:

$$(-g)t^{\alpha\beta}_{\rm LL} := \frac{c^4}{16\pi G} \Big\{ \partial_\lambda \mathfrak{g}^{\alpha\beta} \partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda} \partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu} \partial_\rho \mathfrak{g}^{\lambda\nu} \partial_\nu \mathfrak{g}^{\mu\rho}$$
$$- g^{\alpha\lambda} g_{\mu\nu} \partial_\rho \mathfrak{g}^{\beta\nu} \partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda} g_{\mu\nu} \partial_\rho \mathfrak{g}^{\alpha\nu} \partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu} g^{\nu\rho} \partial_\nu \mathfrak{g}^{\alpha\lambda} \partial_\rho \mathfrak{g}^{\beta\mu}$$
$$+ \frac{1}{8} \big(2 g^{\alpha\lambda} g^{\beta\mu} - g^{\alpha\beta} g^{\lambda\mu} \big) \big(2 g_{\nu\rho} g_{\sigma\tau} - g_{\rho\sigma} g_{\nu\tau} \big) \partial_\lambda \mathfrak{g}^{\nu\tau} \partial_\mu \mathfrak{g}^{\rho\sigma} \Big\}$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$$

$$\partial_j h^{\alpha\beta} = -\frac{1}{c} N_j \partial_\tau h^{\alpha\beta} + O(R^{-2})$$

$$\partial_\beta h^{\alpha\beta} = 0 = \frac{1}{c} \left(\partial_\tau h^{\alpha 0} - N_j \partial_\tau h^{\alpha j} \right) + O(R^{-2})$$

$$\langle\langle n^j \rangle\rangle = 0\,,$$

$$\langle\langle n^j n^k \rangle\rangle = \frac{1}{3} \delta^{jk}\,,$$

$$\langle\langle n^j n^k n^n \rangle\rangle = 0\,,$$

$$\langle\langle n^j n^k n^n n^p \rangle\rangle = \frac{1}{15} \left(\delta^{jk}\delta^{np} + \delta^{jn}\delta^{kp} + \delta^{jp}\delta^{kn} \right)$$

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Wave Zone Physics: Energy flux

$$
\begin{aligned}
\frac{dE}{dt} &= c \int \partial_0 \tau^{00} d^3x \\
&= -c \oint (-g) t^{0j}_{\mathrm{LL}} dS_j \\
&= -\frac{c^3 R^2}{16\pi G} \oint \left[(\partial_t h_+)^2 + (\partial_t h_\times)^2 \right] d\Omega \\
&= -\frac{G}{5c^5} \dddot{I}^{\langle pq \rangle} \dddot{I}^{\langle pq \rangle} + O(c^{-7})
\end{aligned}
$$

- ❑ Called the quadrupole formula for energy flux
- ❑ Also known as the "Newtonian" order contribution
- ❑ Also a flux of angular momentum d**J**/dt and of linear momentum d**P**/dt

For a 2-body system:

$$
\frac{dE}{dt} = -\frac{8}{15} \eta^2 \frac{c^3}{G} \left(\frac{Gm}{c^2 r} \right)^4 (12 v^2 - 11 \dot{r}^2)
$$

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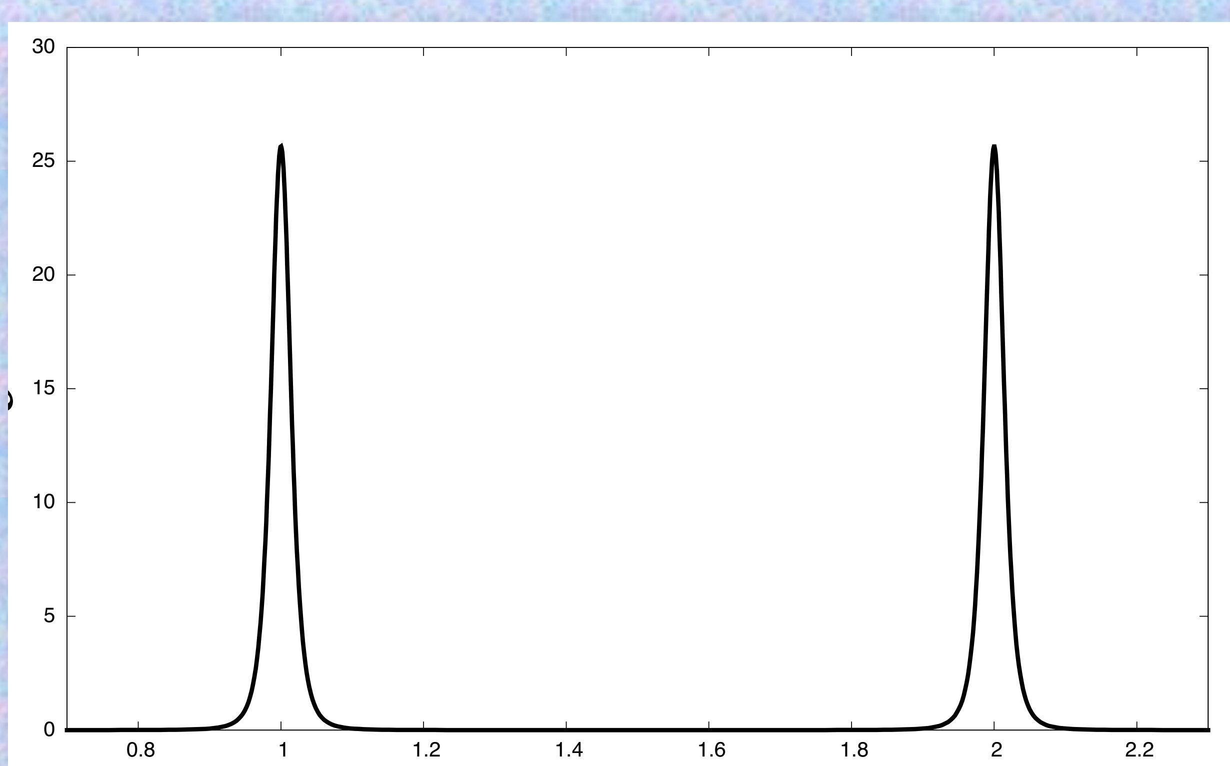
Energy flux: eccentric orbit

$$\mathcal{F} = \frac{32}{5}\eta^2 \frac{c^5}{G}\left(\frac{Gm}{c^2 p}\right)^5 (1+e\cos\phi)^4 \left[1 + 2e\cos\phi + \frac{1}{12}e^2(1+11\cos^2\phi)\right]$$

$$\frac{dE}{dt}$$



$$t/P$$

Energy flux and binary pulsars

Orbit-averaged flux

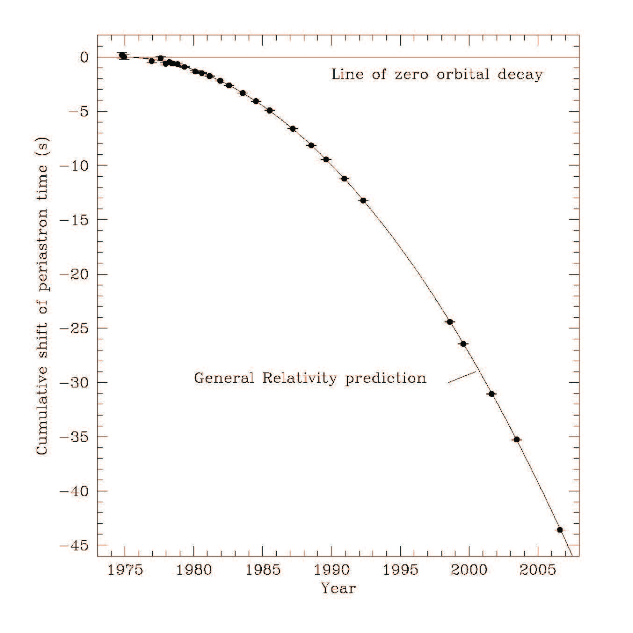
$$\frac{dE}{dt} = -\frac{32}{5}\eta^2 \frac{c^5}{G}\left(\frac{Gm}{c^2 a}\right)^5 F(e) \qquad F(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}$$

Period decrease $E \propto a^{-1} \propto P^{-2/3}$

$$\frac{dP}{dt} = -\frac{192\pi}{5}\left(\frac{G\mathcal{M}}{c^3}\frac{2\pi}{P}\right)^{5/3} F(e)$$

$$\mathcal{M} \equiv \eta^{3/5} m = \text{chirp mass}$$

- "Newtonian" GW flux
- 2.5 PN correction to Newtonian equations of motion
- PN corrections can be calculated, now reaching 4 PN order



PSR 1913+16
Hulse-Taylor binary pulsar

Energy flux and GW interferometers

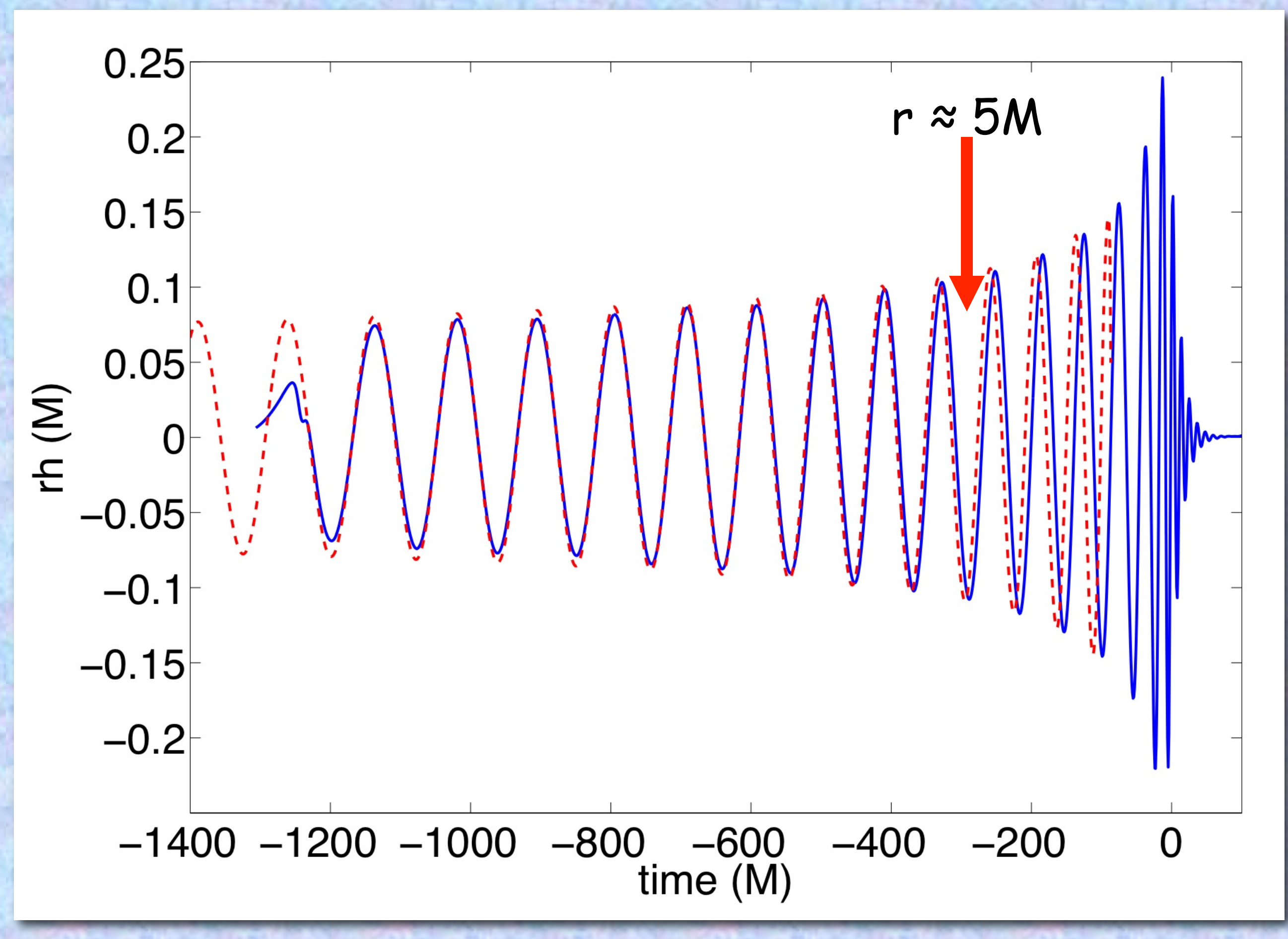
For a circular orbit, to 3.5 PN order:

$$\nu = \eta = M_1 M_2/(M_1 + M_2)^2 \qquad x = \beta^{2/3} = (G m \Omega/c^3)^{2/3} \sim (v/c)^2$$

$$\begin{aligned}
\frac{dE}{dt} = -\frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 &+ \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2} \\
&+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
&+ \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_{\mathrm{E}} - \frac{856}{105}\ln(16x) \right. \\
&\qquad \left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3 \\
&+ \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.
\end{aligned}$$

From Blanchet, *Living Reviews in Relativity* 17, 2 (2014)

Energy flux and GW interferometers



Baker *et al.* gr-qc/0612024

Wave Zone Physics: Momentum flux

$$\frac{dP^j}{dt} = -\frac{c^2 R^2}{32\pi G}\int N^j \left[\left(\partial_\tau h_+\right)^2 + \left(\partial_\tau h_\times\right)^2 \right] d\Omega$$

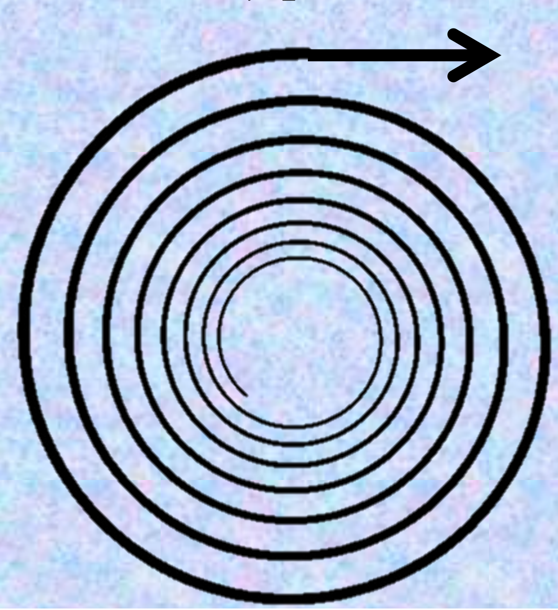
$$= -\frac{G}{c^7}\left(\frac{2}{63} \overset{(4)}{I}{}^{\langle jpq\rangle} \dddot{I}^{\langle pq\rangle} - \frac{16}{45}\epsilon^j{}_{pq} \dddot{J}^{\langle pr\rangle} \dddot{I}^{\langle qr\rangle} \right)$$

$$\Delta = \frac{m_1 - m_2}{m_1 + m_2}$$

$$\eta = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$\frac{dP^j}{dt} = \frac{8}{105}\Delta\eta^2 \frac{c}{G}\left(\frac{Gm}{c^2 r}\right)^4 \left[v^j \left(50v^2 - 38\dot{r}^2 + 8\frac{Gm}{r} \right) - \dot{r} n^j \left(55v^2 - 45\dot{r}^2 + 12\frac{Gm}{r} \right) \right]$$

- Note: recoil is opposite to velocity of lightest particle
- Final v is that when flux turns off (eg merger of 2 BH)



Radiation of momentum and the recoil of massive black holes

General Relativity

- Interference between quadrupole and higher moments

  Peres (62), Bonnor & Rotenberg (61), Papapetrou (61), Thorne (80)
- “Newtonian effect” for binaries

  Fitchett (83), Fitchett & Detweiler (84)
- 1 PN & 2PN correction terms

  Wiseman (92), Blanchet, Qusailah & CMW (05)

J0927+2943 - 2600 km/s?
Komossa et al 2008

Astrophysics

- MBH formation by mergers is affected if BH ejected from early galaxies

  Schnittman (07)
- Ejection from dwarf galaxies or globular clusters
- Displacement from center could affect galactic core

  Merritt, Milosavljevic, Favata, Hughes & Holz (04)

  Favata, Hughes & Holz (04)

How black holes get their kicks

Favata, et al

Getting a kick out of numerical relativity

Centrella, et al

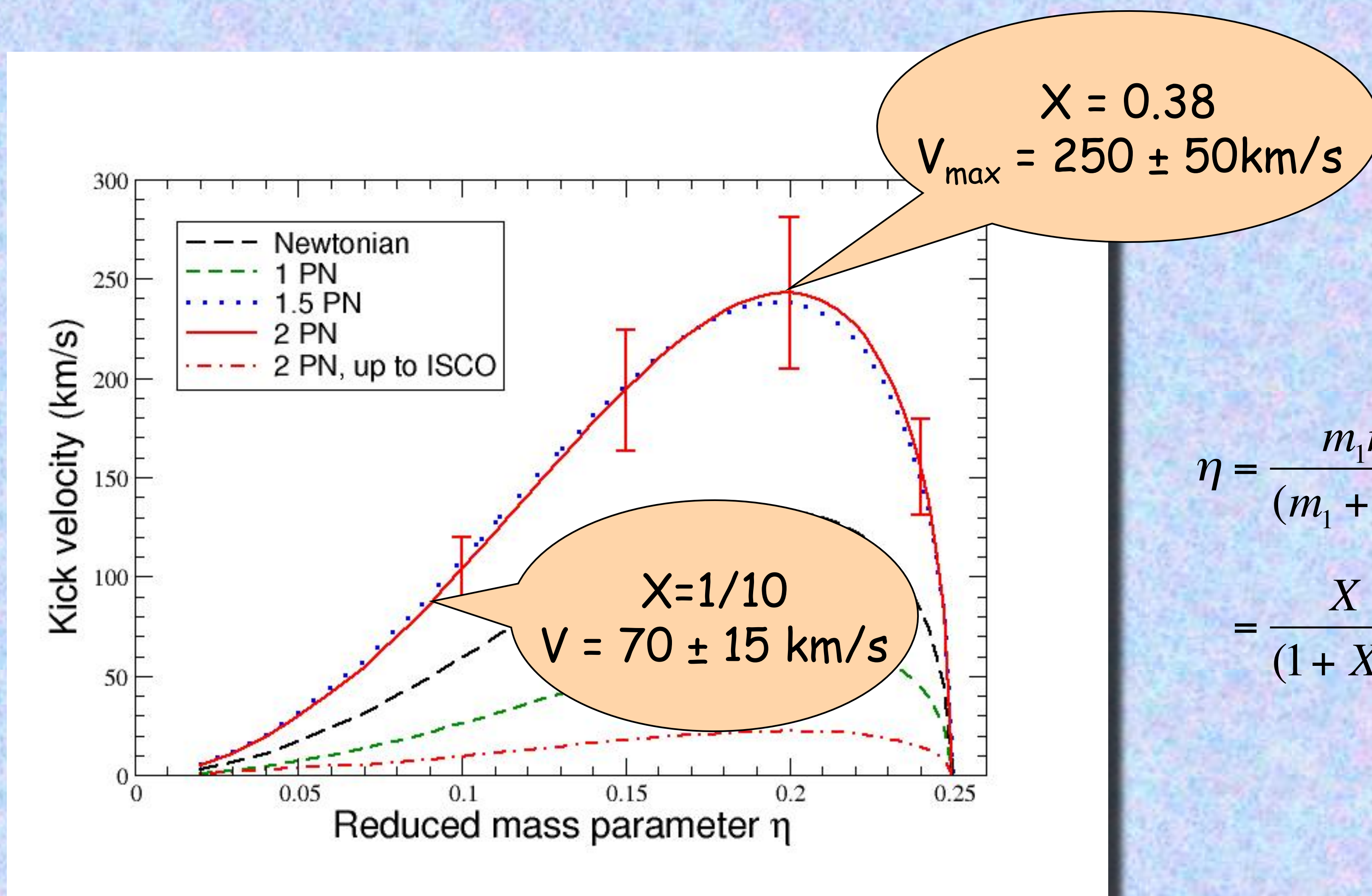
Total recoil: the maximum kick....

Gonzalez et al

A swift kick in the astrophysical compact object

Boot, Foot, et al

Recoil velocity as a function of mass ratio

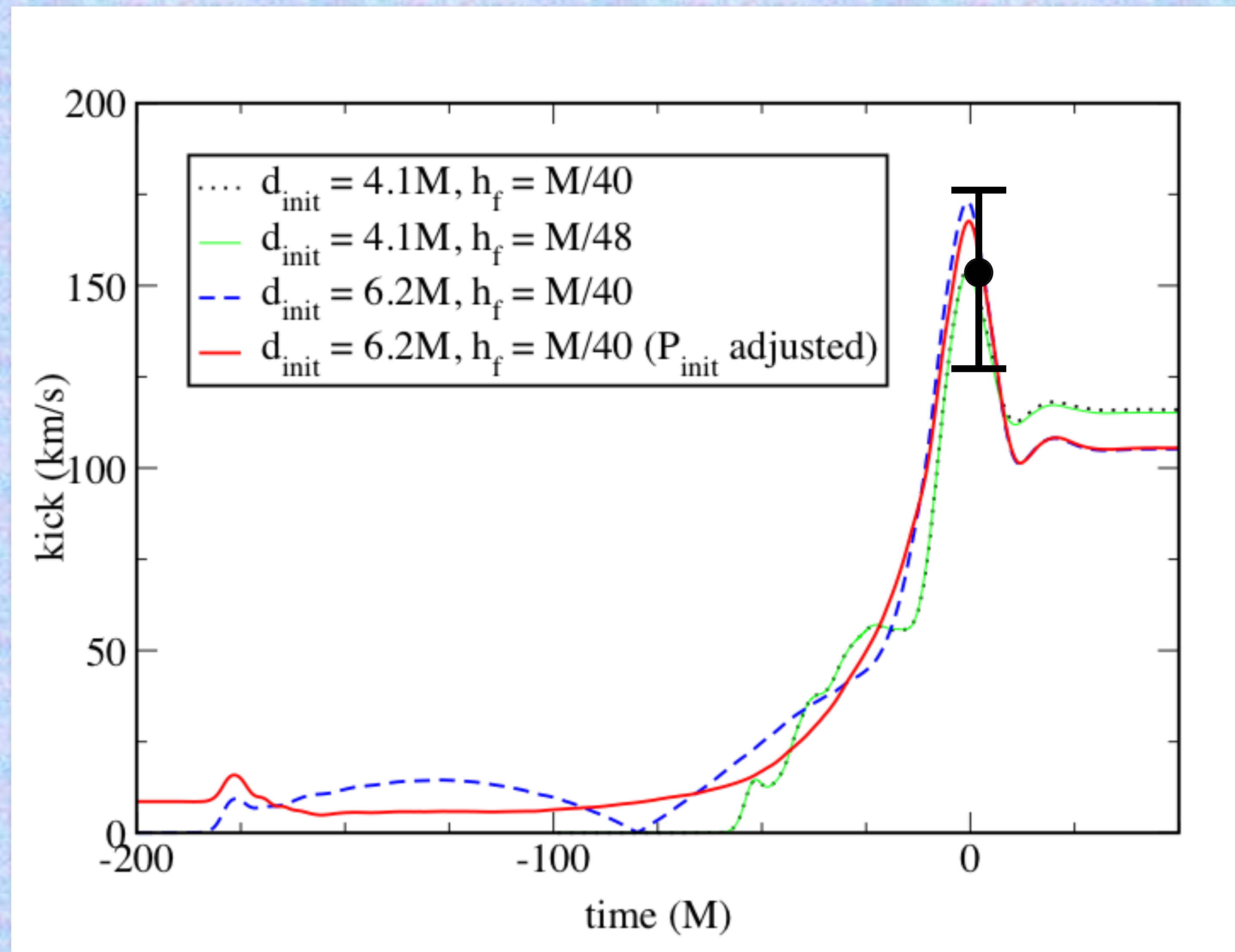


$$\eta = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$= \frac{X}{(1+X)^2}$$

Blanchet, Qusailah & CW (2005)

Comparing 2PN kick with numerical relativity



Baker *et al* (GSFC), gr-qc/0603204

Wave Zone Physics: Radiation reaction

Loss of energy at order c^{-5} implies that the dynamics of a system cannot be conservative at 2.5 PN order

There must be a **radiation reaction force** **F** that dissipates energy according to

$$\sum_A \boldsymbol{F}_A \cdot \boldsymbol{v}_A = \frac{dE}{dt}$$

To find this force, we return to the near-zone and iterate the relaxed Einstein equations **3 times** to find the metric to 2.5 PN order

That metric is inserted into the equations of motion $\nabla_\beta T^{\alpha\beta} = 0$

There are Newtonian, 1 PN, 2 PN, 2.5 PN, …. terms (no 1.5 PN!)
Happily, to find the leading 2.5 PN contributions, it is not necessary to calculate the 2 PN terms explicitly (though that has been done)

Wave Zone Physics: Radiation reaction

Newtonian plus corrections up to 2.5 PN order within τ^{00}

pand h^{00} in the near zone:

No 0.5 PN term: conservation of M

1 PN correction d^2X/dt^2

$$
\begin{aligned}
h^{00}_{\mathcal{N}} = \frac{4G}{c^4}\Bigg\{ &\int_{\mathcal{M}} \frac{\tau^{00}}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|\, d^3x' \\
&\cancel{- \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^2\, d^3x'} + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^3\, d^3x' \\
&- \frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x}'|^4\, d^3x' \Big] + O(c^{-6}) \Bigg\}
\end{aligned}
$$

2 PN term

Pure function of time - a coordinate effect

2.5 PN term

Wave Zone Physics: Radiation reaction

Pulling all the contributions together, we find the equations of hydrodynamics to 2.5 PN order

$$\rho^* \frac{d\boldsymbol{v}}{dt} = \rho^* \boldsymbol{\nabla} U - \boldsymbol{\nabla} p + O(c^{-2}) + O(c^{-4}) + \boldsymbol{f}$$

Where f is a radiation reaction force density. For body A

$$\boldsymbol{F}_A = \int_A \rho^* \boldsymbol{f} d^3x$$

For a 2-body system, this leads to a radiation-reaction contribution

$$\boldsymbol{a}[\mathrm{rr}] = \frac{8}{5}\eta \frac{(GM)^2}{c^5 r^3}\left[\left(3v^2 + \frac{17}{3}\frac{GM}{r}\right)\dot{r}\,\boldsymbol{n} - \left(v^2 + 3\frac{GM}{r}\right)\boldsymbol{v}\right]$$

This is harmonic gauge (also called Damour-Deruelle gauge)

Wave Zone Physics: Radiation reaction

Alternative gauge: the Burke-Thorne gauge. All RR effects embodied in a modification of the Newtonian potential

$$U \to U - \frac{G}{5c^5}\frac{d^5 I^{\langle jk\rangle}}{dt^5} x^j x^k$$

For a two body system

$$\boldsymbol{a}[\mathrm{rr}] = \frac{8}{5}\eta\frac{(GM)^2}{c^5 r^3}\left[\left(18v^2 + \frac{2}{3}\frac{GM}{r} - 25\dot r^2\right)\dot r\,\boldsymbol{n} - \left(6v^2 - 2\frac{GM}{r} - 15\dot r^2\right)\boldsymbol{v}\right]$$

In any gauge, orbital damping precisely matches wave-zone fluxes:

$$\begin{aligned}
\frac{dE}{dt} &= -\frac{8}{15}\eta^2\frac{c^3}{G}\left(\frac{Gm}{c^2 r}\right)^4 \left(12v^2 - 11\dot r^2\right),\\
\frac{dJ^j}{dt} &= -\frac{8}{5}\eta^2\frac{c}{G}\left(\frac{Gm}{c^2 r}\right)^3 h^j\left(2v^2 - 3\dot r^2 + 2\frac{Gm}{r}\right),\\
\frac{dP^j}{dt} &= \frac{8}{105}\Delta\eta^2\frac{c}{G}\left(\frac{Gm}{c^2 r}\right)^4\left[v^j\left(50v^2 - 38\dot r^2 + 8\frac{Gm}{r}\right)\right.\\
&\qquad\left. - \dot r n^j\left(55v^2 - 45\dot r^2 + 12\frac{Gm}{r}\right)\right]
\end{aligned}$$

Wave Zone Physics: Radiation reaction

Check dE/dt:

$$\frac{dE}{dt} = \mu \boldsymbol{v} \cdot \boldsymbol{a}[\mathrm{rr}] = \frac{8}{5}\eta^2 m \frac{(Gm)^2}{c^5 r^3}\left[\left(3v^2 + \frac{17}{3}\frac{Gm}{r}\right)\dot{r}^2 - \left(v^2 + 3\frac{Gm}{r}\right)v^2\right]$$

But

$$\frac{d}{dt}\left(\frac{v^2\dot{r}}{r^2}\right) = \frac{1}{r^3}\left(v^4 - v^2\frac{GM}{r} - 3v^2\dot{r}^2 - 2\dot{r}^2\frac{GM}{r}\right)$$

Thus

$$\frac{dE^*}{dt} = -\frac{8}{15}\eta^2\frac{c^3}{G}\left(\frac{Gm}{c^2 r}\right)^4 (12v^2 - 11\dot{r}^2)$$

$$E^* = E + \frac{8}{5}\eta\frac{G\mu m}{r}\left(\frac{Gm}{c^2 r}\frac{v^2\dot{r}}{c^3}\right)$$

Wave Zone Physics: Radiation reaction

Inserting $\mathbf{a}_{RR}$ into the Lagrange planetary equation as a disturbing force and integrating over an orbit

$p = a(1-e^2)$

$$\frac{dp}{dt} = -\frac{64}{5}\eta c\left(\frac{GM}{c^2 p}\right)^3 (1-e^2)^{3/2}\left(1+\frac{7}{8}e^2\right),$$

$$\frac{de}{dt} = -\frac{304}{15}\eta c\frac{e}{p}\left(\frac{GM}{c^2 p}\right)^3 (1-e^2)^{3/2}\left(1+\frac{121}{304}e^2\right)$$

Radiation reaction causes 2-body orbits to inspiral and circularize

$$\frac{e}{e_0} \sim \left(\frac{p}{p_0}\right)^{19/12}$$

- The Hulse-Taylor binary pulsar will circularize and merge within 300 Myr; the double pulsar within 85 Myr
- This is short compared to the age of galaxies (5 – 10 Gyr)
- There must be NS-NS binaries merging today (possibly even NS-BH and BH-BH binaries)
- The inspiral of compact binaries is a leading potential source of GW for interferometers

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