"Curved spacetime tells matter how to move"

Continuous matter, stress energy tensor

Perfect fluid: $T^{\alpha\beta} = (\rho c^2 + \epsilon + p) u^\alpha u^\beta / c^2 + p g^{\alpha\beta}$

$$j^\alpha = \rho u^\alpha$$

$$\boxed{\nabla_\beta T^{\alpha\beta} = 0\,,\ \nabla_\alpha j^\alpha = 0}$$

ρ = rest mass density
ε = energy density
p = pressure
u^α = four velocity

1st law of Thermodynamics

$$u_\alpha \nabla_\beta T^{\alpha\beta} = 0 = \frac{d\varepsilon}{d\tau} + (\varepsilon + p)\nabla \cdot \vec{u}$$

$$\boxed{d(\varepsilon \mathcal{V}) + p\, d\mathcal{V} = 0}$$

Relativistic Euler equation

$$(\mu + p)\frac{Du^\alpha}{d\tau} = -c^2\left(g^{\alpha\beta} + u^\alpha u^\beta / c^2\right)\nabla_\beta p$$

Compare with Newton

$$\rho \frac{d\boldsymbol{v}}{dt} - \boldsymbol{\nabla} U = -\boldsymbol{\nabla} p$$

"Matter tells spacetime how to curve"

Riemann tensor $$R^{\alpha}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\alpha}_{\beta\gamma}$$

Ricci tensor $$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$$

Ricci scalar $$R = g^{\alpha\beta} R_{\alpha\beta}$$

Einstein tensor $$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Bianchi identities $$\nabla_\beta G^{\alpha\beta} = 0$$

Action $$S = \frac{c^3}{16\pi G} \int \sqrt{-g} R d^4x + S_{\mathrm{matter}}$$

Einstein's equations: $$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}$$

Landau-Lifshitz Formulation of GR

Post-Newtonian and post-Minkowskian theory start with the Landau-Lifshitz formulation

Define the "gothic" metric density $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta}$

Then Einstein's equations can be written in the form

$$\boxed{\begin{aligned} \partial_{\mu\nu} H^{\alpha\mu\beta\nu} &= \frac{16\pi G}{c^4}(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right) \\ H^{\alpha\mu\beta\nu} &\equiv \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \\ t^{\alpha\beta}_{\mathrm{LL}} &\sim \partial\mathfrak{g} \cdot \partial\mathfrak{g} \end{aligned}}$$

Antisymmetry of $H^{\alpha\mu\beta\nu}$ implies the conservation equation

$$\partial_\beta \left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}}\right)\right] = 0 \quad \Longleftrightarrow \quad \nabla_\beta T^{\alpha\beta} = 0$$

Landau-Lifshitz Formulation of GR

The Landau-Lifshitz pseudotensor

$$
\begin{aligned}
(-g)t_{\rm LL}^{\alpha\beta} := \frac{c^4}{16\pi G}\Bigg\{ & \partial_\lambda \mathfrak{g}^{\alpha\beta}\partial_\mu \mathfrak{g}^{\lambda\mu} - \partial_\lambda \mathfrak{g}^{\alpha\lambda}\partial_\mu \mathfrak{g}^{\beta\mu} + \frac{1}{2} g^{\alpha\beta} g_{\lambda\mu}\partial_\rho \mathfrak{g}^{\lambda\nu}\partial_\nu \mathfrak{g}^{\mu\rho} \\
& - g^{\alpha\lambda} g_{\mu\nu}\partial_\rho \mathfrak{g}^{\beta\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} - g^{\beta\lambda} g_{\mu\nu}\partial_\rho \mathfrak{g}^{\alpha\nu}\partial_\lambda \mathfrak{g}^{\mu\rho} + g_{\lambda\mu} g^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\lambda}\partial_\rho \mathfrak{g}^{\beta\mu} \\
& + \frac{1}{8}\big(2\mathfrak{g}^{\alpha\lambda}\mathfrak{g}^{\beta\mu} - \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\lambda\mu}\big)\big(2\mathfrak{g}_{\nu\rho}\mathfrak{g}_{\sigma\tau} - \mathfrak{g}_{\rho\sigma}\mathfrak{g}_{\nu\tau}\big)\partial_\lambda \mathfrak{g}^{\nu\tau}\partial_\mu \mathfrak{g}^{\rho\sigma} \Bigg\}
\end{aligned}
$$

Landau-Lifshitz Formulation of GR

Conservation equation allows the formulation of global conservation laws:

$$E \equiv \int (-g)\left(T^{00} + t^{00}_{\rm LL}\right) d^3x$$

$$\frac{dE}{dt} = \oint (-g) t^{0j}_{\rm LL}\, d^2 S_j$$

Similar conservation laws for linear momentum, angular momentum, and motion of a center of mass, with

$$P^j \equiv \frac{1}{c} \int (-g)\left(T^{j0} + t^{j0}_{\rm LL}\right) d^3x$$

$$J^j \equiv \frac{1}{c} \epsilon^{jkl} \int (-g) x^k \left(T^{l0} + t^{l0}_{\rm LL}\right) d^3x$$

$$X^j \equiv \frac{1}{E} \int (-g) x^j \left(T^{00} + t^{00}_{\rm LL}\right) d^3x$$

The "relaxed" Einstein equations

Define potentials $\boxed{h^{\alpha\beta} \equiv \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}}$

Impose a coordinate condition (gauge): Harmonic or deDonder gauge

$$\partial_\beta h^{\alpha\beta} = 0 \qquad\qquad \Box_g x^{(\alpha)} = 0$$

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\tau^{\alpha\beta} \equiv (-g)\left(T^{\alpha\beta}[\mathsf{m}, g] + t^{\alpha\beta}_{\mathrm{LL}}[h] + t^{\alpha\beta}_{\mathrm{H}}[h]\right)$$

$$g)t^{\alpha\beta}_{\mathrm{H}} := \frac{c^4}{16\pi G}\left(\partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} - h^{\mu\nu}\partial_{\mu\nu}h^{\alpha\beta}\right)$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Matter tells spacetime how to curve

Spacetime tells matter how to move

Still equivalent to the exact Einstein equations

The "relaxed" Einstein equations

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\partial_\beta \tau^{\alpha\beta} = 0$$

Solve for h as a functional of matter variables

Solve for evolution of matter variables to give h(t,x)

Iterating the "Relaxed" Einstein Equations

Assume that $h^{\alpha\beta}$ is "small", and iterate the relaxed equation:

$$\Box h^{\alpha\beta}_{N+1} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}(h_N)$$

$$h^{\alpha\beta}_{N+1} = \frac{4G}{c^4}\int \frac{\tau^{\alpha\beta}(h_N)(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}d^3x'$$

Start with h_0 = 0 and truncate at a desired N

Yields an expansion in powers of G, called a post-Minkowskian expansion

Find the motion of matter using

$$\partial_\beta \tau^{\alpha\beta}(h_N) = 0$$

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Solving the "Relaxed" Einstein Equations

$$\Box\psi = -4\pi\mu \Longrightarrow \psi = \int_{\mathcal{C}} \frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{x'})}{|\boldsymbol{x} - \boldsymbol{x'}|} d^3x'$$

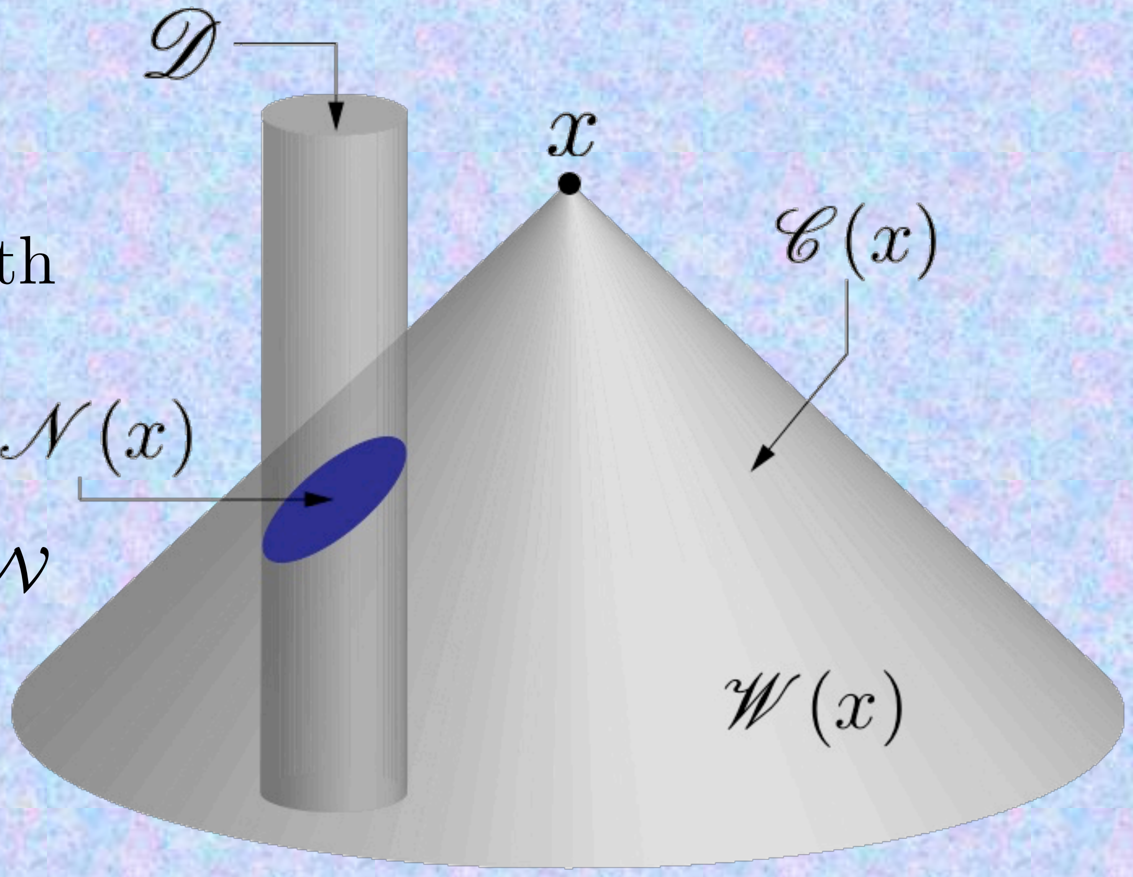
$\mathcal{N} : r' < \mathcal{R}\,,$

$\mathcal{W} : r' > \mathcal{R}$

$\mathcal{R} \sim$ wavelength

$\sim s/v$

$$\psi = \psi_{\mathcal{N}} + \psi_{\mathcal{W}}$$



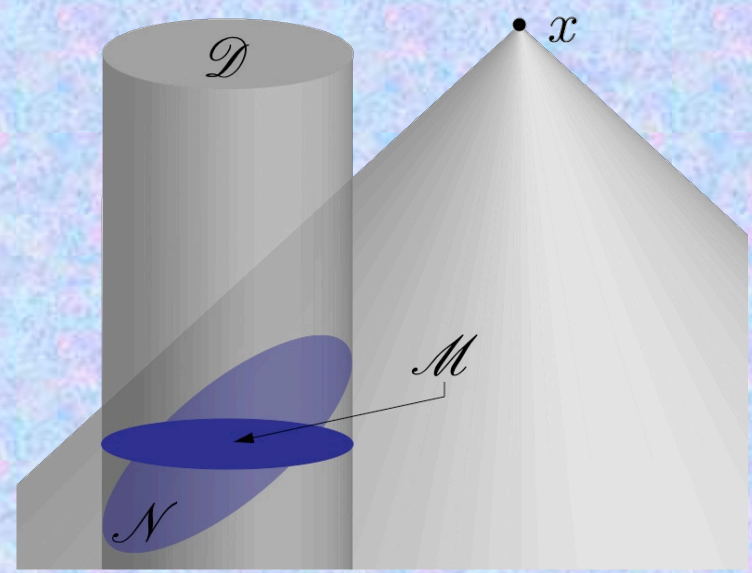
Solving the "Relaxed" Einstein Equations: Far zone

Near zone integral: $\psi_{\mathcal{N}}$

For x >> x′, Taylor expand |x-x′|

$$\frac{\mu(t-|\boldsymbol{x}-\boldsymbol{x'}|/c,\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{x'}|} = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}x'^L\partial_L\frac{\mu(t-r/c,\boldsymbol{y})}{r}.$$

$$\psi_{\mathcal{N}}(t,\boldsymbol{x}) = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\mu(\tau,\boldsymbol{x'})x'^L\,d^3x'\right]$$



A multipole expansion

$$\tau = t - R/c$$

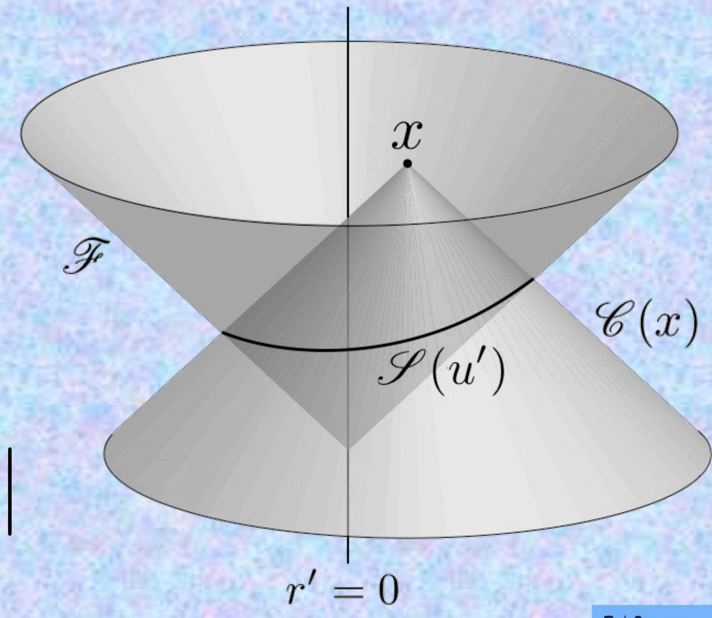
Integrals depend on $\mathcal{R}$

Solving the "Relaxed" Einstein Equations: Far zone

Far zone integral: $\psi_{\mathcal{W}}$

Since contributions to μ in the far zone come from retarded fields, they have the generic form

$$\mu \sim f(\tau', \theta', \phi')/r'^n$$

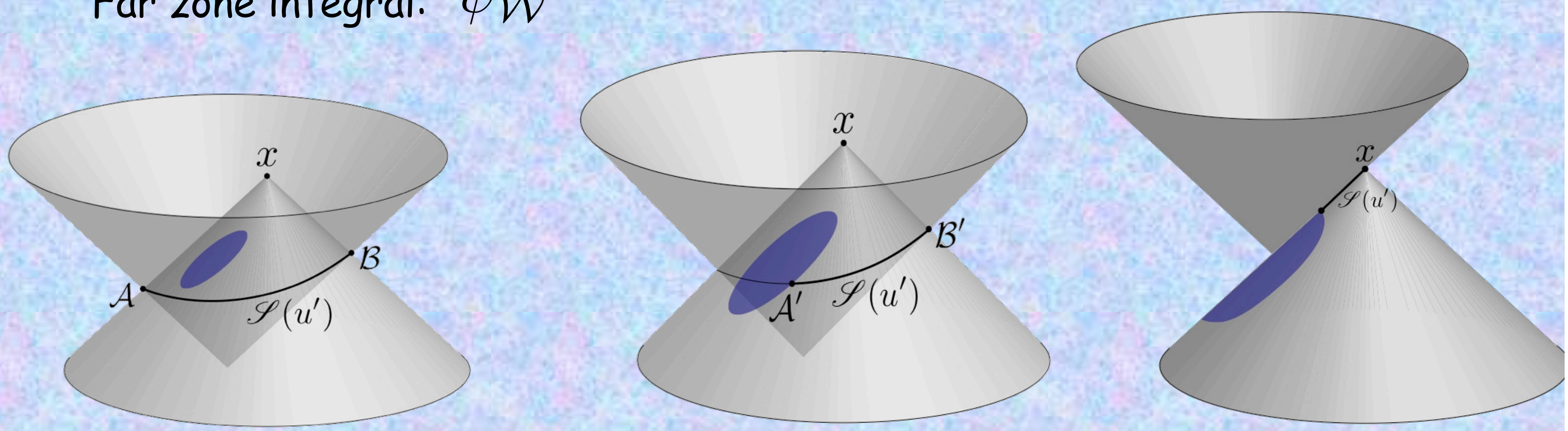


Change variables from (r′, θ′, φ′) to (u′, θ′, φ′), where u′ = cτ′ = ct′-r′

$$u' + r' = ct - |\boldsymbol{x} - \boldsymbol{x}'|$$

Solving the "Relaxed" Einstein Equations: Far zone

Far zone integral: $\psi_{\mathcal{W}}$

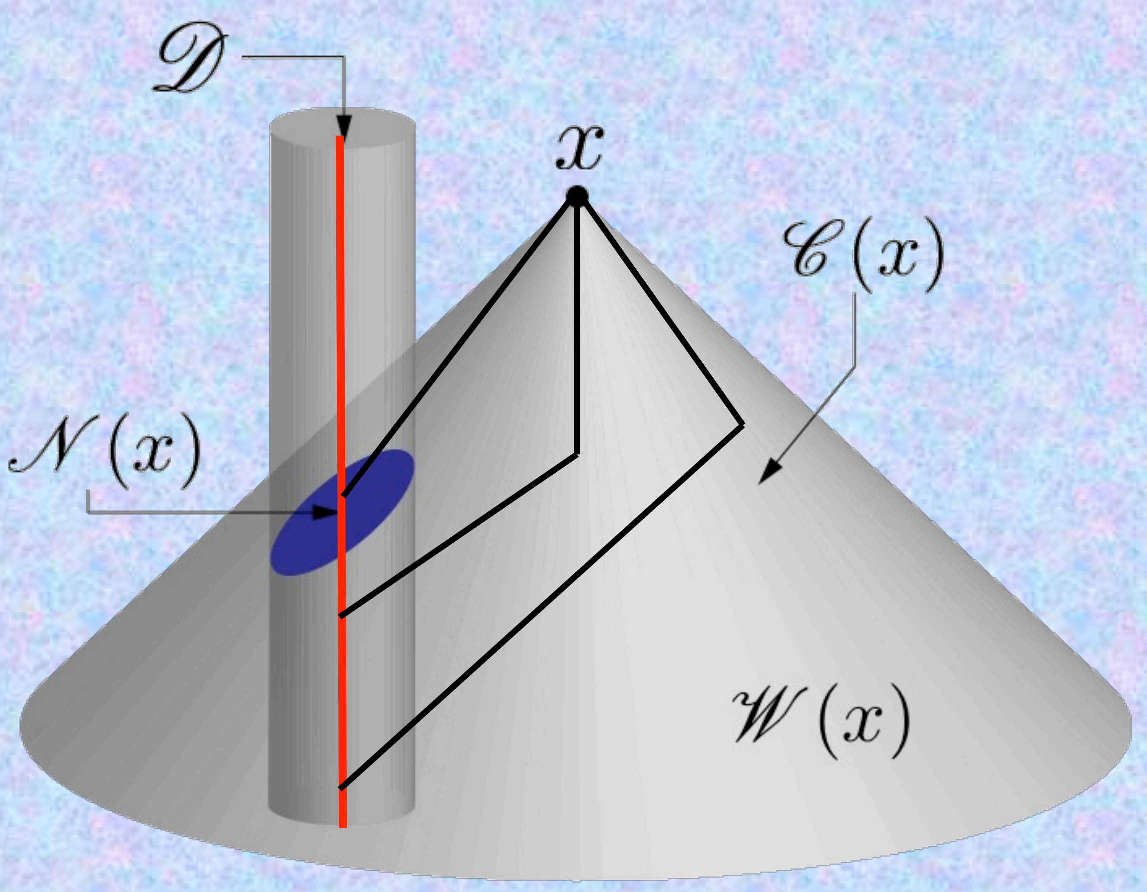


$$\psi_{\mathcal{W}} = \frac{1}{4\pi} \int_{-\infty}^{u} du' \oint_{\mathcal{S}(u')} \frac{f(u'/c, \theta', \phi')}{r'(u', \theta', \phi')^{n-2}} \frac{d\Omega'}{ct - u' - \boldsymbol{n'} \cdot \boldsymbol{x}}$$

Integral also depends on $\mathcal{R}$

But $\psi = \psi_{\mathcal{N}} + \psi_{\mathcal{W}}$ is independent of $\mathcal{R}$

Gravity as a source of gravity and gravitational "tails"



Solving the "Relaxed" Einstein Equations: Near zone

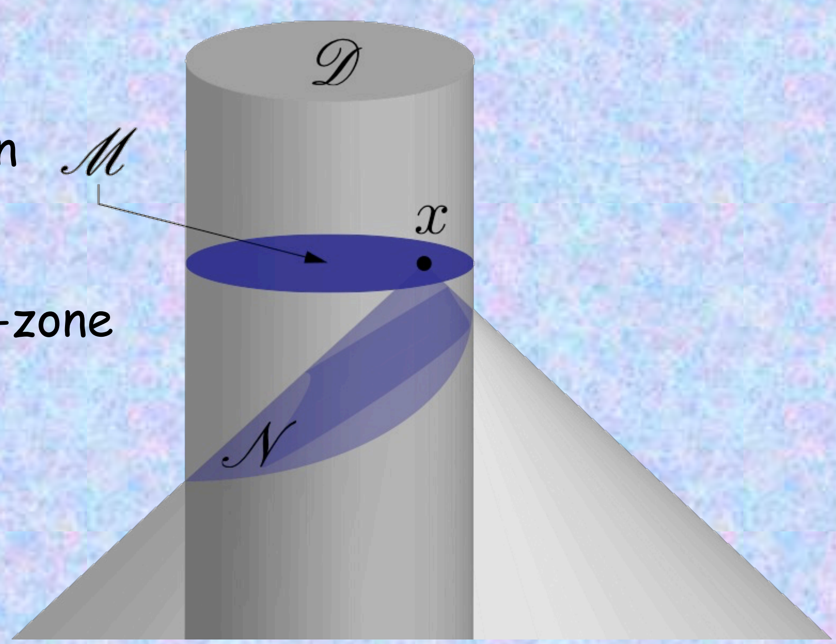
Near zone integral: $\psi_{\mathcal{N}}$

For x ~ x′, Taylor expand about t

$$\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t}\right)^\ell \mu(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^\ell$$

$$\psi_{\mathcal{N}}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell! c^\ell} \left(\frac{\partial}{\partial t}\right)^\ell \int_{\mathcal{M}} \mu(t, \boldsymbol{x'}) |\boldsymbol{x} - \boldsymbol{x'}|^{\ell-1} \, d^3x'$$

- A post-Newtonian expansion in powers of 1/c
- Instantaneous potentials
- Must also calculate the far-zone integral $\psi_{\mathcal{W}}$



Post-Newtonian approximation: Near zone

pand h^{00} in the near zone:

Newtonian plus corrections up to 2.5 PN order within τ^{00}

No 0.5 PN term: conservation of M

1 PN correction d^2X/dt^2

$$
\begin{aligned}
h^{00}_{\mathcal{N}} = \frac{4G}{c^4}\Bigg\{ \int_{\mathcal{M}} \frac{\tau^{00}}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x' + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x' \\
\cancel{- \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^2\, d^3x'} + \frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^3\, d^3x' \\
- \frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int_{\mathcal{M}} \tau^{00}|\boldsymbol{x}-\boldsymbol{x'}|^4\, d^3x' \Big] + O(c^{-6}) \Bigg\}
\end{aligned}
$$

2 PN term

Pure function of time – a coordinate effect

2.5 PN term

$$\boxed{\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \sim \epsilon}$$