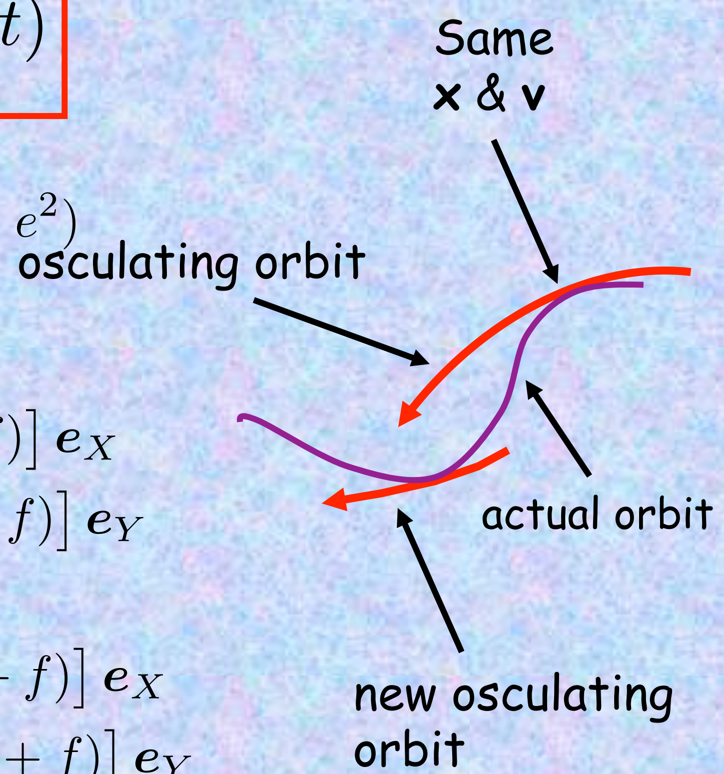
Osculating orbit elements and the perturbed Kepler problem

$$\boxed{\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} + \boldsymbol{f}(\boldsymbol{r},\boldsymbol{v},t)}$$



Define:

$$\boldsymbol{r} := r\boldsymbol{n}\,, \quad r := p/(1+e\cos f)\,, \quad p = a(1-e^2)$$

$$\boldsymbol{v} := \frac{he\sin f}{p}\boldsymbol{n} + \frac{h}{r}\boldsymbol{\lambda}, \quad h := \sqrt{Gmp}$$

$$\begin{aligned}
\boldsymbol{n} := &\left[\cos\Omega\cos(\omega+f) - \cos\iota\sin\Omega\sin(\omega+f)\right]\boldsymbol{e}_X \\
&+ \left[\sin\Omega\cos(\omega+f) + \cos\iota\cos\Omega\sin(\omega+f)\right]\boldsymbol{e}_Y \\
&+ \sin\iota\sin(\omega+f)\,\boldsymbol{e}_Z
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\lambda} := &\left[-\cos\Omega\sin(\omega+f) - \cos\iota\sin\Omega\cos(\omega+f)\right]\boldsymbol{e}_X \\
&+ \left[-\sin\Omega\sin(\omega+f) + \cos\iota\cos\Omega\cos(\omega+f)\right]\boldsymbol{e}_Y \\
&+ \sin\iota\cos(\omega+f)\,\boldsymbol{e}_Z
\end{aligned}$$

$$\hat{\boldsymbol{h}} := \boldsymbol{n}\times\boldsymbol{\lambda} = \sin\iota\sin\Omega\,\boldsymbol{e}_X - \sin\iota\cos\Omega\,\boldsymbol{e}_Y + \cos\iota\,\boldsymbol{e}_Z$$

e, a, ω, Ω, i, T may be functions of time

Perturbed Kepler problem

$$\boxed{\boldsymbol{a} = -\frac{Gm\boldsymbol{r}}{r^3} + \boldsymbol{f}(\boldsymbol{r}, \boldsymbol{v}, t)}$$

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \implies \frac{d\boldsymbol{h}}{dt} = \boldsymbol{r} \times \boldsymbol{f}$$

$$\boldsymbol{A} = \frac{\boldsymbol{v} \times \boldsymbol{h}}{Gm} - \boldsymbol{n} \implies Gm\frac{d\boldsymbol{A}}{dt} = \boldsymbol{f} \times \boldsymbol{h} + \boldsymbol{v} \times (\boldsymbol{r} \times \boldsymbol{f})$$

Decompose: $\boldsymbol{f} = \mathcal{R}\boldsymbol{n} + \mathcal{S}\boldsymbol{\lambda} + \mathcal{W}\hat{\boldsymbol{h}}$

$$\frac{d\boldsymbol{h}}{dt} = -r\mathcal{W}\,\boldsymbol{\lambda} + r\mathcal{S}\,\hat{\boldsymbol{h}}$$

$$Gm\frac{d\boldsymbol{A}}{dt} = 2h\mathcal{S}\,\boldsymbol{n} - \left(h\mathcal{R} + r\dot{r}\mathcal{S}\right)\boldsymbol{\lambda} - r\dot{r}\mathcal{W}\,\hat{\boldsymbol{h}}.$$

Example: $\dot{h} = r\mathcal{S}$

$$\frac{d}{dt}(h\cos\iota) = \dot{\boldsymbol{h}} \cdot \boldsymbol{e}_Z$$

$$\dot{h}\cos\iota - h\frac{d\iota}{dt}\sin\iota = -r\mathcal{W}\cos(\omega + f)\sin\iota + r\mathcal{S}\cos\iota$$

Perturbed Kepler problem

"Lagrange planetary equations"

$$\frac{dp}{dt} = 2\sqrt{\frac{p^3}{Gm}}\frac{1}{1+e\cos f}\,\mathcal{S},$$

$$\frac{de}{dt} = \sqrt{\frac{p}{Gm}}\left[\sin f\,\mathcal{R} + \frac{2\cos f + e(1+\cos^2 f)}{1+e\cos f}\,\mathcal{S}\right],$$

$$\frac{d\iota}{dt} = \sqrt{\frac{p}{Gm}}\frac{\cos(\omega+f)}{1+e\cos f}\,\mathcal{W},$$

$$\sin\iota\,\frac{d\Omega}{dt} = \sqrt{\frac{p}{Gm}}\frac{\sin(\omega+f)}{1+e\cos f}\,\mathcal{W},$$

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S} - e\cot\iota\frac{\sin(\omega+f)}{1+e\cos f}\,\mathcal{W}\right]$$

An alternative pericenter angle:

$$\varpi := \omega + \Omega\cos\iota$$

$$\frac{d\varpi}{dt} = \frac{1}{e}\sqrt{\frac{p}{Gm}}\left[-\cos f\,\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\,\mathcal{S}\right]$$

Perturbed Kepler problem

Comments:

- these six 1st-order ODEs are exactly equivalent to the original three 2nd-order ODEs
- if **f** = 0, the orbit elements are constants
- if **f** << Gm/r^2, use perturbation theory
- yields both periodic and secular changes in orbit elements
- can convert from d/dt to d/df using

$$\frac{df}{dt} = \left(\frac{df}{dt}\right)_{\rm Kepler} - \left(\frac{d\omega}{dt} + \cos\iota \frac{d\Omega}{dt}\right)$$

Drop if working to 1st order

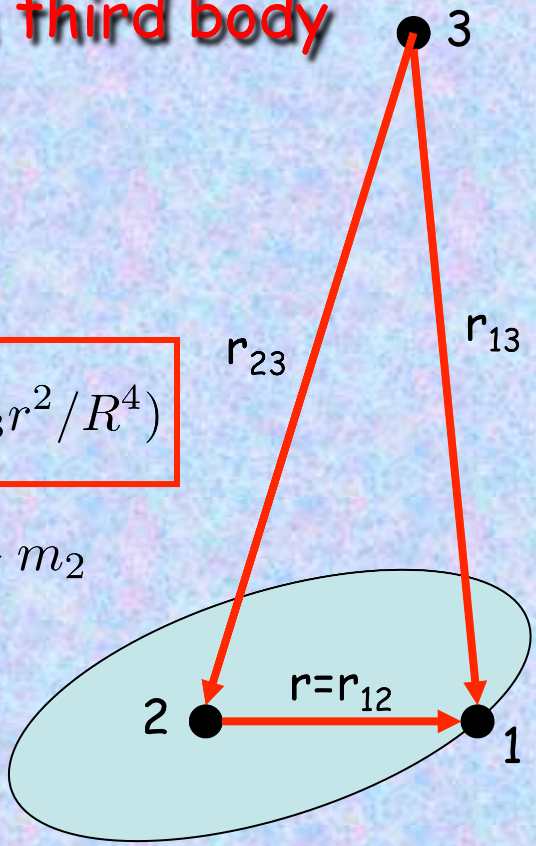
Perturbed Kepler problem

Worked example: perturbations by a third body

$$\boldsymbol{a}_1 = -Gm_2\frac{\boldsymbol{r}_{12}}{r_{12}^3} - Gm_3\frac{\boldsymbol{r}_{13}}{r_{13}^3},$$

$$\boldsymbol{a}_2 = +Gm_1\frac{\boldsymbol{r}_{12}}{r_{12}^3} - Gm_3\frac{\boldsymbol{r}_{23}}{r_{23}^3}$$

$$\boxed{\boldsymbol{a} = \frac{Gm\boldsymbol{r}}{r^3} - \frac{Gm_3 r}{R^3}\Big[\boldsymbol{n} - 3(\boldsymbol{n}\cdot\boldsymbol{N})\boldsymbol{N}\Big] + O(Gm_3 r^2/R^4)}$$



$$R := |\boldsymbol{r}_{23}|\,,\; N := \boldsymbol{r}_{23}/|\boldsymbol{r}_{23}|\,,\; m := m_1 + m_2$$

$$\mathcal{R} := \boldsymbol{f}\cdot\boldsymbol{n} = -\frac{Gm_3 r}{R^3}\big[1 - 3(\boldsymbol{n}\cdot\boldsymbol{N})^2\big],$$

$$\mathcal{S} := \boldsymbol{f}\cdot\boldsymbol{\lambda} = 3\frac{Gm_3 r}{R^3}(\boldsymbol{n}\cdot\boldsymbol{N})(\boldsymbol{\lambda}\cdot\boldsymbol{N}),$$

$$\mathcal{W} := \boldsymbol{f}\cdot\hat{\boldsymbol{h}} = 3\frac{Gm_3 r}{R^3}(\boldsymbol{n}\cdot\boldsymbol{N})(\hat{\boldsymbol{h}}\cdot\boldsymbol{N})$$

Put third body on a circular orbit

$$\boldsymbol{N} = \boldsymbol{e}_X\cos F + \boldsymbol{e}_Y\sin F\,, \qquad \frac{dF}{dt} = \sqrt{\frac{G(m+m_3)}{R^3}} \ll \frac{df}{dt}$$

Perturbed Kepler problem

Worked example: perturbations by a third body

Integrate over f from 0 to 2π holding F fixed, then average over F from 0 to 2π:

$$\langle \Delta a \rangle = 0$$

$$\langle \Delta e \rangle = \frac{15\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 e(1-e^2)^{1/2} \sin^2 \iota \sin \omega \cos \omega$$

$$\langle \Delta \omega \rangle = \frac{3\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 (1-e^2)^{-1/2} \left[5 \cos^2 \iota \sin^2 \omega + (1-e^2)(5 \cos^2 \omega - 3) \right]$$

$$\langle \Delta \iota \rangle = -\frac{15\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 e^2 (1-e^2)^{-1/2} \sin \iota \cos \iota \sin \omega \cos \omega$$

$$\langle \Delta \Omega \rangle = -\frac{3\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 (1-e^2)^{-1/2} (1 - 5e^2 \cos^2 \omega + 4e^2) \cos \iota$$

Also:

$$\langle \Delta \varpi \rangle = \frac{3\pi}{2} \frac{m_3}{m} \left(\frac{a}{R} \right)^3 (1-e^2)^{1/2} \left[1 + \sin^2 \iota (1 - 5 \sin^2 \omega) \right]$$

Perturbed Kepler problem

Worked example: perturbations by a third body

Case 1: coplanar 3rd body and Mercury's perihelion (i = 0)

$$\langle \Delta\varpi \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{1/2}$$

Planet	Semi-major axis (AU)	Orbital period (yr)	Eccentricity	Inclination to ecliptic $^\circ . ' . ''$	Inverse mass $1/M_\odot = 1$
Mercury	0.387099	0.24085	0.205628	7.0.15	6010000
Venus	0.723332	0.61521	0.006787	3.23.40	408400
Earth	1.000000	1.00004	0.016722	0.0.0	328910
Mars	1.523691	1.88089	0.093377	1.51.0	3098500
Jupiter	5.202803	11.86223	0.04845	1.18.17	1047.39
Saturn	9.53884	29.4577	0.05565	2.29.22	3498.5

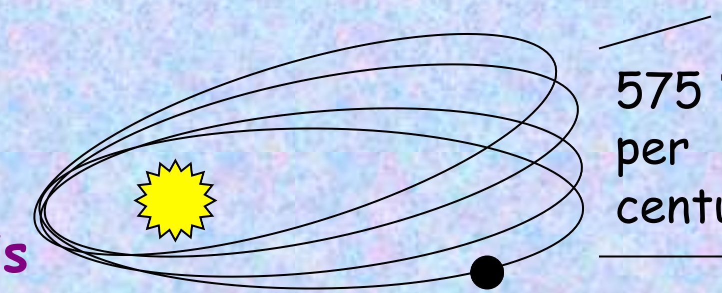
For Jupiter:
$d\varpi/dt = 154 \text{ as per century } (153.6)$

For Earth
$d\varpi/dt = 62 \text{ as per century } (90)$

FLORIDA GRAVITY

Mercury's Perihelion: Trouble to Triumph

- 1687 Newtonian triumph
- 1859 Leverrier's conundrum
- 1900 A turn-of-the century crisis



Planet	Advance
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total	531.2
Discrepancy	42.9
Modern measured value	42.98 ± 0.001
General relativity prediction	42.98

Perturbed Kepler problem

Worked example: perturbations by a third body

Case 2: the Kozai-Lidov mechanism

$$\langle \Delta a \rangle = 0$$

$$\langle \Delta e \rangle = \frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e(1-e^2)^{1/2}\sin^2\iota \sin\omega\cos\omega$$

$$\langle \Delta \omega \rangle = \frac{3\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 (1-e^2)^{-1/2}\left[5\cos^2\iota\sin^2\omega + (1-e^2)(5\cos^2\omega - 3)\right]$$

$$\langle \Delta \iota \rangle = -\frac{15\pi}{2}\frac{m_3}{m}\left(\frac{a}{R}\right)^3 e^2(1-e^2)^{-1/2}\sin\iota\cos\iota\sin\omega\cos\omega$$

Stationary point:

$$\omega_c = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$1 - e_c^2 = \frac{3}{5}\cos^2\iota_c$$

A conserved quantity:

$$\frac{e}{1-e^2}\cos\iota\langle\Delta e\rangle + \sin\iota\langle\Delta\iota\rangle = 0$$

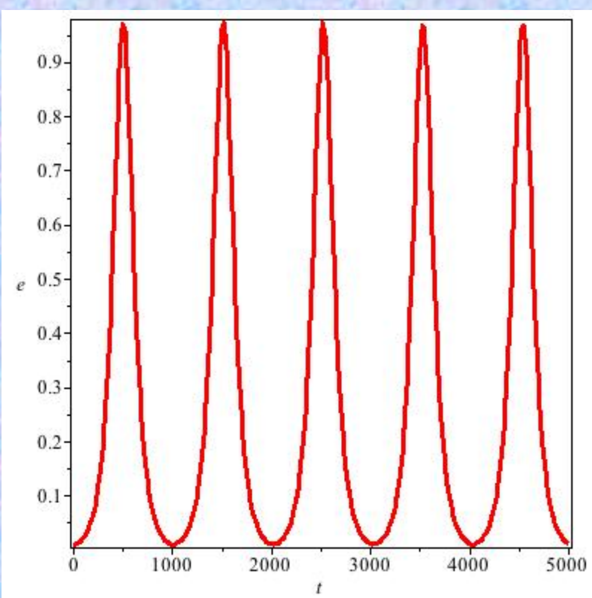
$$\Longrightarrow \sqrt{1-e^2}\cos\iota = \text{constant}$$

L_z!

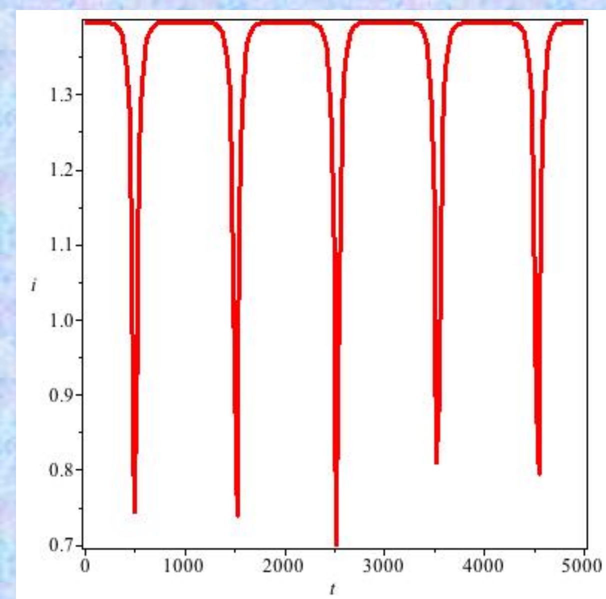
Perturbed Kepler problem

Worked example: perturbations by a third body

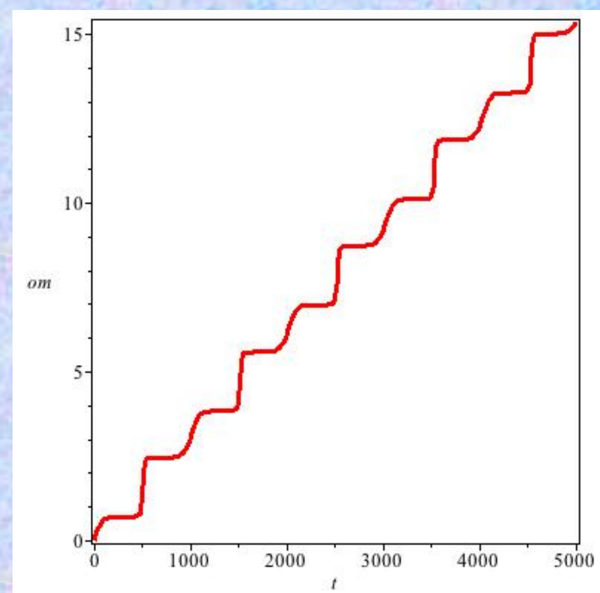
Case 2: the Kozai-Lidov mechanism



Eccentricity

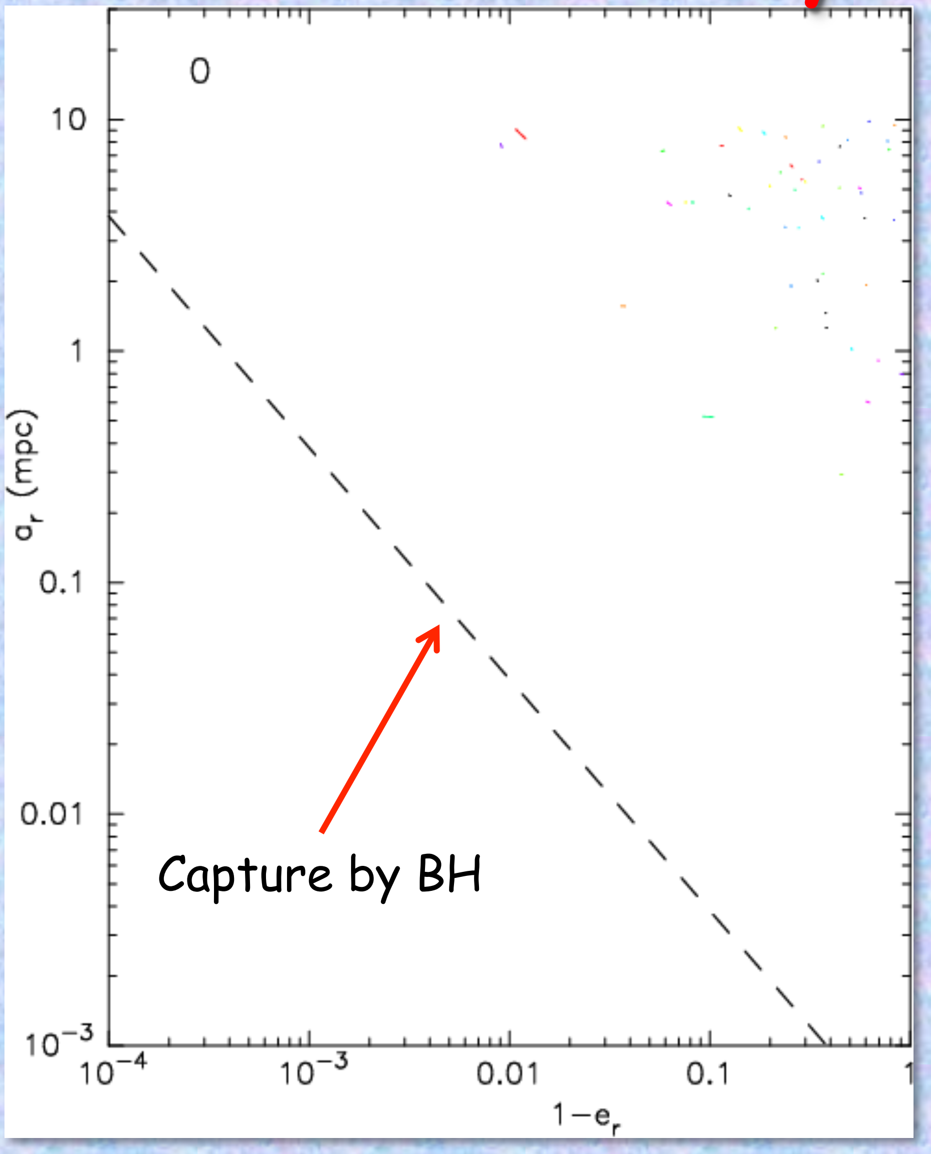


Inclination



Pericenter

Incorporating post-Newtonian effects in N-body dynamics



Merritt, Alexander, Mikkola & Will, PRD **84**, 044024 (2011)

Animations courtesy David Merritt

Perturbed Kepler problem

Worked example: body with a quadrupole moment

$$\boldsymbol{a} = \frac{Gm\boldsymbol{r}}{r^3} - \frac{3}{2}J_2\frac{GmR^2}{r^4}\left\{\left[5(\boldsymbol{e}\cdot\boldsymbol{n})^2 - 1\right]\boldsymbol{n} - 2(\boldsymbol{e}\cdot\boldsymbol{n})\boldsymbol{e}\right\},$$

$$\Delta a = 0,\ \Delta e = 0,\ \Delta\iota = 0$$

$$\Delta\omega = 6\pi J_2\left(\frac{R}{p}\right)^2\left(1 - \frac{5}{4}\sin^2\iota\right)$$

$$\Delta\Omega = -3\pi J_2\left(\frac{R}{p}\right)^2\cos\iota$$

For Mercury (J_2 = 2.2 X 10^{-7})

$$\frac{d\varpi}{dt} = 0.03\,\mathrm{as/century}$$

For Earth satellites (J_2 = 1.08 X 10^{-3})

$$\frac{d\Omega}{dt} = -3639\cos\iota\left(\frac{R}{a}\right)^{7/2}\ \mathrm{deg/yr}$$

- LAGEOS (a=1.93 R, i = $109^\circ.8$): 120 deg/yr !
- Sun synchronous: a= 1.5 R, i = 65.9