N-body equations of motion:
Worked example: 2 bodies and the perihelion shift

Components of the disturbing force

$$\mathcal{R} = \frac{Gm}{c^2r^2}\left[-(1+3\eta)v^2 + \frac{1}{2}(8-\eta)\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right],$$

$$\mathcal{S} = \frac{Gm}{c^2r^2}\left[2(2-\eta)\dot{r}(r\dot{\phi})\right],$$

$$\mathcal{W} = 0$$

Integrate the Lagrange planetary equations:

$$\Delta e = \Delta a = 0$$

$$\Delta\Omega = \Delta\iota = 0$$

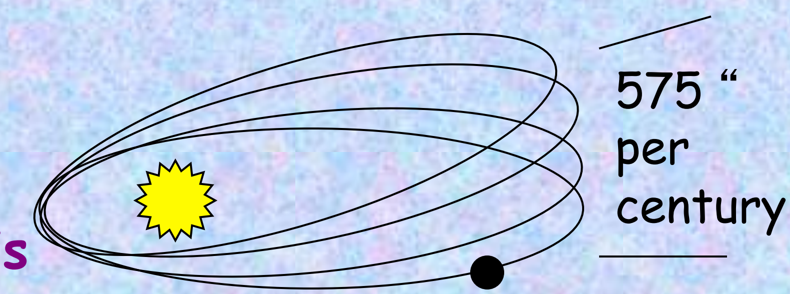
$$\Delta\omega = \frac{6\pi G(M_1+M_2)}{a(1-e^2)c^2}$$

42.98 "/c for Mercury

4.226598 °/yr for PSR 1913+16

Mercury's Perihelion: Trouble to Triumph

- 1687 Newtonian triumph
- 1859 Leverrier's conundrum
- 1900 A turn-of-the century crisis



Planet	Advance
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total	531.2
Discrepancy	42.9
Modern measured value	42.98 ± 0.001
General relativity prediction	42.98

Deflection of light

$$\boldsymbol{v} = c\left(1 - \frac{2}{c^2}U\right)\boldsymbol{n} \qquad \frac{dn^j}{dt} = \frac{2}{c}\left(\delta^{jk} - n^j n^k\right)\partial_k U$$

$$\boldsymbol{r}(t) = \boldsymbol{r}_e + c\boldsymbol{k}(t - t_e) + O(c^{-2})$$

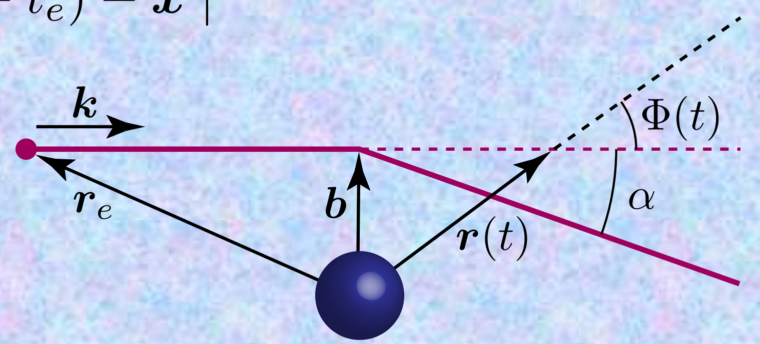
$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{\alpha} + O(c^{-4})$$

$$\boldsymbol{s}_e = \boldsymbol{r}_e - \boldsymbol{x}'$$

$$\boldsymbol{s} = \boldsymbol{r}(t) - \boldsymbol{x}'$$

$$\boldsymbol{b} = \boldsymbol{s}_e - \boldsymbol{k}(\boldsymbol{s}_e \cdot \boldsymbol{k})$$

$$\begin{aligned}\frac{d\alpha^j}{dt} &= -\frac{2G}{c}\left(\delta^{jk} - k^j k^k\right)\int \rho' \frac{r_e^k + ck^k(t - t_e) - x'}{|\boldsymbol{r}_e + c\boldsymbol{k}(t - t_e) - \boldsymbol{x}'|^3} d^3x' \\ &= -\frac{2G}{c}\int \rho' \frac{\boldsymbol{b}}{s^3} d^3x' \\ &= -\frac{2G}{c^2}\int \rho' \frac{\boldsymbol{b}}{b^2}\frac{d}{dt}\left(\frac{\boldsymbol{s}\cdot\boldsymbol{k}}{s}\right) d^3x'\end{aligned}$$



$$\boxed{\boldsymbol{\alpha}(t) = -\frac{2G}{c^2}\int \rho(t, \boldsymbol{x}')\frac{\boldsymbol{b}}{b^2}\left(\frac{\boldsymbol{s}\cdot\boldsymbol{k}}{s} - \frac{\boldsymbol{s}_e\cdot\boldsymbol{k}}{s_e}\right) d^3x'}$$

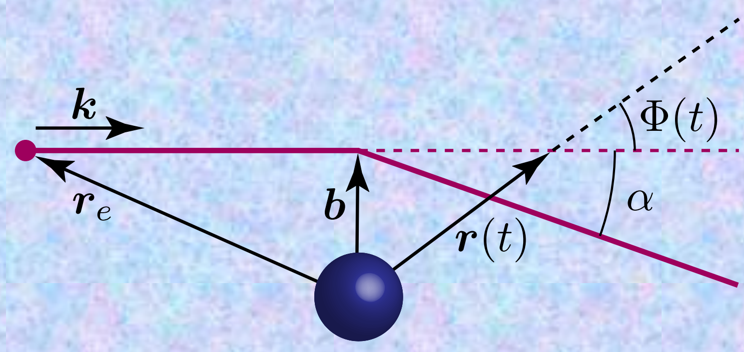
Deflection of light

For a point mass at the origin

$$\boldsymbol{\alpha}(t) = -\frac{2GM}{c^2}\frac{\boldsymbol{b}}{b^2}\left[\frac{\boldsymbol{r}(t)\cdot\boldsymbol{k}}{r(t)} - \frac{\boldsymbol{r}_e\cdot\boldsymbol{k}}{r_e}\right]$$
$$\approx -\frac{4GM}{c^2 b}\hat{\boldsymbol{b}}\left[\frac{\cos\Phi(t)+1}{2}\right]$$

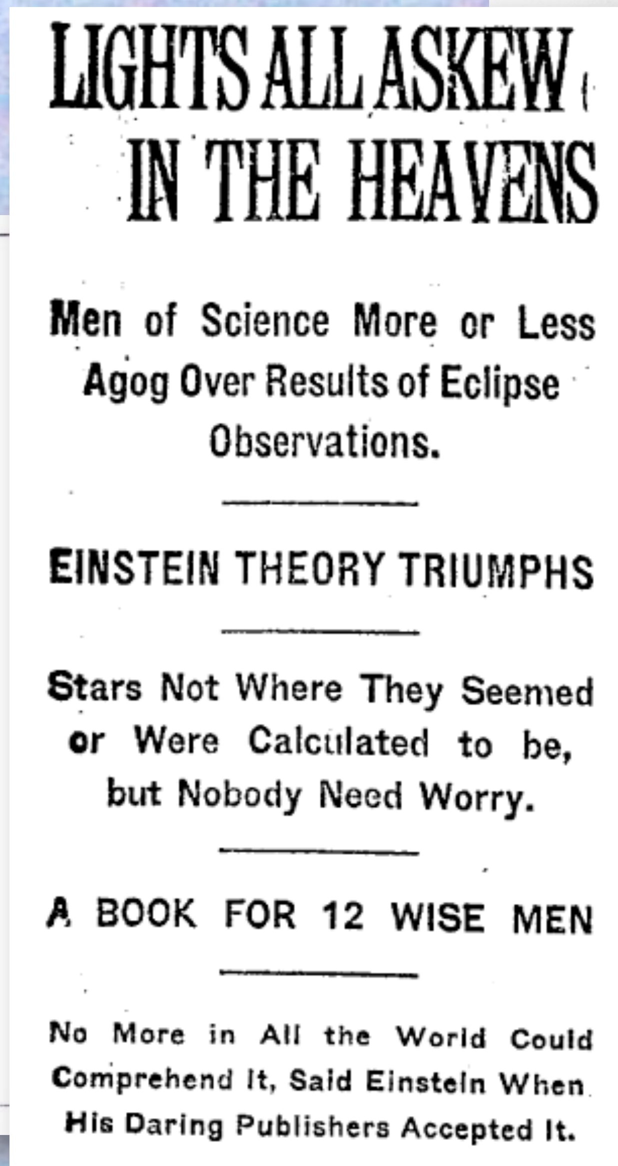
For $\Phi \approx 0$

$$\Delta\theta \approx |\boldsymbol{\alpha}|$$
$$= \frac{4GM}{c^2 b}$$
$$= 1''.7505\left(\frac{M/M_\odot}{b/R_\odot}\right)$$



Deflection of light: The 1919 Eclipse

A. S. Eddington



LIGHTS ALL ASKEW IN THE HEAVENS

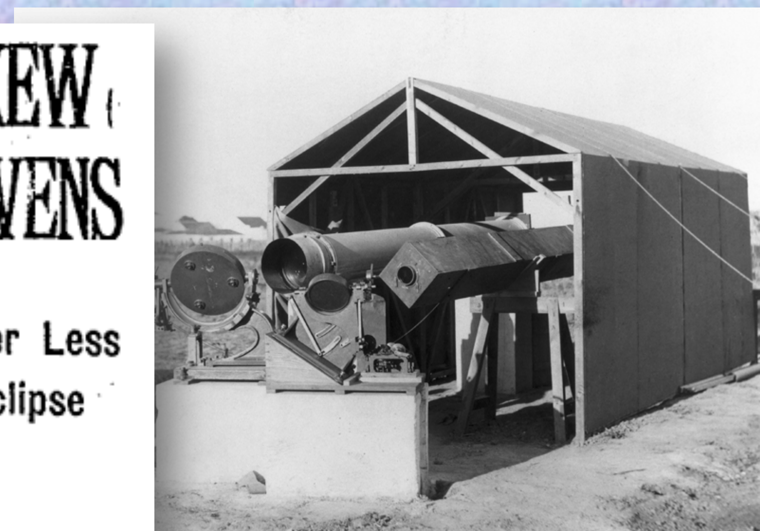
Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

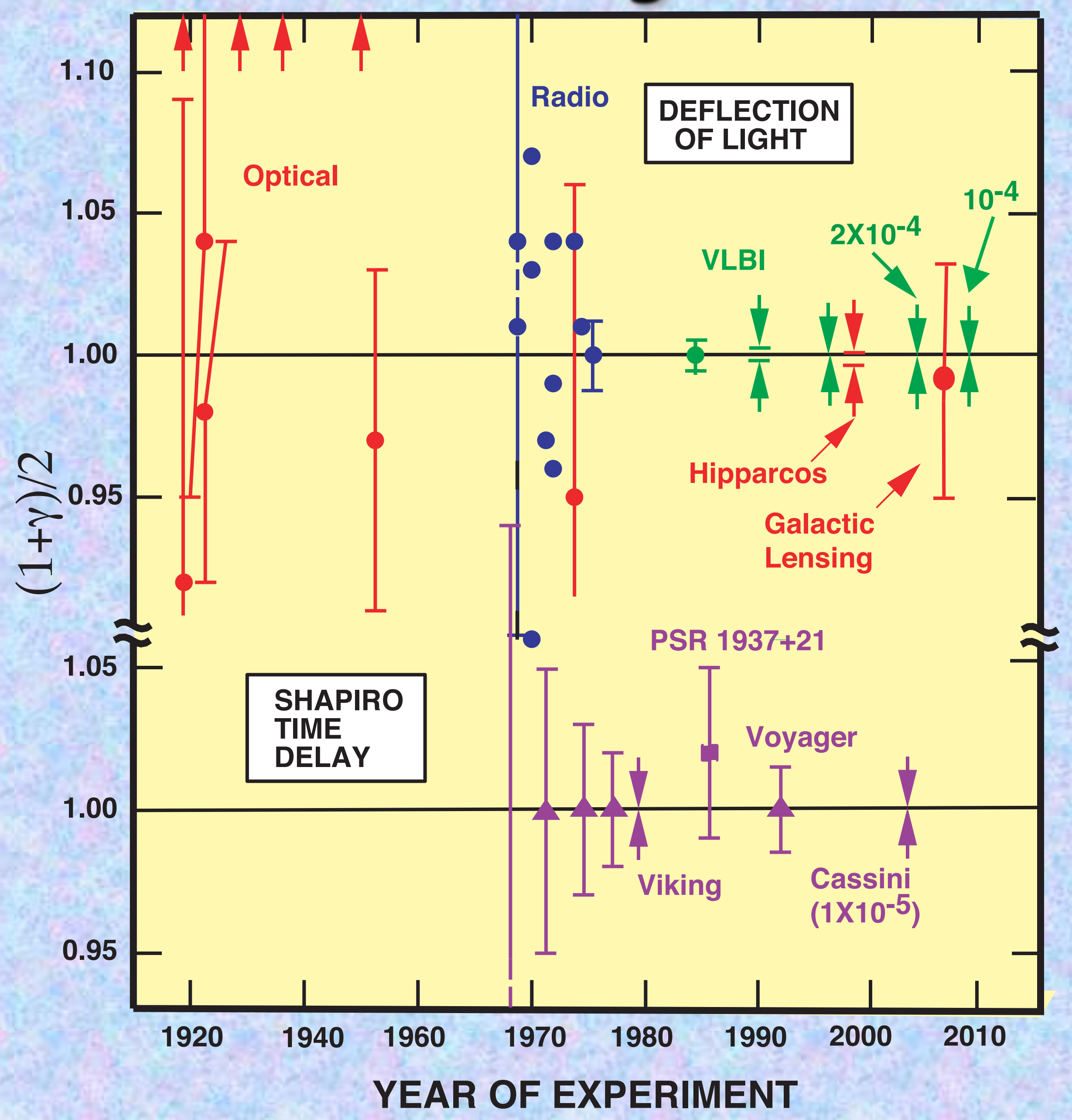
No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.



Sobral site

Photo from Principe

Deflection of light: Results

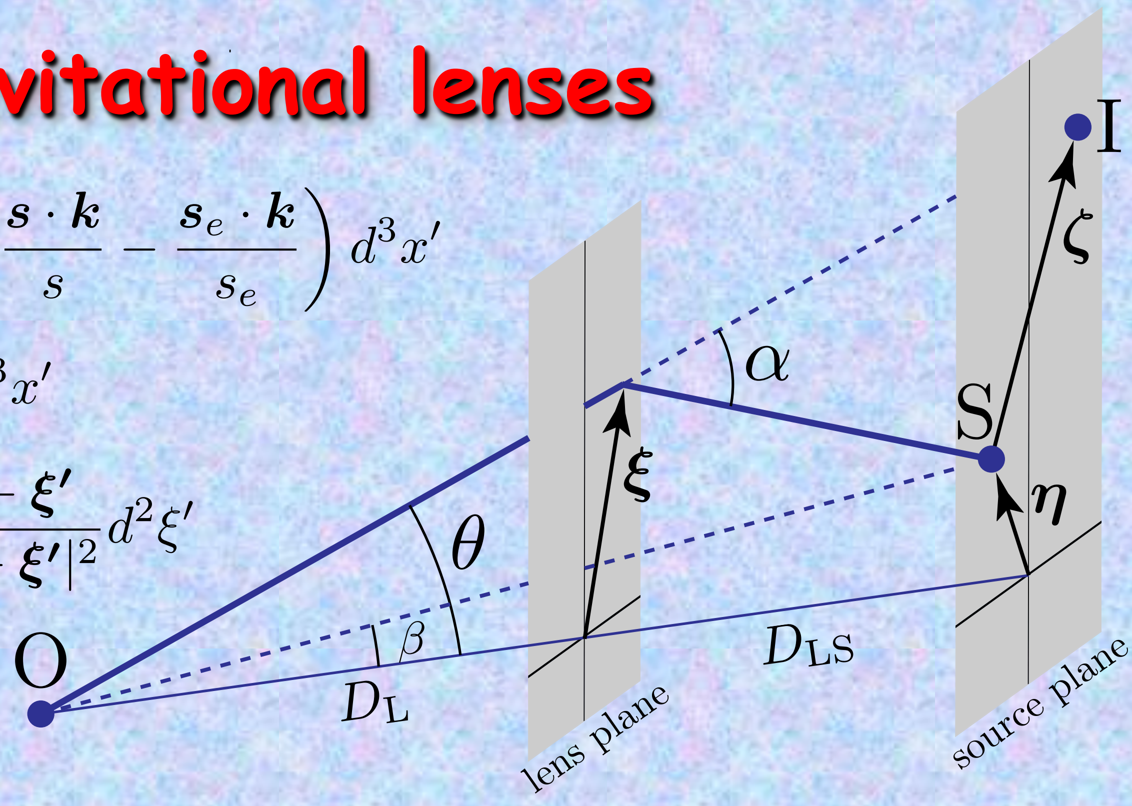


Gravitational lenses

$$\boldsymbol{\alpha}(t) = -\frac{2G}{c^2}\int \rho(t,\boldsymbol{x}')\frac{\boldsymbol{b}}{b^2}\left(\frac{\boldsymbol{s}\cdot\boldsymbol{k}}{s} - \frac{\boldsymbol{s}_e\cdot\boldsymbol{k}}{s_e}\right)d^3x'$$

$$= -\frac{4G}{c^2}\int \rho(t,\boldsymbol{x}')\frac{\boldsymbol{b}}{b^2}d^3x'$$

$$= -\frac{4G}{c^2}\int \Sigma(t,\boldsymbol{\xi}')\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}d^2\xi'$$



Lens equation

$$\boldsymbol{\theta} + \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\boldsymbol{\alpha} = \boldsymbol{\beta}$$

$$\boldsymbol{\xi} = D_{\mathrm{L}}\boldsymbol{\theta}$$

$$D_{\mathrm{S}}\boldsymbol{\theta} = \boldsymbol{\eta} + \boldsymbol{\zeta} = D_{\mathrm{S}}\boldsymbol{\beta} - D_{\mathrm{LS}}\boldsymbol{\alpha}$$

Schwarzschild lens

$$\boldsymbol{\alpha}(\boldsymbol{\xi}) = -\frac{4GM}{c^2}\frac{\boldsymbol{\xi}}{\xi^2}$$

$$\theta - \frac{\theta_{\mathrm{E}}^2}{\theta} = \beta$$

$$\theta_{\mathrm{E}}^2 := \frac{4GM}{c^2}\frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}D_{\mathrm{L}}}$$

Gravitational lenses

$$\boxed{\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\mathrm{E}}^2}\right)}$$

$$\theta_{\mathrm{E}} \sim 2\,\mathrm{as} \qquad M \sim 10^{12} M_\odot$$

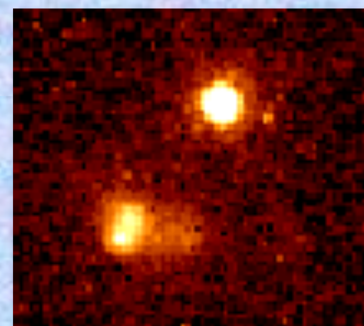
$$\sim 0.5\,\mathrm{mas} \qquad M \sim M_\odot$$

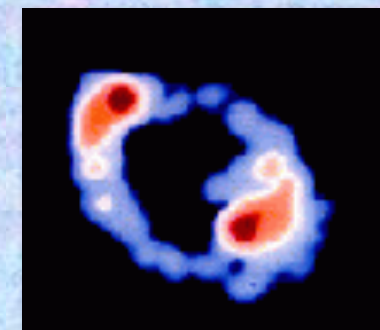
$$\beta \ll \theta_{\mathrm{E}}, \quad \theta_\pm = \pm\theta_{\mathrm{E}} + \beta/2$$

$$\beta = 0, \quad \theta_\pm = \pm\theta_{\mathrm{E}} \quad \text{Einstein ring}$$

$$\beta \gg \theta_{\mathrm{E}}, \quad \theta_\pm = \begin{cases} \beta + \theta_{\mathrm{E}}^2/\beta \\ -\theta_{\mathrm{E}}^2/\beta \end{cases}$$

Gravitational lenses





Gravitational lenses: Magnification

$$\boxed{\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_{\mathrm{E}}^2}\right)}$$

Brightness proportional to $\int \beta d\beta d\varphi$ at the source, $\int \theta d\theta d\varphi$ at the observer

$$\mu_\pm = \frac{\theta_\pm d\theta_\pm}{\beta d\beta} = \pm\frac{1}{4}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_{\mathrm{E}}^2}} + \frac{\sqrt{\beta^2 + 4\theta_{\mathrm{E}}^2}}{\beta} \pm 2\right).$$

Microlensing

$$|\mu_+| + |\mu_-| = \frac{1}{2}\left(\frac{\beta}{\sqrt{\beta^2 + 4\theta_{\mathrm{E}}^2}} + \frac{\sqrt{\beta^2 + 4\theta_{\mathrm{E}}^2}}{\beta}\right) > 1$$

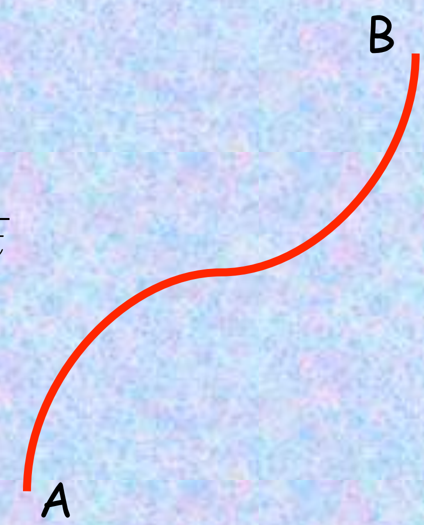
Post-Newtonian time

Clock synchronization on the Earth

$$d\tau = \left[1 - \frac{1}{c^2}\left(\frac{1}{2}\bar{v}^2 + U\right) + O(c^{-4})\right] d\bar{t}$$

$$= \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + \boldsymbol{v}\cdot(\boldsymbol{\omega}\times\boldsymbol{r}) + \Phi\right) + O(c^{-4})\right] d\bar{t}$$

$$\Phi = U + \frac{1}{2}\omega^2(x^2+y^2)$$ Geoid potential



Two clocks at rest

$$\tau_A = \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right)\bar{t}$$

$$\tau_B = \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right)\bar{t} + \tau_B^0$$

For travelling clock with v << c

$$\tau_{\rm trav} = \tau_A(\bar{t}_1) + \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right)(\bar{t}_2 - \bar{t}_1) - \frac{1}{c^2}\int_A^B (\boldsymbol{\omega}\times\boldsymbol{r})\cdot d\boldsymbol{r}$$

$$\boxed{\tau_B^0 = -\frac{1}{c^2}\int_A^B (\boldsymbol{\omega}\times\boldsymbol{r})\cdot d\boldsymbol{r}}$$

Sagnac effect:
207 ns for closed path
at the equator

Post-Newtonian time: GPS

$$
\begin{aligned}
d\tau_{\rm ground} &= \left(1 - \frac{\Phi_{\rm geoid}}{c^2}\right) d\bar{t} \\
d\tau_{\rm sat} &= \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 + U\right)\right] d\bar{t} \\
&= \left(1 - \frac{3}{2}\frac{Gm}{c^2 a}\right) d\bar{t}
\end{aligned}
$$

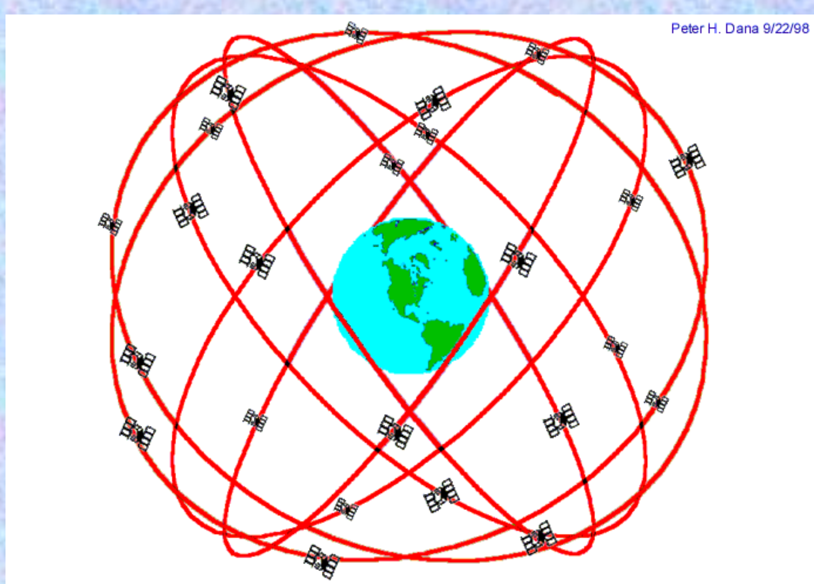
$$\boxed{d\tau_{\rm sat} \left(1 + \frac{\Phi_{\rm geoid}}{c^2} - \frac{3}{2}\frac{Gm}{c^2 a}\right) d\tau_{\rm ground}}$$

$$\sim \frac{Gm}{c^2 a_\oplus} \qquad\qquad \sim \frac{3}{2}\frac{Gm}{c^2(4.2 a_\oplus)} \quad \text{for GPS satellites}$$

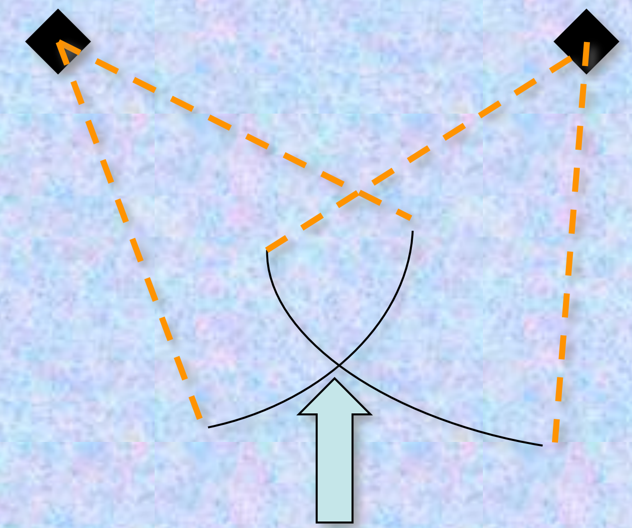
$$\sim 6.9 \times 10^{-10} \qquad\qquad \sim 2.5 \times 10^{-10}$$

General Relativity and Daily Life

The Global Positioning System (GPS)



GPS Nominal Constellation
24 Satellites in 6 Orbital Planes
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination



Navigation Requirement: 15 m ⇨ 50ns

Relativistic effects: 39,000 ns per day!

GR = 46,000 **SR = -7,000**

Relativity **must** be taken into account, for GPS to function

PN theory: Far-zone physics

Recall the wave-zone solution to $\Box\psi = -4\pi\mu$

$$\frac{\mu(t - |\boldsymbol{x} - \boldsymbol{x'}|/c, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{x'}|} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L \frac{\mu(t - r/c, \boldsymbol{y})}{r}.$$

$$\psi_{\mathcal{N}}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \int_{\mathcal{M}} \mu(\tau, \boldsymbol{x'}) x'^L \, d^3x' \right]$$

Must add the integral over the wave zone $\psi_{\mathcal{W}}$

For the field $h^{\alpha\beta}$:

$$h^{\alpha\beta}_{\mathcal{N}}(t, \boldsymbol{x}) = \frac{4G}{c^4} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \partial_L \left[\frac{1}{r} \int_{\mathcal{M}} \tau^{\alpha\beta}(\tau, \boldsymbol{x'}) x'^L \, d^3x' \right]$$

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