Near zone physics; Motion of extended fluid bodies

Matter variables:

$$\boxed{\begin{aligned} \text{rescaled mass density} &: \rho^* \equiv \rho\sqrt{-g}(u^0/c) \\ \text{proper pressure} &: p \\ \text{internal energy per unit mass} &: \Pi \\ \text{four} - \text{velocity of fluid element} &: u^\alpha = u^0(1, \boldsymbol{v}/c) \end{aligned}}$$

$$\nabla_\alpha(\rho u^\alpha) = 0 \Longleftrightarrow \frac{\partial \rho^*}{\partial t} + \boldsymbol{\nabla}(\rho^* \boldsymbol{v}) = 0$$

Slow-motion assumption v/c << 1:

$$T^{0j}/T^{00} \sim v/c\,, \qquad T^{jk}/T^{00} \sim (v/c)^2$$

$$h^{0j}/h^{00} \sim v/c\,, \qquad h^{jk}/h^{00} \sim (v/c)^2$$

Post-Newtonian approximation: Near zone

Recall the action for a geodesic

$$\boxed{\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \sim \epsilon}$$

$$
\begin{aligned}
S &= -mc^2 \int_1^2 d\tau \\
&= -mc \int_1^2 \sqrt{-g_{\alpha\beta} \frac{dr^\alpha}{dt} \frac{dr^\beta}{dt}}\, dt \\
&= -mc \int_1^2 \left(1 - \underbrace{2\frac{U}{c^2}}_{\varepsilon} - \underbrace{\delta g_{00}}_{\varepsilon^2} - \underbrace{2\frac{v^j}{c}\delta g_{0j}}_{\varepsilon^2} - \underbrace{\frac{v^2}{c^2}}_{\varepsilon} - \underbrace{\frac{v^i v^j}{c^2}\delta g_{ij}}_{\varepsilon^2}\right)^{1/2} dt
\end{aligned}
$$

We need to calculate

$$
\begin{aligned}
&\delta g_{00} \quad \text{to} \quad O(\epsilon^2) \\
&\delta g_{0j} \quad \text{to} \quad O(\epsilon^{3/2}) \\
&\delta g_{ij} \quad \text{to} \quad O(\epsilon)
\end{aligned}
$$

<u>Two</u> iterations of the relaxed equations required

Post-Newtonian approximation: Near zone

Conversion between h and g

$$(-g) = 1 - h + \frac{1}{2}h^2 - \frac{1}{2}h^{\mu\nu}h_{\mu\nu} + O(G^3)\,,$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^{\mu}{}_{\beta} - \frac{1}{2}hh_{\alpha\beta}$$
$$+ \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3)\,,$$

To 1PN order:

$$g_{00} = -1 + \frac{1}{2}h^{00} + \frac{1}{2}h^{kk} - \frac{3}{8}\left(h^{00}\right)^2 + O(c^{-6})\,,$$

$$g_{0j} = -h^{0j} + O(c^{-5})\,,$$

$$g_{jk} = \delta_{jk}\left[1 + \frac{1}{2}h^{00}\right] + O(c^{-4})\,,$$

Post-Newtonian limit of general relativity

$$\boxed{\begin{aligned}
g_{00} &= -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\psi + \frac{1}{2}\partial_{tt}X - U^2\right) + O(c^{-6})\,, \\
g_{0j} &= -\frac{4}{c^3}U_j + O(c^{-5})\,, \\
g_{jk} &= \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4})\,,
\end{aligned}}$$

$$U(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'\,,$$

$$\psi(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'\,,$$

$$X(t,\boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x'}|\, d^3x'\,,$$

$$U^j(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x'$$

Post-Newtonian Hydrodynamics

From $\nabla_\beta T^{\alpha\beta} = 0$

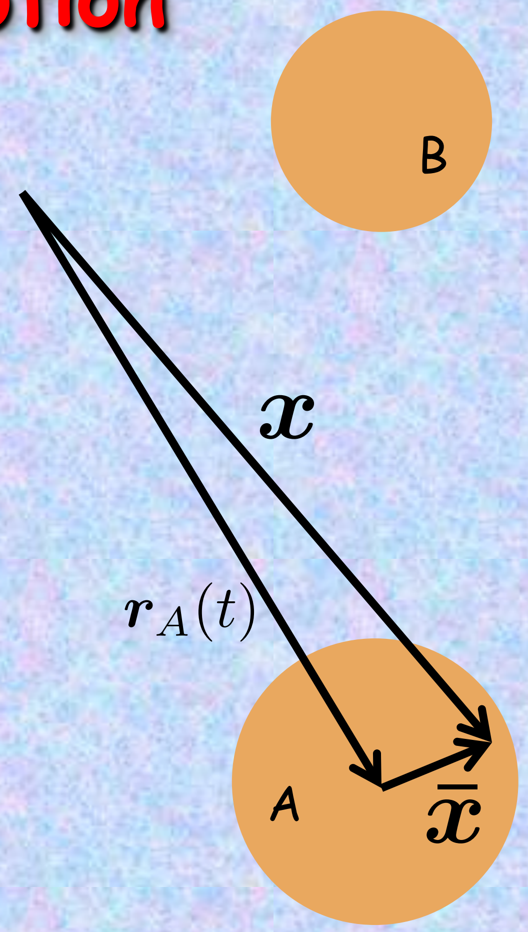
Post-Newtonian equation of hydrodynamics

$$\boxed{\begin{aligned}
\rho^* \frac{dv^j}{dt} = & -\partial_j p + \rho^* \partial_j U \\
& + \frac{1}{c^2}\left[\left(\frac{1}{2}v^2 + U + \Pi + \frac{p}{\rho^*}\right)\partial_j p - v^j \partial_t p\right] \\
& + \frac{1}{c^2}\rho^*\Big[(v^2 - 4U)\partial_j U - v^j\big(3\partial_t U + 4v^k \partial_k U\big) \\
& \qquad + 4\partial_t U_j + 4v^k\big(\partial_k U_j - \partial_j U_k\big) + \partial_j \Psi\Big] \\
& + O(c^{-4})
\end{aligned}}$$

N-body equations of motion

Main assumptions:

- Bodies small compared to typical separation (R << r)
- “isolated” -- no mass flow
- ignore contributions that scale as R^n
- assume bodies are reflection symmetric



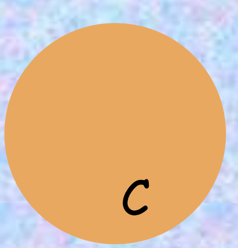
$$\text{mass}: \quad m_A \equiv \int_A \rho^* d^3x$$

$$\text{position}: \quad \boldsymbol{r}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{x} d^3x$$

$$\text{velocity}: \quad \boldsymbol{v}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{v} d^3x = \frac{d\boldsymbol{r}_A}{dt}$$

$$\text{acceleration}: \quad \boldsymbol{a}_A(t) \equiv \frac{1}{m_A}\int_A \rho^* \boldsymbol{a} d^3x = \frac{d\boldsymbol{v}_A}{dt}$$

$$\boldsymbol{x} \equiv \boldsymbol{r}_A(t) + \bar{\boldsymbol{x}}$$



N-body equations of motion

Dependence on internal structure?

$$\mathcal{T}_A \equiv \frac{1}{2}\int_A \rho^* \bar{v}^2\, d^3\bar{x}, \qquad P_A \equiv \int_A p\, d^3\bar{x}\,,$$

$$\Omega_A \equiv -\frac{1}{2}G\int_A \frac{\rho^*\rho^{*\prime}}{|\bar{\boldsymbol{x}}-\bar{\boldsymbol{x}}'|}\, d^3\bar{x}' d^3\bar{x}, \qquad E_A^{\rm int} \equiv \int_A \rho^*\Pi\, d^3\bar{x}$$

Use the virial theorem:

$$2\mathcal{T}_A + \Omega_A + 3P_A = 0$$

Then all structure integrals can be absorbed into a single "total" mass:

$$M_A \equiv m_A + \frac{1}{c^2}\left(\mathcal{T}_A + \Omega_A + E_A^{\rm int}\right) + O(c^{-4})$$

This is a manifestation of the **Strong Equivalence Principle,** satisfied by GR, but not by most alternative theories.
The motions of **all** bodies, including NS and BH, are independent of their internal structure – in GR!

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N-body equations of motion

$$
\begin{aligned}
\boldsymbol{a}_A = &-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\boldsymbol{n}_{AB} \\
&+\frac{1}{c^2}\Bigg(-\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\bigg[v_A^2-4(\boldsymbol{v}_A\cdot\boldsymbol{v}_B)+2v_B^2-\frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2 \\
&\qquad\qquad -\frac{5GM_A}{r_{AB}}-\frac{4GM_B}{r_{AB}}\bigg]\boldsymbol{n}_{AB} \\
&+\sum_{B\neq A}\frac{GM_B}{r_{AB}^2}\Big[\boldsymbol{n}_{AB}\cdot(4\boldsymbol{v}_A-3\boldsymbol{v}_B)\Big](\boldsymbol{v}_A-\boldsymbol{v}_B) \\
&+\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}^2}\bigg[\frac{4}{r_{AC}}+\frac{1}{r_{BC}}-\frac{r_{AB}}{2r_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\bigg]\boldsymbol{n}_{AB} \\
&-\frac{7}{2}\sum_{B\neq A}\sum_{C\neq A,B}\frac{G^2M_BM_C}{r_{AB}r_{BC}^2}\boldsymbol{n}_{BC}\Bigg)+O(c^{-4}).
\end{aligned}
$$

Post-Newtonian limit of general relativity

$$g_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}\left(\Psi - U^2\right) + O(c^{-6}),$$

$$g_{0j} = -\frac{4}{c^3}U_j + O(c^{-5}),$$

$$g_{jk} = \delta_{jk}\left(1 + \frac{2}{c^2}U\right) + O(c^{-4}),$$

$$\Psi = \psi + \frac{1}{2}\partial_{tt}X$$

$$U(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',$$

$$\psi(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}\left(\frac{3}{2}v'^2 - U' + \Pi' + 3p'/\rho^{*\prime}\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x',$$

$$X(t,\boldsymbol{x}) := G\int \rho^{*\prime}|\boldsymbol{x}-\boldsymbol{x}'|\,d^3x',$$

$$U^j(t,\boldsymbol{x}) := G\int \frac{\rho^{*\prime}v'^j}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'$$

Post-Newtonian geodesic equation

$$\frac{d^2 r^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dr^\beta}{d\lambda}\frac{dr^\gamma}{d\lambda} = 0$$

λ = proper time (timelike)
= affine parameter (null)

Use 0 component $d^2t/d\lambda^2$ to change from λ to t

$$\frac{dv^\alpha}{dt} = -\left(\Gamma^\alpha_{\beta\gamma} - \frac{v^\alpha}{c}\Gamma^0_{\beta\gamma}\right)v^\beta v^\gamma$$

$v^\alpha = (c, \mathbf{v})$

Timelike test body $v^2 \sim U$

$$\boxed{\begin{aligned}\frac{dv^j}{dt} = \partial_j U + \frac{1}{c^2}\Big[&(v^2 - 4U)\partial_j U - \left(4v^k\partial_k U + 3\partial_t U\right)v^j \\ &- 4v^k\left(\partial_j U_k - \partial_k U_j\right) + 4\partial_t U_j + \partial_j \Psi\Big] + O(c^{-4})\end{aligned}}$$

Post-Newtonian geodesic equation

Lightlike test body $v^2 \sim c^2$

$$ds^2 = 0 = -\left(1 - \frac{2}{c^2}U + O(c^{-4})\right)c^2dt^2 + \left(1 + \frac{2}{c^2}U + O(c^{-4})\right)v^2dt^2 - \frac{8}{c^2}U^jv_jdt^2$$

$$v^2 = c^2\left(1 - \frac{4}{c^2}U + O(c^{-3})\right) \Rightarrow \boxed{\boldsymbol{v} = c\left(1 - \frac{2}{c^2}U\right)\boldsymbol{n}}$$

Geodesic equation with v ~ c

$$\frac{dv^j}{dt} = \left(1 + \frac{v^2}{c^2}\right)\partial_j U - \frac{4}{c^2}v^jv^k\partial_k U + O(c^{-3})$$

$$\boxed{\frac{dn^j}{dt} = \frac{2}{c}(\delta^{jk} - n^jn^k)\partial_k U + O(c^{-2})}$$

These will be used for the deflection of light, Shapiro time delay and gravitational lenses

PN 2-body equations of motion

Define:

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad m \equiv M_1 + M_2$$

$$\boldsymbol{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2 \qquad \eta \equiv \frac{M_1 M_2}{(M_1 + M_2)^2}$$

$$\boldsymbol{a} \equiv \boldsymbol{a}_1 - \boldsymbol{a}_2$$

$$\boldsymbol{n} \equiv \boldsymbol{r}/r$$

$$\dot{r} \equiv dr/dt = \boldsymbol{n} \cdot \boldsymbol{v}$$

$$\boxed{\begin{aligned}\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} - \frac{Gm}{c^2 r^2}\Bigg\{\bigg[&(1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\bigg]\boldsymbol{n} \\ &- 2(2-\eta)\dot{r}\boldsymbol{v}\Bigg\} + O(c^{-4}),\end{aligned}}$$

- Geodesic equation: η = 0, more general potentials U, ψ, X
- PN 2-body equation: general masses, ignore moments

PN 2-body equations of motion: conserved quantities

Conserved energy: dot v into the equation of motion

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}v^2 &= -\frac{Gm}{r^2}\dot{r} + \frac{Gm}{c^2r^2}\dot{r}\left[(3-5\eta)v^2 + \frac{3}{2}\eta\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right] \\ &= \frac{d}{dt}\left(\frac{Gm}{r}\right) - \frac{d}{dt}\left\{\frac{Gm}{c^2r}\left[(3-4\eta)v^2 - \left(1-\frac{9}{2}\eta\right)\frac{Gm}{r} + \frac{1}{2}\eta\dot{r}^2\right]\right\}\end{aligned}$$

$$\boxed{\begin{aligned}\frac{(Gm)^2}{r^3}\dot{r} &= -\frac{1}{2}\frac{d}{dt}\left(\frac{Gm}{r}\right)^2 \\ \frac{Gm}{r^2}\dot{r}v^2 &= \frac{d}{dt}\left[\frac{Gm}{r}\left(\frac{Gm}{r} - v^2\right)\right] \\ \frac{Gm}{r^2}\dot{r}^3 &= \frac{1}{3}\frac{d}{dt}\left[\frac{Gm}{r}\left(3\frac{Gm}{r} - 2v^2 - \dot{r}^2\right)\right]\end{aligned}}$$

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PN 2-body equations of motion: conserved quantities

Conserved energy: dot v into the equation of motion

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}v^2 &= -\frac{Gm}{r^2}\dot{r} + \frac{Gm}{c^2r^2}\dot{r}\left[(3-5\eta)v^2 + \frac{3}{2}\eta\dot{r}^2 + 2(2+\eta)\frac{Gm}{r}\right] \\ &= \frac{d}{dt}\left(\frac{Gm}{r}\right) - \frac{d}{dt}\left\{\frac{Gm}{c^2r}\left[(3-4\eta)v^2 - \left(1-\frac{9}{2}\eta\right)\frac{Gm}{r} + \frac{1}{2}\eta\dot{r}^2\right]\right\} \\ &\qquad -\frac{3}{2c^2}(1-3\eta)\frac{d}{dt}\left(\frac{1}{2}v^2 - \frac{Gm}{r}\right)^2\end{aligned}$$

$$\boxed{\varepsilon := \frac{1}{2}v^2 - \frac{Gm}{r} + \frac{1}{c^2}\left\{\frac{3}{8}(1-3\eta)v^4 + \frac{Gm}{2r}\left[(3+\eta)v^2 + \eta\dot{r}^2 + \frac{Gm}{r}\right]\right\} + O(c^{-4})}$$

$$E_{\mathrm{SR}} = \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2$$

PN 2-body equations of motion: conserved quantities

Conserved angular momentum: cross x with the equation of motion

$$\frac{d}{dt}(\boldsymbol{r}\times\boldsymbol{v}) = (4-2\eta)\frac{Gm}{c^2r^2}\dot{r}(\boldsymbol{r}\times\boldsymbol{v})$$
$$= -\frac{d}{dt}\left[(4-2\eta)\frac{Gm}{c^2r}(\boldsymbol{r}\times\boldsymbol{v})\right]$$
$$-\frac{d}{dt}\left[\frac{1-3\eta}{c^2}\left(\frac{1}{2}v^2-\frac{Gm}{c^2r}\right)(\boldsymbol{r}\times\boldsymbol{v})\right]$$

$$\boxed{\boldsymbol{h} = \left\{1+\frac{1}{c^2}\left[\frac{1}{2}(1-3\eta)v^2+(3+\eta)\frac{Gm}{r}\right]\right\}(\boldsymbol{r}\times\boldsymbol{v})+O(c^{-4})}$$

$$\boldsymbol{L}_{\rm SR} = \gamma_1 m_1(\boldsymbol{x}_1\times\boldsymbol{v}_1)+\gamma_2 m_2(\boldsymbol{x}_2\times\boldsymbol{v}_2)$$

PN circular orbits

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n} - \frac{Gm}{c^2 r^2}\left\{\left[(1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\right]\boldsymbol{n} - 2(2-\eta)\dot{r}\boldsymbol{v}\right\} + O(c^{-4}),$$

$$\boxed{\begin{aligned} d\boldsymbol{n}/dt &= \dot{\phi}\boldsymbol{\lambda} \\ d\boldsymbol{\lambda}/dt &= -\dot{\phi}\boldsymbol{n} \\ \boldsymbol{v} &= \dot{r}\boldsymbol{n} + r\dot{\phi}\boldsymbol{\lambda} \\ \boldsymbol{a} &= (\ddot{r} - r\dot{\phi}^2)\boldsymbol{n} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\boldsymbol{\lambda} \end{aligned}}$$

$$\dot{r} = 0 \Longrightarrow \begin{cases} \ddot{r} = 0 \\ \dot{\phi} = \Omega = \text{const} \end{cases}$$

$$\Omega^2 = \frac{Gm}{r^3}\left[1 - (3-\eta)\frac{Gm}{c^2 r} + O(c^{-4})\right]$$

$$\varepsilon = -\frac{Gm}{2r}\left[1 - \frac{1}{4}(7-\eta)\frac{Gm}{c^2 r} + O(c^{-4})\right]$$

$$h = \sqrt{Gmr}\left[1 + 2\frac{Gm}{c^2 r} + O(c^{-4})\right]$$

Remark: test body in Schwarzschild coords (η = 0, r= r_S - Gm/c^2)

$$\Omega^2 = \frac{Gm}{r_S^3}$$