Motion of extended fluid bodies

Main assumptions:

- Bodies small compared to typical separation ($R \ll r$)
- "isolated" -- no mass flow
- $T_{int} \sim (R^3/Gm)^{1/2} \ll T_{orb} \sim (r^3/Gm)^{1/2}$ -- quasi equilibrium
- adiabatic response to tidal deformations -- nearly spherical

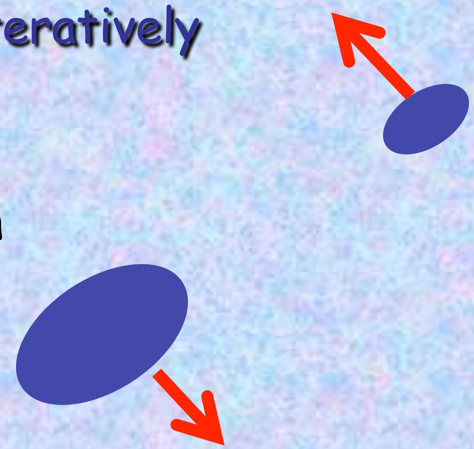
External problem:

- determine motions of bodies as functions (or functionals) of internal parameters

Internal problem:

- given motions, determine evolution of internal parameters

Solve the two problems self-consistently or iteratively



Example: Earth-Moon system -- orbital motion raises tides, tidally deformed fields affect motions

Motion of extended fluid bodies

Basic definitions

$$m_A := \int_A \rho(t,\boldsymbol{x})\, d^3x$$

$$\boldsymbol{r}_A(t) := \frac{1}{m_A}\int_A \rho(t,\boldsymbol{x})\boldsymbol{x}\, d^3x$$

$$dm_A/dt = 0$$

$$\boldsymbol{v}_A(t) := \frac{d\boldsymbol{r}_A}{dt} = \frac{1}{m_A}\int_A \rho\boldsymbol{v}\, d^3x$$

$$\boldsymbol{a}_A(t) := \frac{d\boldsymbol{v}_A}{dt} = \frac{1}{m_A}\int_A \rho\frac{d\boldsymbol{v}}{dt}\, d^3x$$

Is the center of mass unique?

- pure convenience, should not wander outside the body
- not physically measurable
- almost impossible to define in GR

$$\begin{aligned} m_A\boldsymbol{a}_A &= \cancel{-G\int_A\int_A \rho\rho'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3x d^3x'} \\ &\quad -G\int_A \rho\left[\sum_{B\neq A}\int_B \rho'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3x'\right]d^3x \end{aligned}$$

Define:

$$\boldsymbol{x} := \boldsymbol{r}_A(t) + \bar{\boldsymbol{x}}$$

$$\boldsymbol{x}' := \boldsymbol{r}_B(t) + \bar{\boldsymbol{x}}'$$

$$\boldsymbol{r}_{AB} := \boldsymbol{r}_A - \boldsymbol{r}_B$$

Motion of extended fluid bodies

N-body point mass terms

Moments of other bodies

Effect of body's own moments

Moment-moment interaction terms

$$
\begin{aligned}
a_A^j = G \sum_{B \neq A} \Bigg\{ & -\frac{m_B}{r_{AB}^2} n_{AB}^j \\
& + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \left[(-1)^\ell I_B^{\langle L \rangle} + \frac{m_B}{m_A} I_A^{\langle L \rangle} \right] \partial_{jL}^A \left(\frac{1}{r_{AB}} \right) \\
& + \frac{1}{m_A} \sum_{\ell=2}^{\infty} \sum_{\ell'=2}^{\infty} \frac{(-1)^{\ell'}}{\ell! \ell'!} I_A^{\langle L \rangle} I_B^{\langle L' \rangle} \partial_{jLL'}^A \left(\frac{1}{r_{AB}} \right) \Bigg\}
\end{aligned}
$$

Two-body system with only body 2 having non-zero $I^{\langle L \rangle}$

$$\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2 \,, \qquad r := |\boldsymbol{r}|$$

$$\boldsymbol{R} := (m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2)/m$$

$$m := m_1 + m_2$$

$$\mu := m_1 m_2 / m$$

$$\boxed{a^j = -\frac{Gm}{r^2} n^j + Gm \sum_{\ell=2}^{\infty} \frac{(-1)^\ell}{\ell!} \frac{I_2^{\langle L \rangle}}{m_2} \partial_{jL} \left(\frac{1}{r} \right)}$$

The two-body Kepler problem

- set center of mass at the origin (X = 0)
- ignore all multipole moments (spherical bodies or point masses)
- define $\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2\,,\; r := |\boldsymbol{r}|\,,\; m := m_1 + m_2\,,\; \mu := m_1 m_2/m$
- reduces to effective one-body problem

$$\boldsymbol{a} = -\frac{Gm}{r^2}\boldsymbol{n}$$

Energy and angular momentum conserved:

$$\begin{aligned} E &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - G\frac{m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \\ &= \frac{1}{2}\mu v^2 - G\frac{\mu m}{r} \\ L &= m_1 \boldsymbol{r}_1 \times \boldsymbol{v}_1 + m_2 \boldsymbol{r}_2 \times \boldsymbol{v}_2 \\ &= \mu \boldsymbol{r} \times \boldsymbol{v} \end{aligned}$$

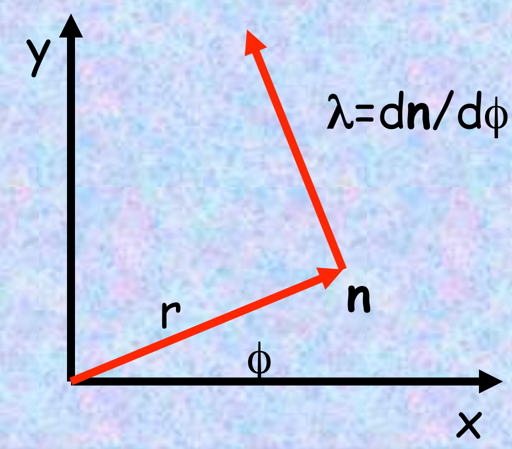
orbital plane
is fixed

Effective one-body problem

Make orbital plane the x-y plane

$$\boldsymbol{r} \times \boldsymbol{v} = r^2 \frac{d\phi}{dt} := h\boldsymbol{e}_z$$

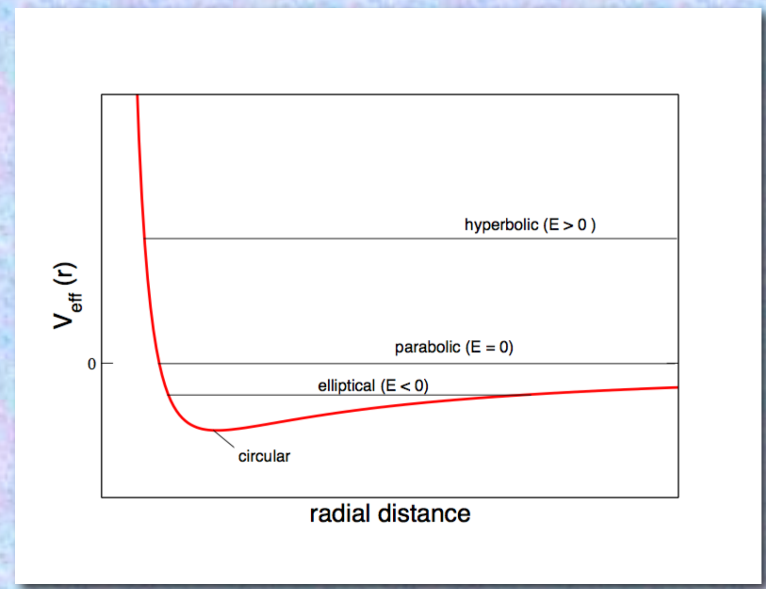
$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{r}\boldsymbol{n} + r\dot{\phi}\boldsymbol{\lambda}$$



From energy conservation: $\varepsilon = E/\mu$

$$\dot{r}^2 = 2\left[\varepsilon - V_{\rm eff}(r)\right]$$

$$V_{\rm eff}(r) = \frac{h^2}{r^2} - \frac{Gm}{r}$$



Reduce to quadratures (integrals)

$$\boxed{\begin{aligned} t - t_i &= \pm \int_{r_i}^{r} \frac{dr'}{\sqrt{2[\varepsilon - V_{\rm eff}(r')]}} \\ \phi - \phi_i &= h \int_{t_i}^{t} \frac{dt'}{r(t')^2} \end{aligned}}$$

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Keplerian orbit solutions

Radial acceleration, or d/dt of energy equation:

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{Gm}{r^2}$$

Find the orbit in space: convert from t to ϕ:

$$d/dt = \dot{\phi} d/d\phi = (h/r^2) d/d\phi$$

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{Gm}{h^2}$$

$$\boxed{\frac{1}{r} = \frac{1}{p}(1 + e\cos f)}$$

$f := \phi - \omega$ true anomaly

$p := h^2/Gm$ semilatus rectum

Elliptical orbits (e < 1, a > 0)

$$r_{\text{peri}} = \frac{p}{1+e}, \quad \phi = \omega$$

$$r_{\text{apo}} = \frac{p}{1-e}, \quad \phi = \omega + \pi$$

$$a := \frac{1}{2}(r_{\text{peri}} + r_{\text{apo}}) = \frac{p}{1-e^2}$$

Hyperbolic orbits (e > 1, a < 0)

$$\phi_{\text{in}} - \phi_{\text{out}} = \pi - 2\arcsin(1/e)$$

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Keplerian orbit solutions

Useful relationships

$$\dot{r} = \frac{he}{p} \sin f$$

$$v^2 = \frac{Gm}{p}(1 + 2e\cos f + e^2) = Gm\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$E = -\frac{G\mu m}{2a}$$

$$e^2 = 1 + \frac{2h^2 E}{\mu (Gm)^2}$$

$$P = 2\pi \left(\frac{a^3}{Gm}\right)^{1/2}$$ for closed orbits

Alternative solution

$$r = a(1 - e\cos u)$$

$$n(t - T) = u - e\sin u$$

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$$

$$n = 2\pi/P$$

u = eccentric anomaly

f = true anomaly

n = mean motion

Dynamical symmetry in the Kepler problem

- a and e are constant (related to E and h)
- orbital plane is constant (related to direction of h)
- ω is constant -- a hidden, dynamical symmetry

Runge-Lenz vector

$$\begin{aligned} \boldsymbol{A} &:= \frac{\boldsymbol{v} \times \boldsymbol{h}}{Gm} - \boldsymbol{n} \\ &= e\big(\cos\omega\, \boldsymbol{e}_x + \sin\omega\, \boldsymbol{e}_y\big) \\ &= \text{constant} \end{aligned}$$

Comments:

- responsible for the degeneracy of hydrogen energy levels
- added symmetry occurs only for $1/r$ and r^2 potentials
- deviation from $1/r$ potential generically causes $d\omega/dt$

Keplerian orbit in space

Six orbit elements:

- i = inclination relative to reference plane:

$$\cos\iota = \hat{\boldsymbol{h}} \cdot \boldsymbol{e}_Z$$

- Ω = angle of ascending node

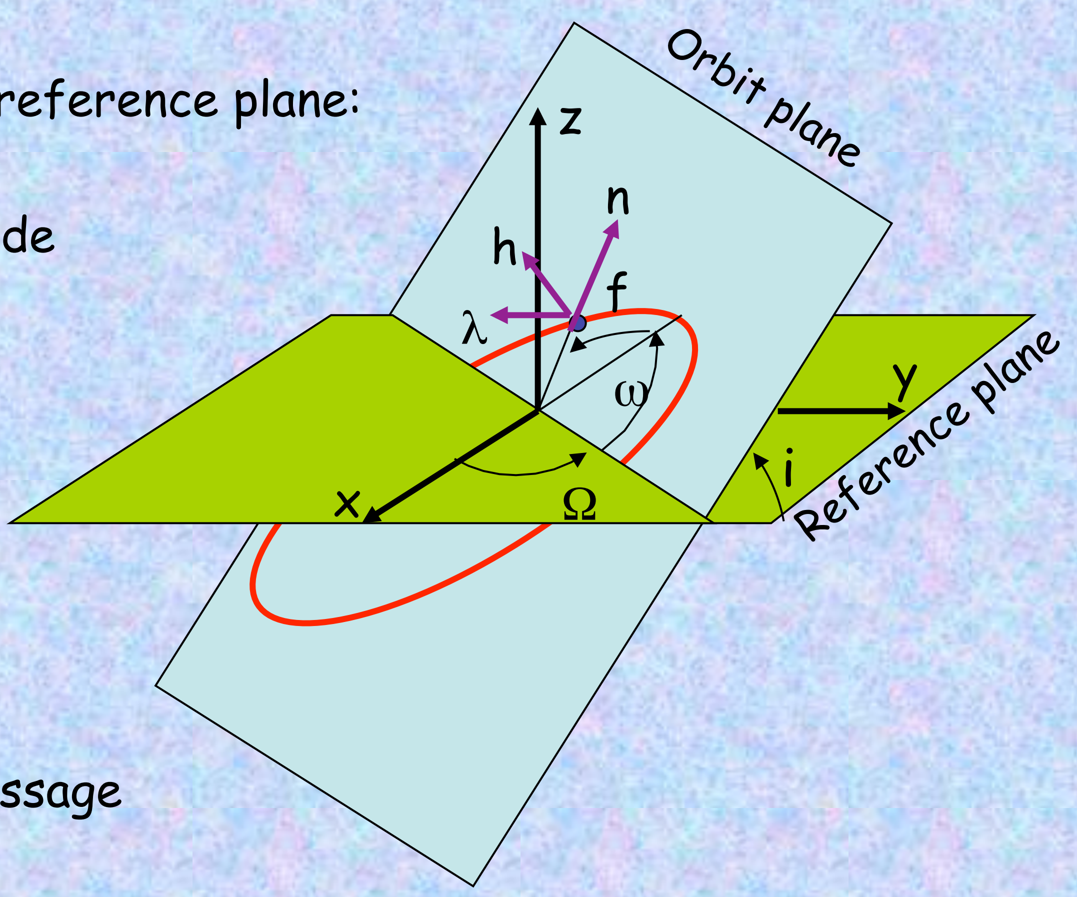
$$\cos\Omega = -\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{e}_Y}{\sin\iota}$$

- ω = angle of pericenter

$$\sin\omega = \frac{\boldsymbol{A} \cdot \boldsymbol{e}_z}{e \sin\iota}$$

- e = |**A**|
- a = h^2/Gm(1-e^2)
- T = time of pericenter passage

$$T = t - \int_0^f \frac{r^2}{h} df$$



Comment: equivalent to the initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$