Gravity: Newtonian, post-Newtonian, Relativistic

Summer School on Gravitational Waves
Indian Institute for Theoretical Sciences
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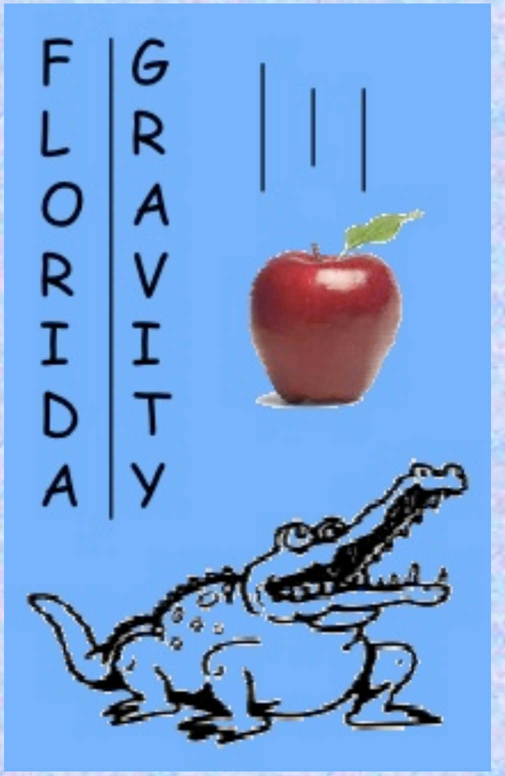
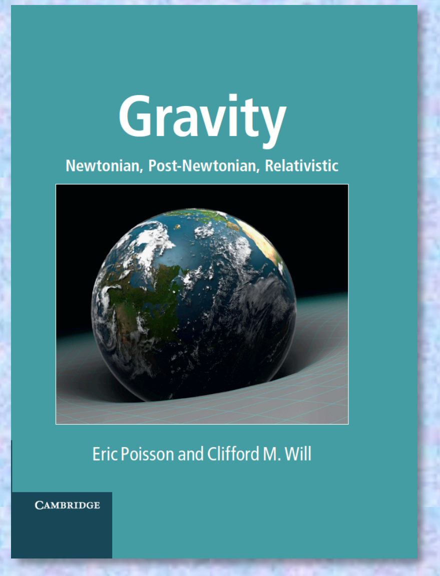


Outline of the Lectures



1. Newtonian gravity (3 lectures)
   Foundations, Isolated gravitating bodies, Orbital dynamics
   PW, Chapters 1, 3
2. Post-Minkowskian theory (1 lecture)
   Formulation, Implementation
   PW, Chapters 6 - 7
3. Post-Newtonian theory: Near-zone physics (3 lectures)
   Implementation, PN celestial mechanics, astrometry & timekeeping
   PW, Chapters 8 - 10
4. Post-Newtonian theory: Far-zone physics (3 lectures)
   Gravitational radiation, Radiation Reaction
   PW, Chapters 11 - 12

Textbook: *Gravity: Newtonian, post-Newtonian, General Relativistic*, by Eric Poisson and Clifford Will (Cambridge U Press, 2014)

Lecture slides (in pdf) and selected PW chapters available at
www.phys.ufl.edu/~cmw/Gravity-Lectures/outline.html

Foundations of Newtonian Gravity

Newton's 2nd law and the law of gravitation:

$$m_I \boldsymbol{a} = \boldsymbol{F}$$

$$\boldsymbol{F} = -G m_G M \boldsymbol{r}/r^3$$

The principle of equivalence:

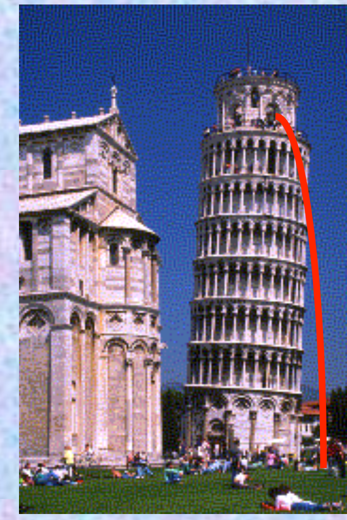
$$\boldsymbol{a} = -\frac{m_G}{m_I}\frac{GM\boldsymbol{r}}{r^3}$$

If $m_G = m_I(1+\eta)$

Then, comparing the acceleration of two different bodies or materials

$$\Delta\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2 = -(\eta_1 - \eta_2)\frac{GM\boldsymbol{r}}{r^3}$$

The Weak Equivalence Principle (WEP)



400 CE Ioannes Philiponus: *"…let fall from the same height two weights of which one is many times as heavy as the other …. the difference in time is a very small one"*

1553 Giambattista Benedetti
proposed equality

1586 Simon Stevin
experiments

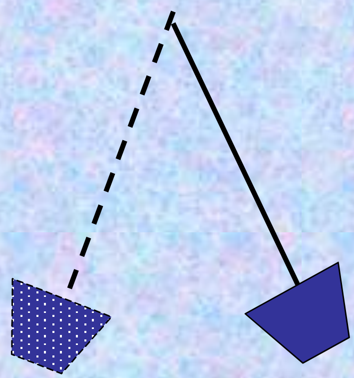
1589-92 Galileo Galilei
Leaning Tower of Pisa?

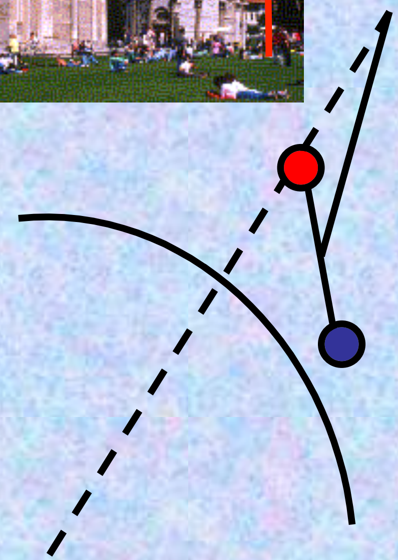
1670-87 Newton
pendulum experiments

1889, 1908 Baron R. von Eötvös
torsion balance experiments (10^{-9})

1990 – 2010 UW (Eöt-Wash)
10^{-13}

2010 Atom inteferometers
matter waves vs macroscopic object

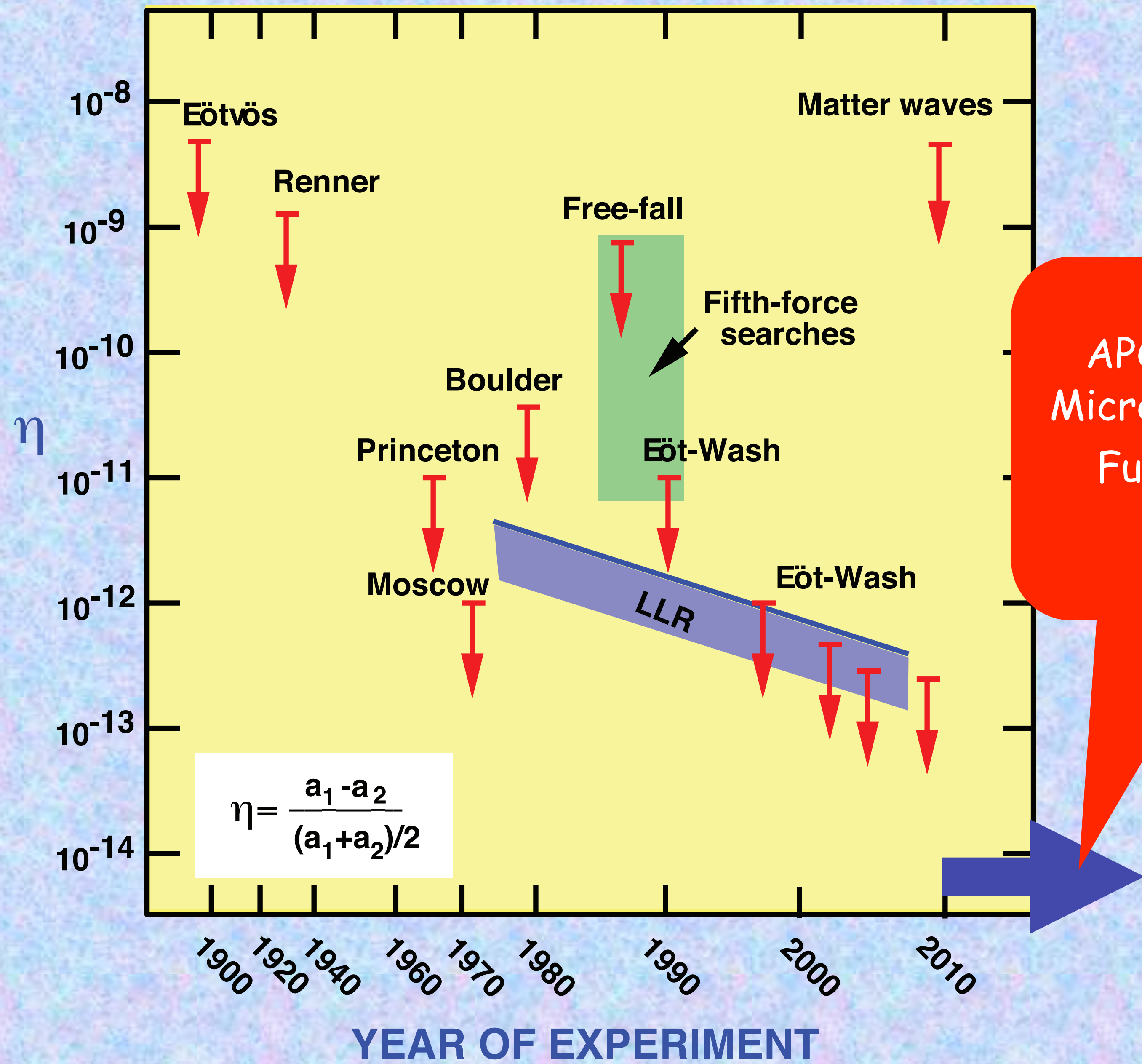




Bodies fall in a gravitational field with an acceleration that is independent of mass, composition or internal structure

FLORIDA GRAVITY

Tests of the Weak Equivalence Principle



APOLLO (LLR) 10^{-13}

Microscope 10^{-15} (2015)

Future: STEP, GG, STE-QUEST

Newtonian equations of Hydrodynamics

Writing $m\boldsymbol{a} = m\boldsymbol{\nabla}U \,,$ Equation of motion

$U = GM/r \,,$ Field equation

Generalize to multiple sources (sum over M's) and continuous matter

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{\nabla} U - \boldsymbol{\nabla} p \,,$$ Euler equation of motion

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \,,$$ Continuity equation

$$\nabla^2 U = -4\pi G \rho \,,$$ Poisson field equation

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \,,$$ Total or Lagrangian derivative

$$p = p(\rho, T, \dots)$$ Equation of state

Formal solution of Poisson's field equation:

Write $U(t, \boldsymbol{x}) = G \int G(\boldsymbol{x}, \boldsymbol{x}')\rho(t, \boldsymbol{x}')d^3x' \,,$

Green function $\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') \Rightarrow G(\boldsymbol{x}, \boldsymbol{x}') = 1/|\boldsymbol{x} - \boldsymbol{x}'|$

$$U(t, \boldsymbol{x}) = G \int \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x'$$

Rules of the road

Consequences of the continuity equation: for any f(x,t):

$$\begin{aligned}
\frac{d}{dt}\int \rho(t,\boldsymbol{x}) f(t,\boldsymbol{x})\, d^3x &= \int \left(\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} \right) d^3x \\
&= \int \left(\rho \frac{\partial f}{\partial t} - f \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \right) d^3x \\
&= \int \left(\rho \frac{\partial f}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} f \right) d^3x - \oint f \rho \boldsymbol{v} \cdot d\boldsymbol{S} \\
&= \int \rho \frac{df}{dt}\, d^3x \,.
\end{aligned}$$

Useful rules:

$$\begin{aligned}
\frac{\partial}{\partial t}\int \rho(t,\boldsymbol{x'}) f(t,\boldsymbol{x},\boldsymbol{x'})\, d^3x' &= \int \rho' \left(\frac{\partial f}{\partial t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla'} f \right) d^3x' \,, \\
\frac{d}{dt}\int \rho(t,\boldsymbol{x'}) f(t,\boldsymbol{x},\boldsymbol{x'})\, d^3x' &= \int \rho' \left(\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f + \boldsymbol{v'} \cdot \boldsymbol{\nabla'} f \right) d^3x' \\
&= \int \rho' \frac{df}{dt}\, d^3x'
\end{aligned}$$

Global conservation laws

$$M := \int \rho(t,\boldsymbol{x})\, d^3x = \text{constant}$$

$$\boldsymbol{P} := \int \rho(t,\boldsymbol{x})\boldsymbol{v}\, d^3x = \text{constant}$$

$$E := \mathcal{T}(t) + \Omega(t) + E_{\text{int}}(t) = \text{constant}$$

$$\boldsymbol{J} := \int \rho\boldsymbol{x}\times\boldsymbol{v}\, d^3x = \text{constant}$$

$$\boldsymbol{R}(t) := \frac{1}{M}\int \rho(t,\boldsymbol{x})\boldsymbol{x}\, d^3x = \frac{\boldsymbol{P}}{M}(t-t_0) + \boldsymbol{R}_0$$

$$d(\epsilon\mathcal{V}) + p\,d\mathcal{V} = 0$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \mathcal{V}^{-1}d\mathcal{V}/dt$$

$$\mathcal{T}(t) := \frac{1}{2}\int \rho v^2\, d^3x$$

$$\Omega(t) := -\frac{1}{2}G\int \frac{\rho\rho'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' d^3x\,,$$

$$E_{\text{int}}(t) := \int \epsilon\, d^3x$$

$$\begin{aligned}\frac{d}{dt}\int \rho\boldsymbol{v}d^3x &= \int (\rho\boldsymbol{\nabla}U - \boldsymbol{\nabla}p)\, d^3x \\ &= -G\int\!\!\int \rho\rho'\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d^3x d^3x' - \oint p\boldsymbol{n}d^2S \\ &= 0\end{aligned}$$

Spherical and nearly spherical bodies

Spherical symmetry

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = -4\pi G\rho(t,r)$$

$$\frac{\partial U}{\partial r} = -\frac{Gm(t,r)}{r^2} \qquad m(t,r) := \int_0^r 4\pi\rho(t,r')r'^2\,dr'$$

$$U(t,r) = \frac{Gm(t,r)}{r} + 4\pi G\int_r^R \rho(t,r')r'\,dr'\,.$$

Outside the body $U = GM/r$

Spherical and nearly spherical bodies

Non-spherical bodies: the external field $|\boldsymbol{x}'| < |\boldsymbol{x}|$

Taylor expansion:

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x'}|} = \frac{1}{r} - x'^j \partial_j \left(\frac{1}{r}\right) + \frac{1}{2} x'^j x'^k \partial_j \partial_k \left(\frac{1}{r}\right) - \cdots$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L \left(\frac{1}{r}\right)$$

Then the Newtonian potential outside the body becomes

$$U_{\text{ext}}(t,\boldsymbol{x}) = G \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} I^{\langle L\rangle} \partial_{\langle L\rangle} \left(\frac{1}{r}\right) ,$$

$$I^{\langle L\rangle}(t) := \int \rho(t,\boldsymbol{x'}) x'^{\langle L\rangle}\, d^3x'$$

$$x^L := x^i x^j \ldots (\mathrm{L\ times})$$

$$\partial_L := \partial_i \partial_j \ldots (\mathrm{L\ times})$$

$$\langle \ldots \rangle := \text{symmetric tracefree product}$$

Symmetric tracefree (STF) tensors

$A^{\langle ijk\ldots\rangle}$ Symmetric on all indices, and $\delta_{ij}A^{\langle ijk\ldots\rangle} = 0$

Example: gradients of 1/r

$$\partial_j r^{-1} = -n_j r^{-2}\,,$$
$$\partial_{jk} r^{-1} = (3n_j n_k - \delta_{jk}) r^{-3}\,,$$
$$\partial_{jkn} r^{-1} = -\Big[15 n_j n_k n_n - 3\big(n_j\delta_{kn} + n_k\delta_{jn} + n_n\delta_{jk}\big)\Big] r^{-4}$$

$$\partial_L r^{-1} = \partial_{\langle L\rangle} r^{-1} = (-1)^\ell (2\ell-1)!!\,\frac{n_{\langle L\rangle}}{r^{\ell+1}}$$

General formula for $n^{\langle L\rangle}$:

$$n^{\langle L\rangle} = \sum_{p=0}^{[\ell/2]} (-1)^p \frac{(2\ell-2p-1)!!}{(2\ell-1)!!}\Big[\delta^{2P} n^{L-2P} + \mathrm{sym}(q)\Big]$$

$$q := \ell!/[(\ell-2p)!(2p)!!]$$

Symmetric tracefree (STF) tensors

Link between $n^{\langle L \rangle}$ and spherical harmonics

$$e_{\langle L \rangle} n^{\langle L \rangle} = \frac{\ell!}{(2\ell - 1)!!} P_\ell(\boldsymbol{e} \cdot \boldsymbol{n})$$

$$n^{\langle L \rangle} := \frac{4\pi \ell!}{(2\ell + 1)!!} \sum_{m=-\ell}^{\ell} \mathcal{Y}^{\langle L \rangle}_{\ell m} Y_{\ell m}(\theta, \phi)$$

$$\mathcal{Y}^{\langle z \rangle}_{10} = \sqrt{\frac{3}{4\pi}}\,, \qquad \mathcal{Y}^{\langle x \rangle}_{11} = -\sqrt{\frac{3}{8\pi}}\,, \qquad \mathcal{Y}^{\langle y \rangle}_{11} = i\sqrt{\frac{3}{8\pi}}\,,$$

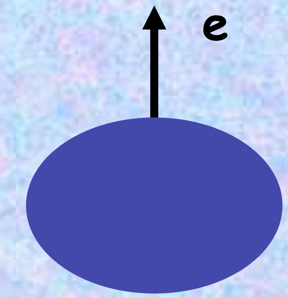
$$\mathcal{Y}^{\langle xx \rangle}_{20} = -\sqrt{\frac{5}{16\pi}}\,, \qquad \mathcal{Y}^{\langle yy \rangle}_{20} = -\sqrt{\frac{5}{16\pi}}\,, \qquad \mathcal{Y}^{\langle zz \rangle}_{20} = 2\sqrt{\frac{5}{16\pi}}\,,$$

Average of $n^{\langle L \rangle}$ over a sphere:

$$\langle\langle n^L \rangle\rangle := \frac{1}{4\pi} \oint n^L d\Omega = \begin{cases} \frac{1}{(\ell+1)!!} \left(\delta^{L/2} + \mathrm{sym}[(\ell - 1)!!]\right) & \ell = \text{even} \\ 0 & \ell = \text{odd} \end{cases}$$

Spherical and nearly spherical bodies

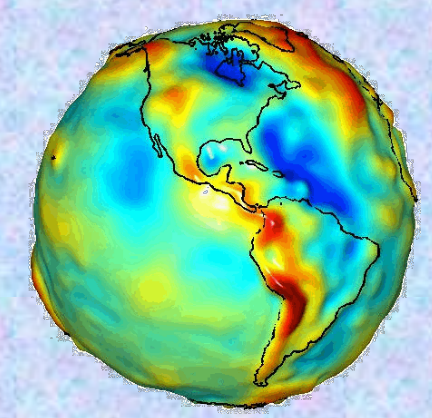
Example: axially symmetric body



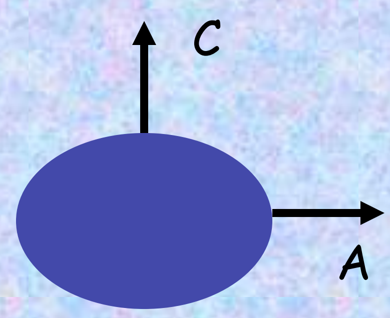
$$I_A^{\langle L \rangle} = -m_A R_A^\ell (J_\ell)_A e^{\langle L \rangle}$$

$$J_\ell := -\sqrt{\frac{4\pi}{2\ell+1}}\frac{1}{MR^\ell}\int \rho(t,\boldsymbol{x}) r^\ell Y^*_{\ell 0}(\theta,\phi)\, d^3x$$

$$U_{\rm ext}(t,\boldsymbol{x}) = \frac{GM}{r}\left[1 - \sum_{\ell=2}^{\infty} J_\ell \left(\frac{R}{r}\right)^\ell P_\ell(\cos\theta)\right]$$



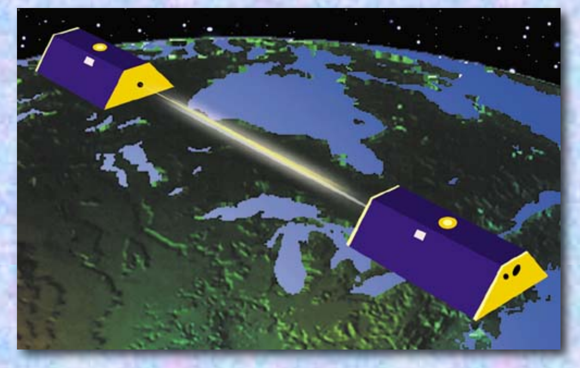
Note that:

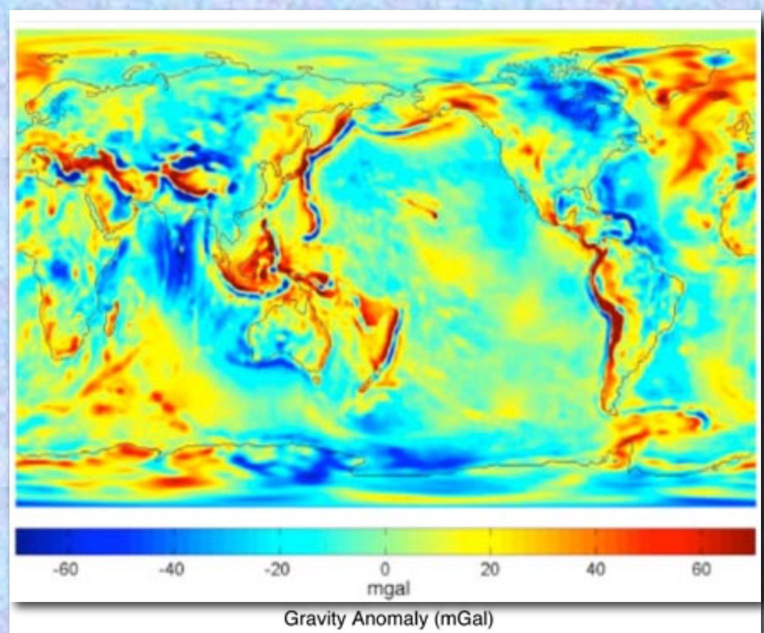


$$J_2 = \frac{C-A}{MR^2}$$

Measuring the Earth's Newtonian hair

Gravity Recovery And Climate Experiment (GRACE)





Earth: $j_2 = -4.8416932(5) \times 10^{-4}$

$j_3 = 9.5721(1) \times 10^{-7}$

$j_4 = 5.3999(4) \times 10^{-7}$

……

……

$j_{360,0} = 3.2\ (5) \times 10^{-10}$