Thermal noise

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Outline

1. Brownian motion and the Fluctuation-Dissipation Theorem
2. Thermal noise in interferometers
3. Internal friction, and how to make it small

Large mechanical noise

How large?

Seismic: $x_{rms} \sim 1\ \mu$m.

Brownian motion:

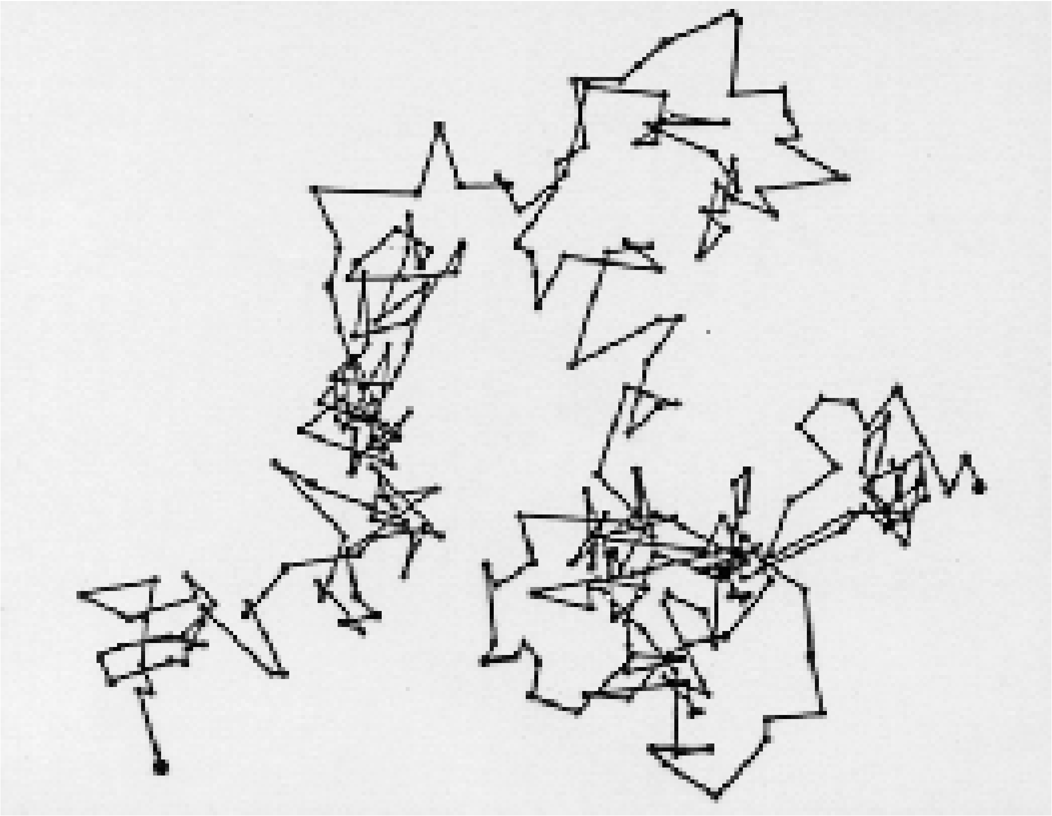
- mirror's CM: ~ 3 x 10^{-12} m.
- mirror's surface: ~ 3 x 10^{-16} m.

If so, can we detect gravitational waves?

Now, we'll focus on Brownian motion.

Brownian motion

In 1827, Robert Brown noticed incessant jiggling of tiny particles suspended in water, as seen through a microscope.



Evidence for a ubiquitous "vital force"?

Not just living cells, but small particles of any material.

He even checked dust ground from a piece of the Great Sphinx!

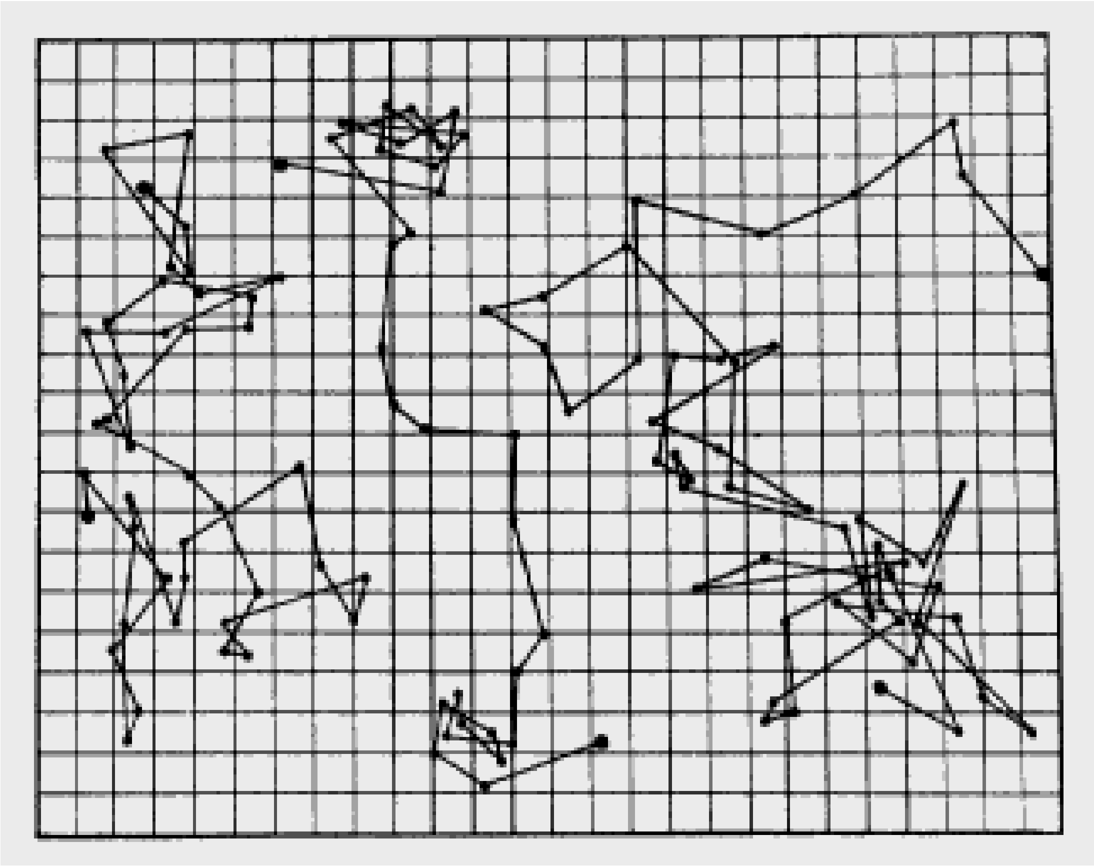
Did this mean that all material was in some sense animate, i.e. alive?

Or was there a physical explanation?

The molecular explanation of Brownian motion

- Water is made of tiny molecules.
- Molecules randomly collide with visible particles.
- Random collisions from all sides yield jiggling motion: a *random walk*.

What experiments showed



Many individual particles were studied.

Keep track of how far a particle drifts in a fixed time interval.

How to find a pattern in all of those random walks?

The graph at right shows <u>endpoints</u> of many random walks.

Scattered all over!

But you can characterize this graph by the typical width of the pattern.

Mathematically, find the average of the square of the distance drifted, or $< x^2 >$.

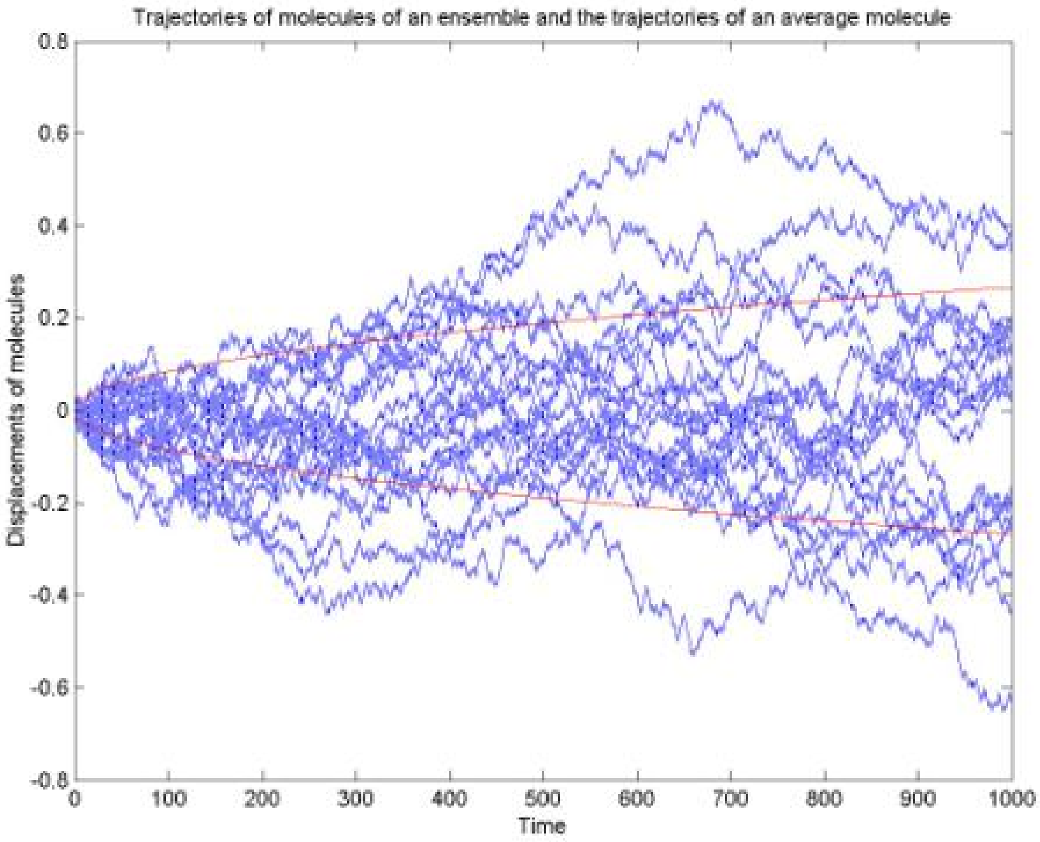
Fig. 10.

How does the typical drift distance grow with time?

Experiments showed that $< x^2 > \propto t$.

Also:

- Drift is greater at higher temperature.
- Drift decreases as drag on particle increases.



Einstein's key contribution

1905: Einstein shows that Brownian particle's random walk obeys

$$\left\langle x^2(t) \right\rangle = 2k_B TBt$$

where B is a coefficient called the *mobility* of the particle.

"Jitter" a result of the existence of atoms.

The first link clear and incontrovertible link between fluctuation and dissipation.

Atoms are real!

Soon, Avogadro's number was determined:

$$N = 6.02 \times 10^{23} \text{ molecules/mole.}$$

Brownian motion is a graphic demonstration that *heat is the microscopic motion of molecules.*

It also suggests an explanation of the relationship between friction and heat.

Thus, Brownian motion gives us a direct window into the microscopic world of atoms.

Other key contributions

- Jean Perrin won the 1926 Nobel Prize for measurements of diffusion, checking Einstein's theory, thereby "proving the existence of atoms".
- Johnson noise, 1928, electrical equivalent of Brownian motion.

Fluctuation-Dissipation Theorem
(Callen *et al.* 1951-2)

For a linear system in thermal equilibrium,

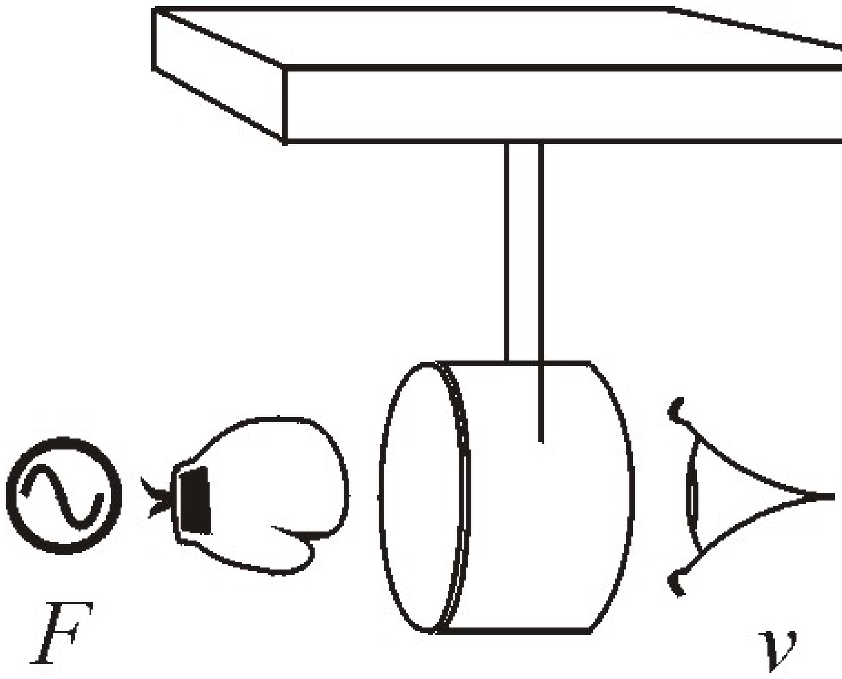
$$S_x(f) = \frac{4k_BT}{(2\pi f)^2}\mathrm{Re}[Y(f)].$$

Y(f) is the admittance *v/F;*
Re[*Y(f)*] is proportional to the dissipation in the system.

For lower noise, we can either:

1. lower the dissipation, or
2. lower the temperature.

Real part of admittance Y determines strength of thermal noise



$$Y = v/F$$

SHO in impedance language

$$ma = F$$

$$m\ddot{x} = -kx - b\dot{x} + F_{ext}$$

$$m\dot{v} = -k\frac{v}{i\omega} - bv + F_{ext}$$

$$mi\omega v = -\frac{k}{i\omega} - bv + F_{ext}$$

$$\left(i\omega m + \frac{k}{i\omega} + b\right)v = F_{ext}$$

$$Z \equiv \frac{F_{ext}}{v} = b + \left(\omega m - \frac{k}{\omega}\right)$$

Admittance of SHO

$$Y \equiv Z^{-1} = \frac{v}{F_{ext}}$$

$$Y = \frac{1}{b + i(\omega m - k/\omega)}$$

$$Y = \frac{\omega^2 b + i(\omega k - m\omega^3)}{(m\omega^2 - k)^2 + \omega^2 b^2}$$

Relation to Equipartition

Equipartition Theorem: each d.o.f. has, on average, an energy of $k_B T/2.$

For a simple harmonic oscillator, this means

$$\frac{1}{2}kx^2 = \frac{1}{2}k_B T,$$

or rms motion of 3×10^{-12} meters, for 10 kg mirror in 1 Hz pendulum at room temp. (!)

Temporal character

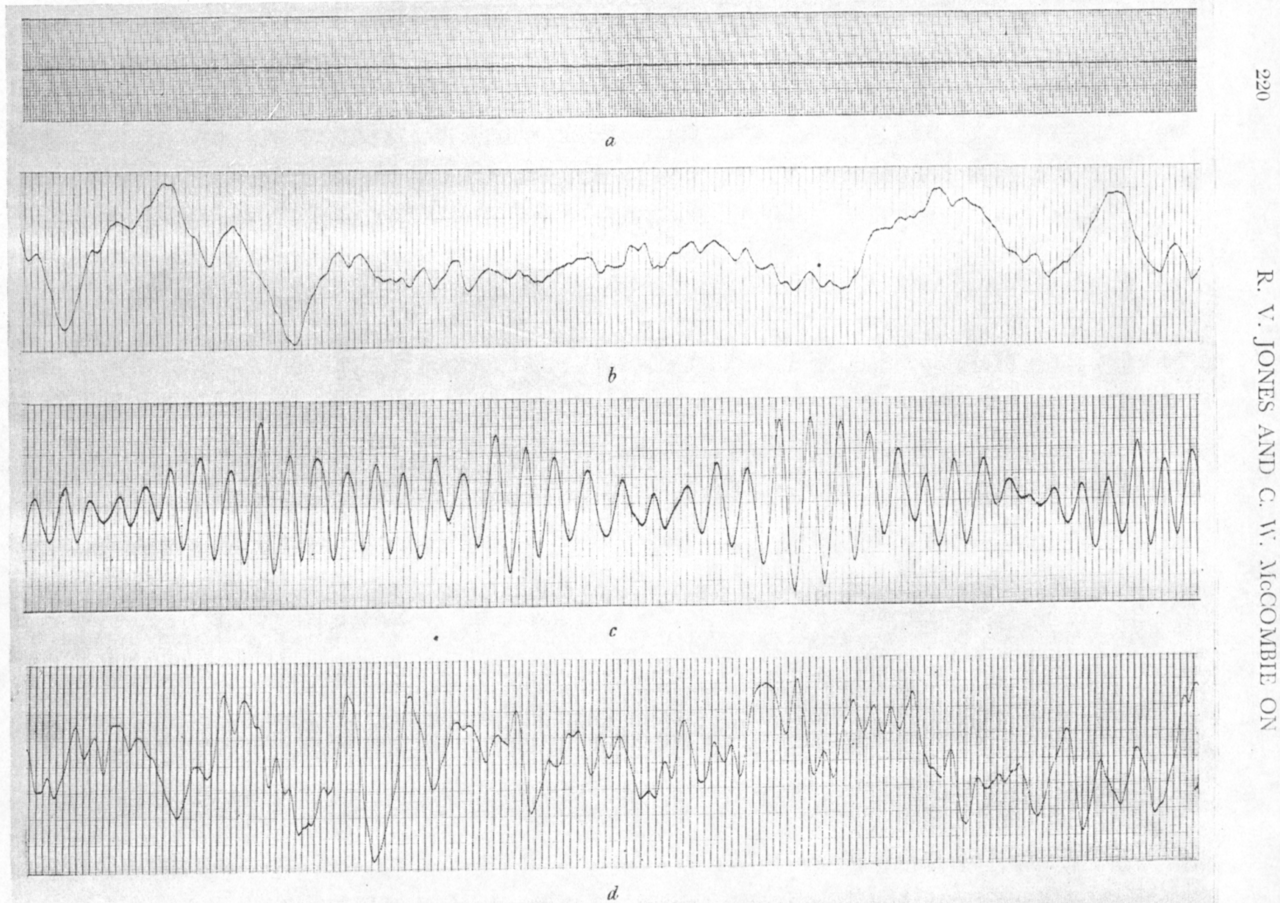
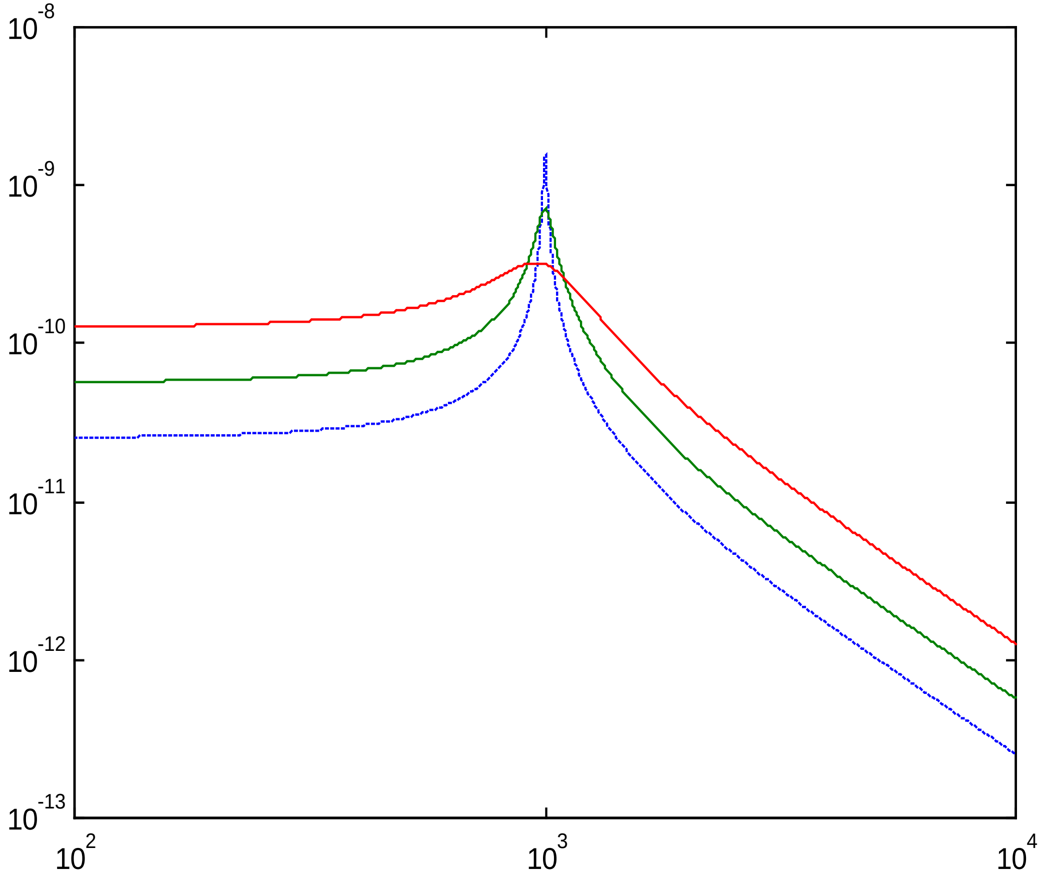


FIGURE 1. *a*. Record of secondary trace, with fixed primary mirror, at approximately twice the sensitivity used in records (*b*), (*c*) and (*d*). Vertical lines at 1 sec. intervals. *b*, *c*, *d*. Records of secondary trace with primary galvanometer (resistance 10·76Ω, free period 2·17 sec.) [illegible] circuited (*b*), on open circuit (*c*), and nearly critically damped with external resistance 133·24 ohms (*d*). Secondary galvanometer (free [illegible] Sensitivity about 3·2 × 10[illegible] amp. per large division. Vertical lines at 0·5 sec. intervals.

Same lesson in frequency domain



Q as a Figure of Merit

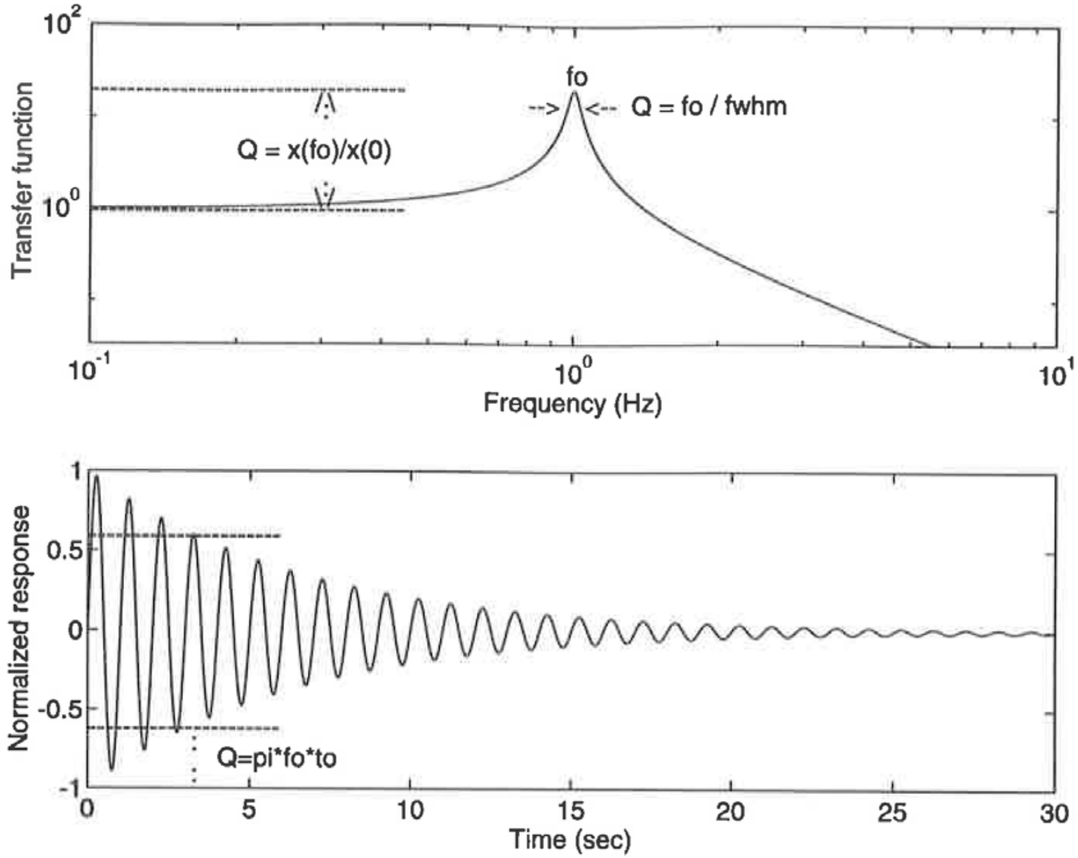
Q: dimensionless measure of the ratio of elastic restoring force to dissipative force.

Measured by width of resonance or by ringdown time of vibration.

Related to "loss angle" ϕ, phase lag between force and displacement, by $\phi = 1/Q$.

In round numbers: garden variety $Q = 10^3$, good $Q = 10^6$, advanced $Q = 10^8$.

Q



Transfer function
Q = x(fo)/x(0)
fo
--> <-- Q = fo / fwhm
Frequency (Hz)
Normalized response
Q=pi*fo*to
Time (sec)

Complex spring constant

It can be convenient to model internal friction in real springs with an extension of Hooke's Law:

$$F = -k[1 + i\phi(\omega)]x$$

If F is sinusoidal, spring's stretch lags the applied force by a phase angle ϕ.

Can model many forms of damping in this way.

Empirically, often find that ϕ = constant. (!)

How does internal friction work?

Not like a simple dashpot.

- No such thing as "internal viscosity".

If it did, high frequency modes would be damped more than grave modes, $Q \propto 1/f$.

Xylophones would be impossible.

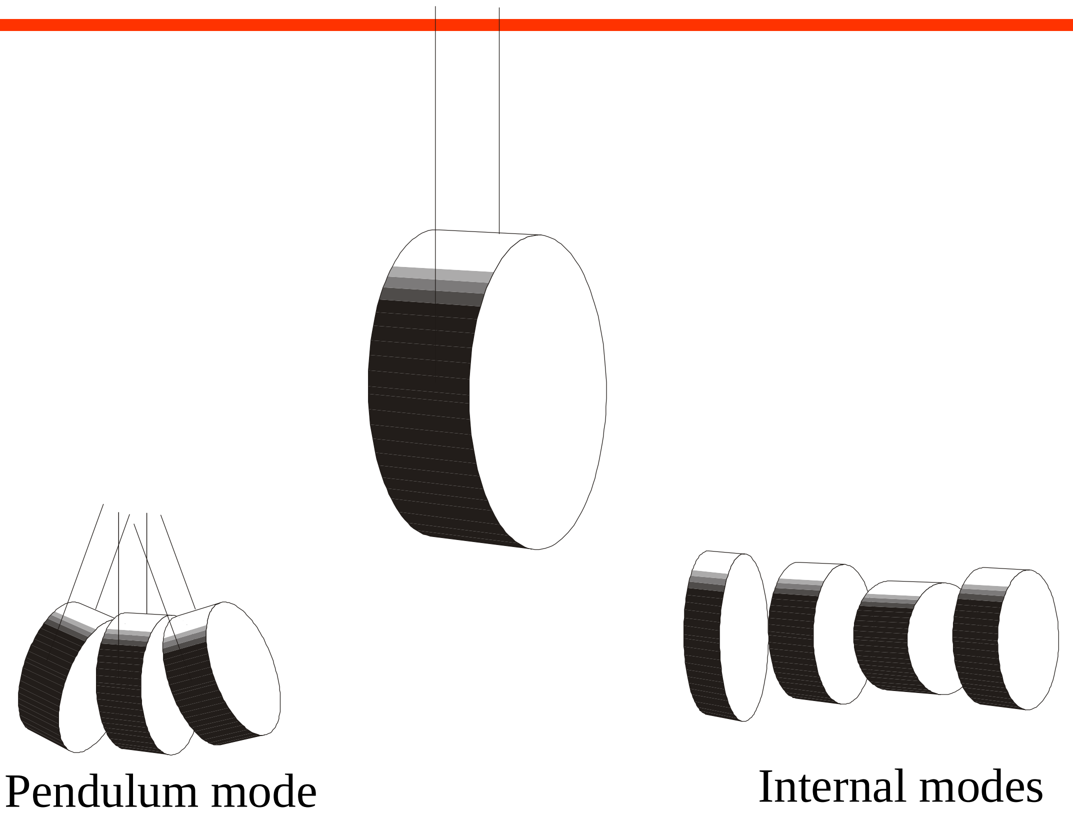
Internal friction typically has $\phi(f) \sim$ constant.

SHO made from such a spring has $Q = 1/\phi(f)$.

Exercise

Calculate the thermal noise spectrum for a simple harmonic oscillator whose spring has internal friction ϕ.

Interferometer suspensions



$$x^2(f) = \frac{2}{\pi}\frac{k_B T}{k}\frac{1}{f\left[\left(1 - f^2/f_0^2\right)^2 + \phi^2\right]}\phi(f)$$

Why pendulums?

Pendulum suspensions are used universally for their low thermal noise.

Why? Let's figure it out.

Pendulum analyzed as two springs side by side

Two springs used "side by side" have a combined spring constant equal to the sum of the individual spring constants.

We can model a pendulum as two springs side by side. One is the regular pendulum spring constant, $k_{grav} = mg/l$. The other is a very weak elastic spring constant from the flexing of the pendulum wire. Only the wire has any internal friction.

Is a pendulum better than a spring for suspending test masses?

$$k_{grav} = \frac{mg}{l}$$

$$k_{el} = \frac{\sqrt{mgEI}}{2l^2}$$

where

$$I = \frac{\pi}{64} d^4$$

and E = 72 GPa, and where $d = 4*10^{-4}$ m.

Assume the elastic spring constant has $Q = 10^6$.

What is the Q of the pendulum?

Q of some useful materials

Rubber: 10 to 30

Aluminum, tungsten, steel: ~ 10^3 to 10^4

Fused silica: up to $2\ 10^8$

Sapphire: 2 to $3\ 10^8$

(all values given at room temperature)

Best practices

- Advanced LIGO:
  - Use fused silica fibers (instead of iLIGO's steel wires).
  - Use "best quality" fused silica for test masses
  - Then, most important thermal noise comes from the high-reflectivity coatings on the mirrors. (!)

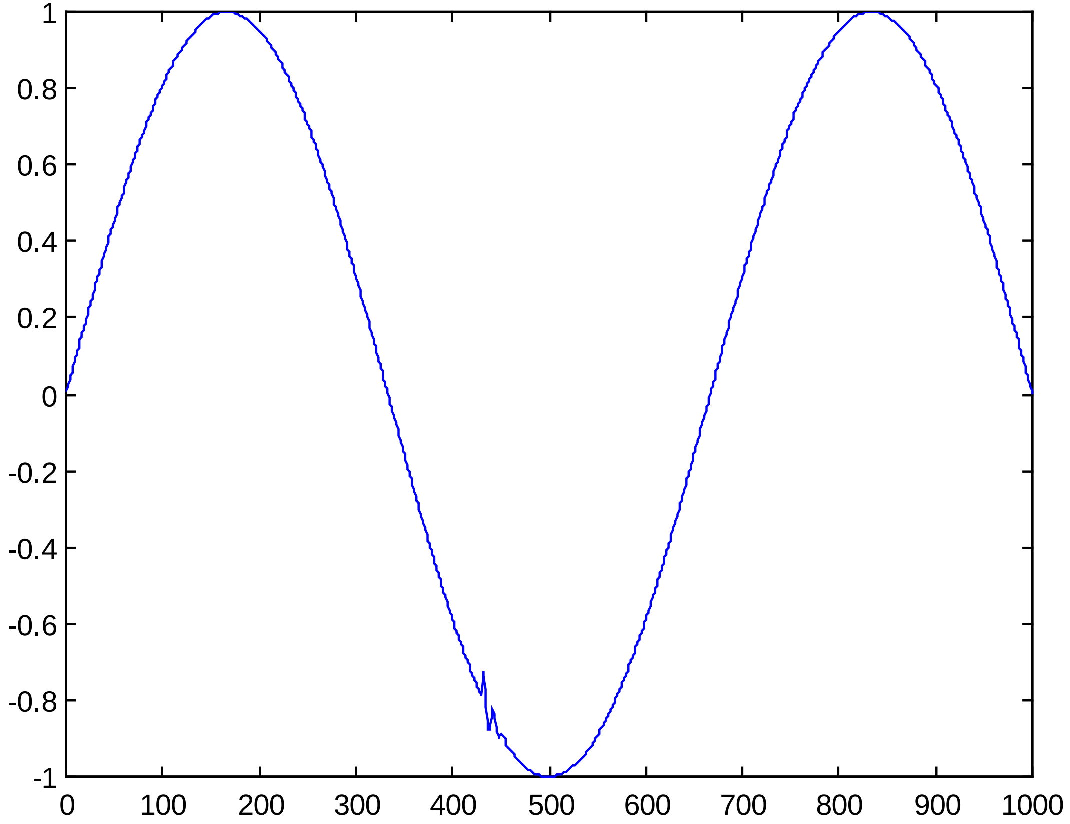
- Longer term:
  - Reduce thermal noise by going to low temperatures

    KAGRA is the pathfinder for this
  - For low *T*, need to replace silica with crystalline material

    all glasses have terrible *Q* at low temperatures, for deep condensed matter physics reasons

Extra slides

Signal detection in strongly colored noise



Signal detection in strongly colored noise (II)