Mathematical methods of experimental physics

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Outline

1. Useful operations on time series

    Fourier transform

    Cross-correlation, autocorrelation, convolution

    Various versions of the power spectrum

2. How to characterize a linear system

    Impulse response

    Frequency response

3. Application to a simple harmonic oscillator

Time series

For many purposes, it is interesting to consider a single-valued function of time, $x(t)$.

May be

- Deterministic
- Random
- A sum of deterministic and random processes

A variety of operations (and combinations of operations) on time series have proved useful.

The Fourier transform

The Fourier transform $X(f)$ of $x(t)$ is defined as

$$X(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

This measures the amount of a sine and cosine of each frequency *f* that it takes to build up the function $x(t)$.

Most useful when $x(t)$ is a deterministic function.

But used all over the place …

Defines the relation between "the time domain" and "the frequency domain."

The cross-correlation

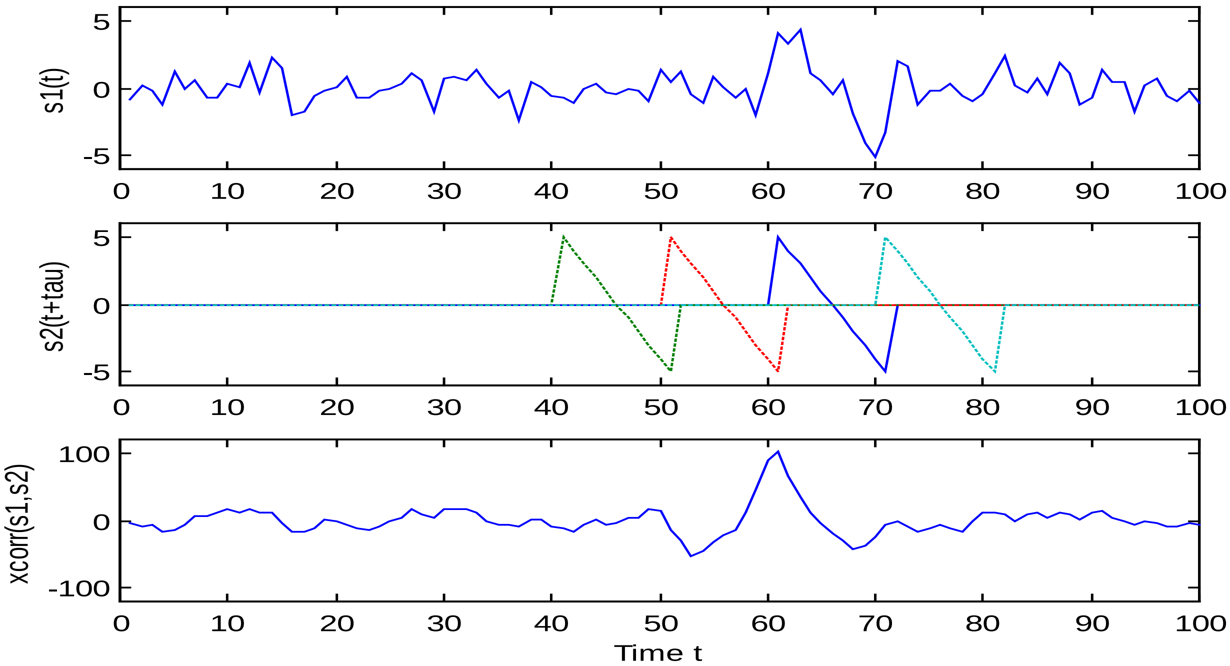
- If we have two time series, $x_1(t)$ and $x_2(t)$, we can form the cross-correlation

$$x_1 \otimes x_2(\tau) \equiv \int_{-\infty}^{\infty} x_1(t) x_2(t+\tau) dt$$

- This measures the extent to which the shape of one time series matches the shape of the other time series, as a function of the time-shift τ between them.
- Often used in a situation in which one time series is deterministic and the other is random.

How to find weak signals in strong noise

Look for something that looks more like the signal than noise, at suspiciously large level.



The autocorrelation

We call the "cross-correlation" of a time series *x(t)* with itself the autocorrelation

$$x \otimes x(\tau) \equiv \int_{-\infty}^{\infty} x(t) x(t+\tau) dt$$

This gives a measure of the time scale over which the time series varies, and well as any scales on which it repeats.

Used in as many and varied cases as the Fourier transform.

The convolution

Two time series $x_1(t)$ and $x_2(t)$ can be combined in a different way, called the convolution

$$x_1 * x_2(\tau) \equiv \int_{-\infty}^{\infty} x_1(t) x_2(\tau - t) dt$$

Looks a lot like the cross-correlation, but note the difference in sign of the time variable t in the second time series.

Plays a key role in describing the action of a linear system (e.g., a filter) on an input. In this application, one of the time series is deterministic, the other may be random.

The power spectrum

If we take the Fourier transform of the auto-correlation of a time series, we form the power spectrum

$$S_x(f) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \otimes x(\tau) e^{-i2\pi f\tau} d\tau$$

Like the Fourier transform, it measures the admixture of sinusoids of all frequencies *f* that make up the time series *x(t)*; however, it throws away the phase information (sines vs. cosines.)

Thus, most often used on random time series.

The single-sided power spectrum

- The power spectrum $S_x(f)$ refers to exponentials of both positive and negative frequencies.
- Experimentalists usually prefer to think in terms of just positive frequencies, so there is a single-sided power spectrum

$$\begin{aligned} x^2(f) &\equiv 2S_x(f), \text{if } f \geq 0 \\ &\equiv 0, \text{otherwise} \end{aligned}$$

- True to form, I like this version of the power spectrum best.

The periodogram

Consider a section (with duration T) of time series $x(t)$, which has Fourier transform $X(f)$.

The periodogram of $x(t)$ is given by $\left|X(f)\right|^2/T$.

Theorem: In the limit that T goes to infinity, the expectation value of the periodogram of $x(t)$ is equal to the power spectrum $S_x(f)$.

This is another reminder that the power spectrum of $x(t)$ measures the "amount" of sinusoids of frequency f in $x(t)$.

This is also the way that power spectra are usually calculated.

More on interpretation of the power spectrum

Conceptual way of measuring the power spectrum:

Apply signal $x(t)$ to a bank of bandpass filters, each with 1 Hz pass band width, band centers at each integer frequency.

Compute the mean-square value of the output of each filter, and display as a function of f.

N.B.: If you sum up all outputs of all filters, then you recover the mean-square value of $x(t)$. Thus, the units of the power spectrum must be [units of x]^2/Hz.

The amplitude spectral density

Experimenters have limited minds, and find it easier to get their minds around something that doesn't square the units of $x(t)$. So we often use the amplitude spectral density

$$x(f) \equiv \sqrt{x^2(f)}$$

Its units are [units of $x(t)$]/$\mathrm{Hz}^{1/2}$.

Why /$\mathrm{Hz}^{1/2}$? Each frequency "bin" of the spectrum of a random time series is independent of the others. So they add in quadrature.

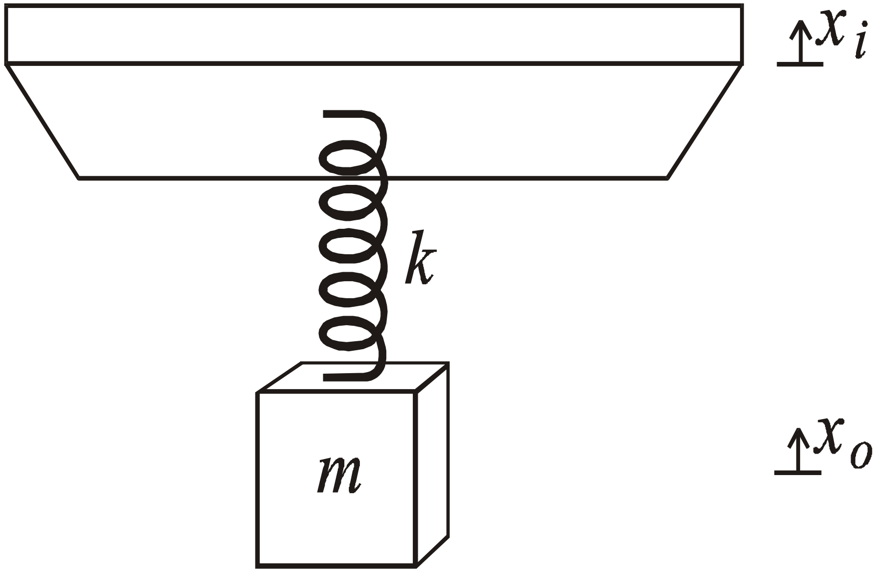
Linear systems

A *linear system* is a physical system that produces a single output from a single input, for which there is a linear relation between the input and the output.

That is, if input $h_1(t)$ causes output $v_1(t)$ and input $h_2(t)$ causes output $v_2(t)$, the application of input $h_1(t) + h_2(t)$ causes the output $v_1(t) + v_2(t)$.

Canonical example

My favorite linear system, the simple harmonic oscillator, a.k.a. a mass on a spring.



The input is the position of the top of the spring, and the output is the position of the mass.

Filters and transducers

More terminology:

A *filter* is a linear system whose input and output have the same units.

- Our canonical example is a filter, since both input and output are positions.

A *transducer* is a linear system whose input and output have different units.

- A gravitational wave interferometer's input is a dimensionless strain, but its output is an optical power in watts.

Equation of motion

The equation of motion of our canonical linear system is

$$m\ddot{x}_o + k\left(x_o - x_i\right) + b\dot{x}_o = 0.$$

(Note, I've included some velocity damping of the motion of the mass.)

The input is applied by moving the top of the spring, thus stretching the spring (the mass has inertia), so a Hooke's Law force is applied to the mass.

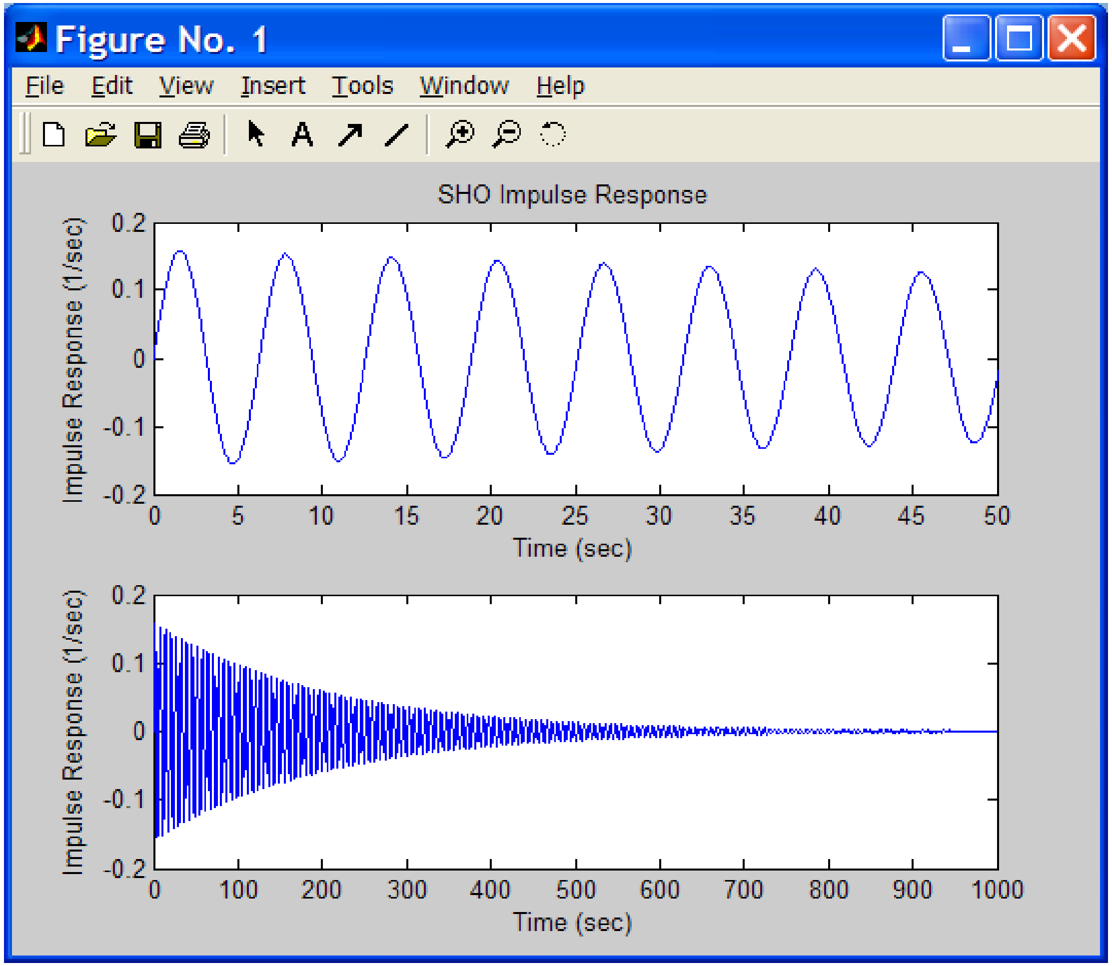
Impulse response

The input-output relationship of a linear system can be encapsulated in its impulse response $g(\tau)$, giving its output due to an input consisting of a single unit impulse (i.e., a delta function) at $\tau = 0$.

Causality requires that

$$g(\tau) = 0 \text{ for } \tau < 0.$$

Impulse response of oscillator



Relation of output to input

Theorem: The output of a linear system in response to an arbitrary input is that input convolved with the system's impulse response.

$$x_0(t) = \int_{-\infty}^{t} x_i(\tau) g(t-\tau) d\tau$$

Intuition: Each instant of the input launches a new impulse response, with height proportional to the input's strength at that moment. Total response is superposition of all of those impulse responses.

Intuition example

Consider the output of the system at $t = 0$.

$$x_0(0) = \int_{-\infty}^{0} x_i(\tau) g(-\tau) d\tau$$

The output consists of the sum of

- $x_i(0)g(0)$
- $x_i(-1)g(1)$
- $x_i(-2)g(2)$
- $x_i(-3)g(3)$
- … and all times between and beyond

Note that present value depends only on past inputs.

Frequency response

Deriving the impulse response from the e.o.m. can be a pain. It is more of a pain to evaluate the convolution of the impulse response with the input.

Alternative: Work with the Fourier transform of the impulse response, which is called the *frequency response*

$$G(f) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau) e^{-i2\pi f\tau} d\tau.$$

As we'll see, the algebra governing the frequency response is much simpler. <u>Building up intuition in the frequency domain is very worthwhile.</u>

Convolution Theorem

The utility of the frequency response can be seen in the Convolution Theorem

$$X_o(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i2\pi ft} dt \int_{-\infty}^{t} x_i(\tau) g(t-\tau) d\tau$$

$$= X_i(f) G(f)$$

Multiplication (in the frequency domain) is a much easier operation than carrying out a convolution integral (in the time domain). This is one of the main reasons for the use of frequency-domain analysis.

Interpretation of the frequency response

- The Convolution Theorem can also be written as

$$G(f) = \frac{X_o(f)}{X_i(f)},$$

- This means that the frequency response of a system is the (complex) ratio of the Fourier transform of its output to the Fourier transform of its input.
- This also suggests a way of measuring the frequency response, by measuring the complex ratio of output to input for a set of sinusoidal inputs.

Frequency response example

Derivation of frequency responses also involves easier math than finding the impulse response. Here's how.

Consider a sinusoidal input of frequency *f*:

$$x_i(t) = X_i(f)e^{i2\pi ft}.$$

Then, the output will also have a sinusoidal form, since the e.o.m. is linear.

$$x_o(t) = X_o(f)e^{i2\pi ft}.$$

Frequency response example (II)

Recall:

$$\frac{d}{dt} e^{i2\pi ft} = i2\pi f e^{i2\pi ft}, \quad \frac{\mathrm{d}^2}{\mathrm{dt}^2} e^{i2\pi ft} = -(2\pi f)^2 e^{i2\pi ft}$$

Plug our *ansatz* into the e.o.m.

$$m\ddot{x}_o + k\left(x_o - x_i\right) + b\dot{x}_o = 0.$$

divide through by $e^{i2\pi ft}$ everywhere, and find

$$-m(2\pi f)^2 X_o + k(X_o - X_i) + i2\pi f b X_o = 0.$$

Finally, solve for $G(f) = X_o(f)/X_i(f)$

$$G(f) = \frac{k}{k + i2\pi f b - m(2\pi f)^2}.$$

Frequency response example (II)

$$G(f) = \frac{k}{k + i2\pi f b - m(2\pi f)^2}.$$

Q: Why are we happy to have done this?

A:

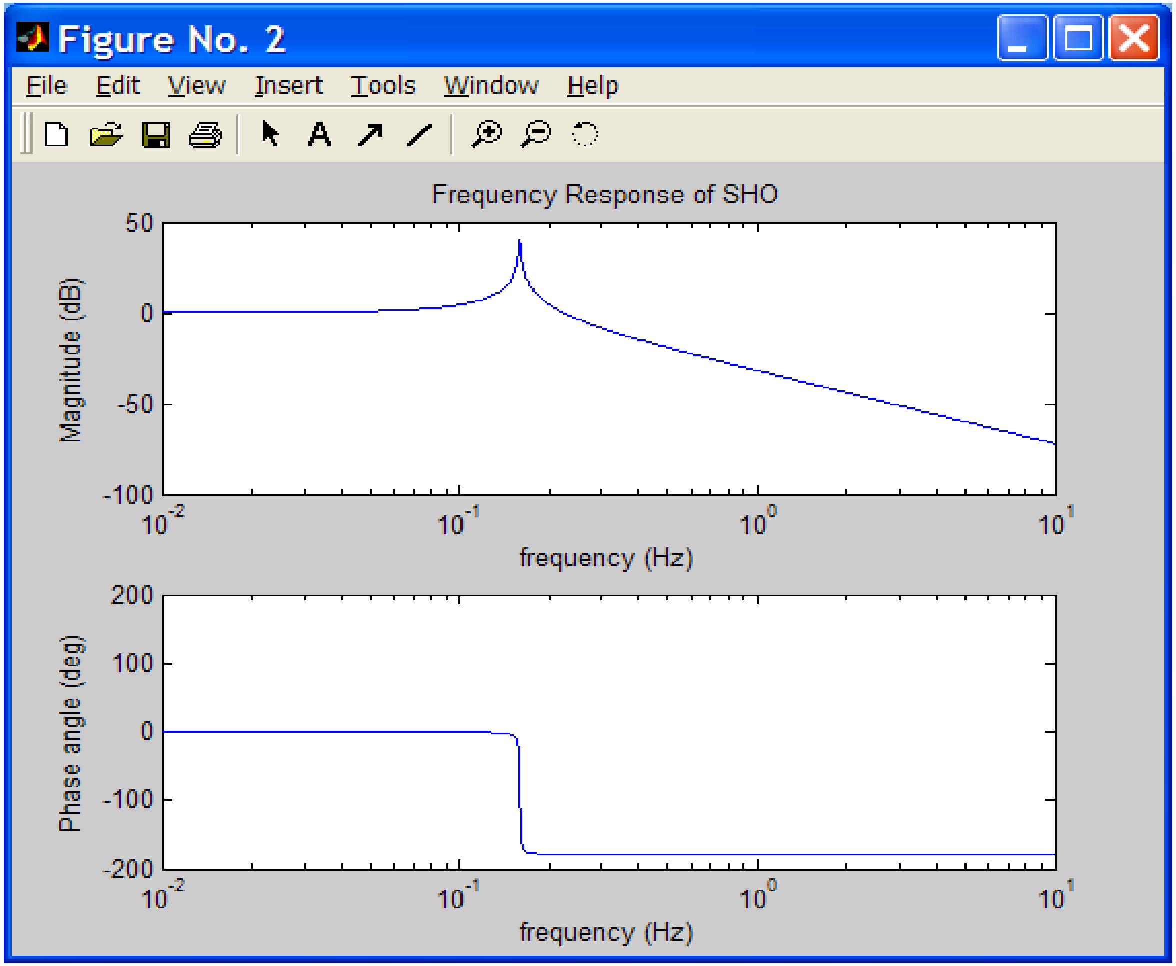
1. Using only simple algebra, we've solved a differential equation.
2. We can gain insight in the frequency domain that is hard to obtain in the time domain.

Bode plots

A frequency response is typically graphed in the form of a Bode plot (actually two graphs on the same logarithmic frequency scale.)

a) The magnitude of the frequency response is plotted on a logarithmic scale. The traditional units are deciBels (dB), given by Mag(dB) = 20 $\log_{10}|G(f)|$.
b) The phase of $G(f)$ is plotted on a linear scale between –180 and +180 degrees.

Bode plot of our example's frequency response



Reading a Bode plot

The resonant frequency stands out as the place where response is largest. It isn't infinite, because of damping.

At $f \ll f_{res}$, response is unity (= 0 dB.) The mass tracks the motion of the top end of the spring. The dynamics is "stiffness controlled."

At $f \gg f_{res}$, the mass moves less at higher frequencies (proportional to $1/f^2$, or –40 dB per decade), due to the inertia of the mass.

The utility of the frequency response

The frequency response is a popular mathematical tool because

- It is a simple way to solve for the dynamics of a system.
- The frequency response of a set of systems cascaded together is simply the product of their individual frequency responses. (In time domain, we'd have to do multiple convolution integrals.) This will be handy when we analyze servos. (More on this in a later lecture.)
- Measurement is straightforward.
- Separation of frequencies lets one easily see dynamics on many scales.
- Intuition is easy to gain, with practice.