Introduction and Background

For the first few problems we will practice some calculations using the Schwarzschild spacetime. In Schwarzschild coordinates, the invariant interval is

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -(1-2M/r)dt^2 + dr^2/(1-2M/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1}$$

Here we have used the *Einstein summation convention*, in which we sum, over all four spacetime components, any time we have the same index repeated: one as a subscript, one as a superscript. Therefore, for example, the square of the four-velocity is $u^\alpha u_\alpha = u_\alpha u^\alpha = u^t u_t + u^r u_r + u^\theta u_\theta + u^\phi u_\phi$. The covariant components (with lowered, i.e., subscript, indices) of the metric tensor g can be read off the line element: $g_{tt} = -(1-2M/r)$, $g_{rr} = 1/(1-2M/r)$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$, and all other elements are zero (e.g., $g_{\phi t} = g_{\theta r} = 0$). The *contravariant* components of the metric tensor can be obtained as a matrix inverse of the covariant components:

$$g^{\alpha\beta}g_{\beta\mu} = \delta^\alpha_{\ \mu} \ , \tag{2}$$

where $\delta^\alpha_{\ \mu}$ is the Kronecker delta: it equals 1 if $\alpha = \mu$, and equals zero otherwise. Note that because we sum over β in this expression, it is a *dummy index*; we can use any symbol we like, as long as it appears once as a superscript and once as a subscript on a single side of the equation. Because the Schwarzschild metric tensor is diagonal, we have simply that $g^{tt} = -1/(1-2M/r)$, $g^{rr} = (1-2M/r)$, $g^{\theta\theta} = 1/r^2$, and $g^{\phi\phi} = 1/(r^2\sin^2\theta)$, with other components being zero.

We use the metric tensor to raise or lower indices. Suppose, for example, that we have a vector v^α, which is therefore in contravariant form. To get the covariant components, we write $v_\beta = v^\alpha g_{\alpha\beta}$. Here, as in any tensor equations, we must ensure that the "free" (unsummed) indices match on either side, including whether they are up or down. As a check, we see that we sum over α, so that doesn't count, and β is a lowered index on both sides. Similarly, we could write $v^\mu = g^{\mu\nu}v_\nu$.

Let's also say a few words about what the covariant and contravariant components mean, at least for the four-velocity $\mathbf{u}$. Suppose we are considering the motion of a particle of nonzero rest mass. Then $u^\alpha = dx^\alpha/d\tau$, where τ is the proper time (i.e., the time as measured by someone riding along with the particle). For example, $u^r = dx^r/d\tau$, which we would usually write as just $dr/d\tau$. Similarly, $u^t = dt/d\tau$, which might look strange, but note that in general time will run differently for an observer at infinity (who sees time intervals of dt) than for a local observer (who sees time intervals of $d\tau$). Those are the contravariant components.

The covariant components are more directly related to the conserved quantities. For example, for the metric signature we have chosen $-u_t$ is the specific energy at infinity. That is, this is the energy per unit mass of the particle, with the reference point being that

$-u_t = 1$ for a particle at rest at infinite distance from the gravitating source. Similarly, u_ϕ is the specific angular momentum; it is the angular momentum per unit mass of the particle. u_r and u_θ do not have particularly important meanings. $-u_t$ and u_ϕ are conserved in the motion of a test particle in the Schwarzschild spacetime (test particle means that it reacts to the spacetime, but does not affect it). Another conserved quantity is the squared four-velocity: for a particle with nonzero rest mass, $u^\alpha u_\alpha = u^t u_t + u^r u_r + u^\theta u_\theta + u^\phi u_\phi = -1$.

Let's do a test problem first to give you an idea of how to do calculations in the Schwarzschild spacetime.

0. Consider a particle in temporarily azimuthal motion, with $u^r = u^\theta = 0$. Using the fact that $u^\alpha u_\alpha = -1$ for a particle with nonzero rest mass, derive the specific energy $-u_t$ as a function of u_ϕ. Note that u_ϕ does not have to be the value for a Keplerian orbit. To test your expression, consider a particle on the surface of a nonrotating star of radius r (such that $u_\phi = 0$). Does your expression make sense in the Newtonian limit $M/r \ll 1$?

Answer:

In spherical coordinates, the general formula for a particle of nonzero rest mass expands to

$$u^t u_t + u^r u_r + u^\theta u_\theta + u^\phi u_\phi = -1 \; . \tag{3}$$

Here $u^r = u^\theta = 0$. Our strategy will be to use the Schwarzschild metric tensor to express u^t and u^ϕ in terms of their covariant counterparts.

$$\begin{aligned} u^t u_t + u^\phi u_\phi &= -1 \\ (g^{tt} u_t) u_t + (g^{\phi\phi} u_\phi) u_\phi &= -1 \\ g^{tt} (u_t)^2 + g^{\phi\phi} (u_\phi)^2 &= -1 \\ -(1 - 2M/r)^{-1} u_t^2 + r^{-2} u_\phi^2 &= -1 \\ u_t^2 &= (1 - 2M/r)(1 + u_\phi^2/r^2) \\ -u_t &= (1 - 2M/r)^{1/2} (1 + u_\phi^2/r^2)^{1/2} \; . \end{aligned} \tag{4}$$

Note that because the metric tensor is diagonal, we can get away with writing $u^t = g^{tt} u_t$. If we were being completely general we would need to write $u^t = g^{t\alpha} u_\alpha = g^{tt} u_t + g^{tr} u_r + g^{t\theta} u_\theta + g^{t\phi} u_\phi$. Our sign choice in the final square root is determined by our overall convention, which indicates that $-u_t$ is positive. When $u_\phi = 0$, this becomes $-u_t = (1 - 2M/r)^{1/2}$. In the Newtonian limit $M/r \ll 1$, this is approximately $1 - M/r$, or $1 - GM/(rc^2)$ when we put the factors of G and c back in. This is the specific energy so to get the actual energy we multiply by mc^2 to get $mc^2 - GMm/r$ for a particle of mass m. This is indeed the weak-gravity expression: the total energy is the rest-mass energy minus the gravitational binding energy.

Now it's your turn. Good luck!

1. The specific angular momentum of a particle of nonzero rest mass in a circular orbit (which, again, has $u^r = u^\theta = 0$) is given by

$$u_\phi^2 = \frac{Mr^2}{r - 3M} \ . \tag{5}$$

Given that, compute the specific energy of a particle in a circular orbit. Does this work in the Newtonian limit $M/r \ll 1$?

2. Find the radius r at which the angular momentum is a minimum, and the value of the minimum angular momentum. By considering a circular orbit that loses a small amount of angular momentum, make a qualitative argument that the radius of minimum u_ϕ is also the radius of the innermost stable circular orbit (ISCO). Find the specific energy for a particle in a circular orbit at the ISCO; what fraction of the particle's rest-mass energy must be released to get to that orbit, and how does that compare with the 0.7% efficiency of hydrogen fusing into helium?

3. Using your expression for the specific angular momentum of a circular orbit, and for the specific energy, to derive the radius of the *marginally bound* orbit, which is where $-u_t = 1$ and hence a slight perturbation outward could send the particle to infinity.

4. The Schwarzschild time coordinate t is the elapsed time as seen at infinity. Therefore, the angular velocity of an orbit as seen from infinity is $d\phi/dt = (d\phi/d\tau)/(dt/d\tau) = u^\phi/u^t$. Use this and your previous expressions to derive the angular velocity of a circular orbit at radius r, as seen at infinity.

5. Dr. Sane plans to explore a black hole more directly. His idea is to free-fall radially to a nonrotating $10\,M_\odot$ black hole, then, just outside the horizon, fire his rockets outward to escape. Ignoring the overwhelming acceleration he would feel when he fired his rockets, estimate the maximum tidal force he would feel during his radial free fall and use that estimate to counsel him on whether his trip is advisable.

6. Stunned by your answer to the previous question, Dr. Sane has resolved to explore black holes in the safe way: theoretically. In doing so, he has proven that black holes cannot exist. Indeed, black holes violate special relativity; this had been missed by all the so-called great minds of the past. To understand his proof:

(a) Consider a particle that is initially at rest at infinity and falls radially into a Schwarzschild black hole: thus $-u_t = 1$ and $u^\theta = u^\phi = 0$. Derive the expression, as a function of r, for the proper radial velocity $u^r = dr/d\tau$.

(b) You should find that inside the event horizon, $u^r > 1$; remembering that this is in units of c, this implies the obviously ridiculous result that the speed is faster than light inside the horizon. To demonstrate that this is not simply a pathology of the coordinates, Dr. Sane points out that the radial velocity of this particle seen by a local static observer gives exactly the same expression that you got for the proper radial speed, so again we would have a contradiction if $r < 2M$, where $2M$ is the radius of the Schwarzschild event horizon. Therefore, all objects must have surfaces with $r > 2M$.

The US National Science Foundation has been called in to investigate a possible case of fraud against the gravitational wave community, which has claimed that they will detect BH-BH mergers. You have been consulted as an external expert to deliver your opinion. What is your evaluation?