Testing general relativity
with
gravitational waves

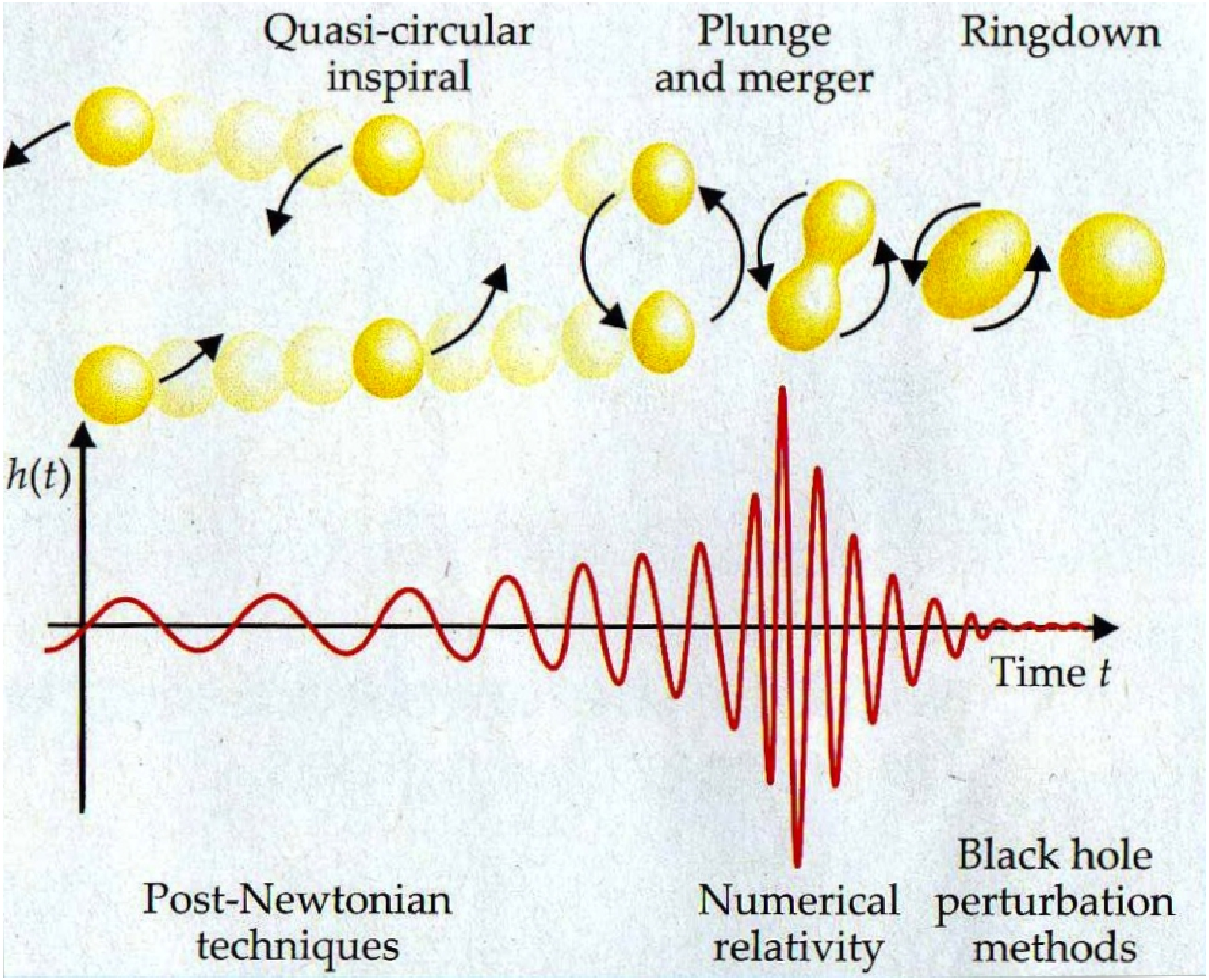
Statement of the problem

- Solar system tests of GR:
  - Perihelium precession of Mercury [weak, static field]
  - Deflection of starlight by the Sun [weak, static field]
  - Shapiro time delay [weak, static field]
  - Gravity Probe B
    - * Geodetic effect [weak, static field]
    - * Frame dragging effect [weak, stationary field]
- Binary neutron star observations (e.g. Hulse-Taylor):
  - Most penetrating tests of GR up to the present time
  - But: dissipative dynamics only to leading order (quadrupole)
  - Dynamical self-interaction of spacetime not being probed

No test of the strong-field dynamics of spacetime

Ideal laboratories: coalescing binary neutron stars and black holes

Coalescence of binary neutron stars and black holes



Inspiral

- Inspiral phase can be expressed as expansion in v/c. Schematically:

$$\boxed{\Phi(v(t)) = \left(\frac{v}{c}\right)^{-5} \sum_{n=0}^{7} \left[\psi_n + \psi_n^{(l)} \ln\left(\frac{v}{c}\right)\right] \left(\frac{v}{c}\right)^n}$$

- The coefficients ψ_n and $\psi_n^{(l)}$ are functions of component *masses* and *spins*

  – ψ_3 incorporates lowest-order non-linear effects:

  scattering of gravitational waves off spacetime

  + lowest-order spin-orbit effects

  – ψ_4 has lowest-order spin-spin effects

  – $\psi_5^{(l)}$ is lowest-order “logarithmic” coefficient

Hulse-Taylor and similar wide binaries only probe the leading order term!

Possible modifications to GR

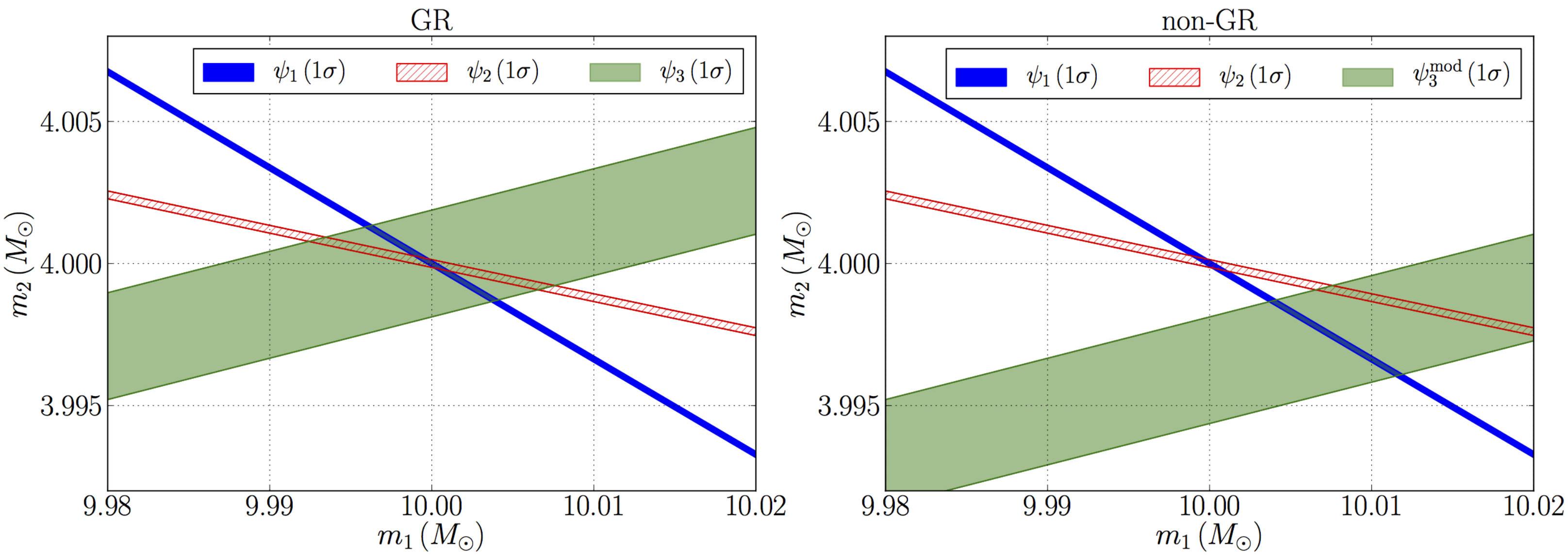
$$\boxed{\Phi(v(t)) = \left(\frac{v}{c}\right)^{-5} \sum_{n=0}^{7} \left[\psi_n + \psi_n^{(l)} \ln\left(\frac{v}{c}\right)\right] \left(\frac{v}{c}\right)^n}$$

- Massive graviton would modify ψ_2
- Scalar-tensor theories introduce $\psi_{ST} v^{-2}$ within the sum above
- Quadratic curvature corrections add $\psi_{QC} v^4$ within the sum
- Dynamical parity violations add $\psi_{CS} v^9$
- Varying G adds $\psi_{G(t)} v^{-8}$

Probing the strong-field dynamics with compact binaries

$$\Phi(v(t)) = \left(\frac{v}{c}\right)^{-5} \sum_{n=0}^{7} \left[\psi_n + \psi_n^{(l)} \ln\left(\frac{v}{c}\right)\right] \left(\frac{v}{c}\right)^n$$

- If no spins, then ψ_n and $\psi_n^{(l)}$ only depend on component masses m_1, m_2
- Measure any two coefficients and see whether a third one is consistent!



- Parameter estimation in this case does not easily allow us to combine information from multiple sources
  - Instead use model selection

Testing the strong-field dynamics with compact binaries

- Recall schematic expression for the gravitational wave phase:

$$\boxed{\Phi(v(t)) = \left(\frac{v}{c}\right)^{-5} \sum_{n=0}^{7} \left[\psi_n + \psi_n^{(l)} \ln\left(\frac{v}{c}\right)\right] \left(\frac{v}{c}\right)^n}$$

- Assume a clear-cut detection has been made: there is a signal in the noise!
- Want to use this detection to compare two hypotheses:

  $\mathcal{H}_{\mathrm{GR}}$ the signal waveform is as predicted by GR

  $\mathcal{H}_{\mathrm{modGR}}$ the signal waveform deviates from the GR prediction

- In practice: not possible to let $\mathcal{H}_{\mathrm{modGR}}$ be the negation of $\mathcal{H}_{\mathrm{GR}}$
- Choice we make:

  $\mathcal{H}_{\mathrm{modGR}}$ is the hypothesis that one or more of the ψ_n , $\psi_n^{(l)}$ are not as predicted by GR, without specifying which

Next problem: there is no single waveform family which can be used to check this hypothesis!

Testing the strong-field dynamics with compact binaries

$\mathcal{H}_{\mathrm{modGR}}$ is the hypothesis that one or more of the ψ_n , $\psi_n^{(l)}$ are not as predicted by GR, without specifying which

- Introduce auxiliary hypotheses:

  $H_{i_1 i_2 \ldots i_k}$ is the hypothesis that the phase coefficients $\{\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k}\}$ do *not* have the dependence on masses and spins as predicted by GR, but all other coefficients ψ_j , $j \notin \{i_1, i_2, \ldots, i_k\}$ do

- Let $\bar{\theta} = \{m_1, m_2, \vec{S}_1, \vec{S}_2, \ldots\}$ be the parameters appearing in the GR waveform. Then the hypothesis $H_{i_1 i_2 \ldots i_k}$ is tested by waveforms in which the parameters $\{\bar{\theta}, \psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k}\}$ are allowed to vary independently
- The hypothesis $\mathcal{H}_{\mathrm{modGR}}$ is the logical "or" of all the $H_{i_1 i_2 \ldots i_k}$:

$$\boxed{\mathcal{H}_{\mathrm{modGR}} = \bigvee_{i_1 < i_2 < \ldots < i_k} H_{i_1 i_2 \ldots i_k}}$$

Testing the strong-field dynamics with compact binaries

$$\boxed{\mathcal{H}_{\mathrm{modGR}} = \bigvee_{i_1 < i_2 < \ldots < i_k} H_{i_1 i_2 \ldots i_k}}$$

- Define the odds ratio

$$O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{P(\mathcal{H}_{\mathrm{modGR}}|d, \mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|d, \mathrm{I})}$$

- Example: Two "testing parameters" ψ_1 , ψ_2

$$\mathcal{H}_{\mathrm{modGR}} = H_1 \vee H_2 \vee H_{12}$$

$$^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{P(H_1 \vee H_2 \vee H_{12}|d, \mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|d, \mathrm{I})}$$

Testing the strong-field dynamics with compact binaries

$$ {}^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{P(H_1 \vee H_2 \vee H_{12}|d,\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|d,\mathrm{I})} $$

- Now note that any two auxiliary hypotheses $H_{i_1 i_2 \ldots i_k}$ and $H_{j_1 j_2 \ldots j_l}$ with $\{i_1, i_2, \ldots, i_k\} \neq \{j_1, j_2, \ldots, j_l\}$ are *logically disjoint*, so that

$$ {}^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{P(H_1|d,\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|d,\mathrm{I})} + \frac{P(H_2|d,\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|d,\mathrm{I})} + \frac{P(H_{12}|d,\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|d,\mathrm{I})} $$

- Use Bayes' theorem in each term:

$$ {}^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{P(H_1|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})}\frac{P(d|H_1,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(H_2|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})}\frac{P(d|H_2,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(H_{12}|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})}\frac{P(d|H_{12},\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} $$

What values to give to ratios of prior probabilities?

Testing the strong-field dynamics with compact binaries

$$ {}^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{P(H_1|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} \frac{P(d|H_1,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(H_2|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} \frac{P(d|H_2,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(H_{12}|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} \frac{P(d|H_{12},\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} $$

- No one auxiliary hypothesis is preferable over any other:

$$ \frac{P(H_1|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} = \frac{P(H_2|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} = \frac{P(H_{12}|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} $$

$$ \frac{P(\mathcal{H}_{\mathrm{modGR}}|\mathrm{I})}{P(\mathcal{H}_{\mathrm{GR}}|\mathrm{I})} = \alpha $$

- Then the odds ratio is proportional to the average of Bayes factors:

$$ {}^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{\alpha}{3} \left[\frac{P(d|H_1,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(d|H_2,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(d|H_{12},\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} \right] $$

Testing the strong-field dynamics with compact binaries

- Example of only two "testing parameters":

$$^{(2)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{\alpha}{3}\left[\frac{P(d|H_1,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(d|H_2,\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})} + \frac{P(d|H_{12},\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})}\right]$$

- Easy to generalize to N_T testing parameters $\{\psi_1, \psi_2, \ldots, \psi_{N_T}\}$:

$$\mathcal{H}_{\mathrm{modGR}} = \bigvee_{i_1<i_2<\ldots<i_k;\, k\leq N_T} H_{i_1 i_2 \ldots i_k}$$

$$^{(N_T)}O_{\mathrm{GR}}^{\mathrm{modGR}} = \frac{\alpha}{2^{N_T}-1}\sum_{k=1}^{N_T}\sum_{i_1<i_2<\ldots<i_k}\frac{P(d|H_{i_1 i_2 \ldots i_k},\mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}},\mathrm{I})}$$

- This applies to only one detected event. Is there a possibility to combine information from multiple detections?

Testing the strong-field dynamics with compact binaries

- Consider a catalog of detections $d_1, d_2, \ldots, d_\mathcal{N}$

$$
\begin{aligned}
\mathcal{O}_{\rm GR}^{\rm modGR} &= \frac{P(\mathcal{H}_{\rm modGR}|d_1, d_2, \ldots, d_\mathcal{N}, \mathrm{I})}{P(\mathcal{H}_{\rm GR}|d_1, d_2, \ldots, d_\mathcal{N}, \mathrm{I})} \\
&= \frac{\sum_{k=1}^{N_T} \sum_{i_1<i_2<\ldots<i_k} P(H_{i_1 i_2 \ldots i_k}|d_1, d_2, \ldots, d_\mathcal{N}, \mathrm{I})}{P(\mathcal{H}_{\rm GR}|d_1, d_2, \ldots, d_\mathcal{N}, \mathrm{I})} \\
&= \sum_{k=1}^{N_T} \sum_{i_1<i_2<\ldots<i_k} \frac{P(H_{i_1 i_2 \ldots i_k}|\mathrm{I})}{P(\mathcal{H}_{\rm GR}|\mathrm{I})} \frac{P(d_1, d_2, \ldots, d_\mathcal{N}|H_{i_1 i_2 \ldots i_k}, \mathrm{I})}{P(d_1, d_2, \ldots, d_\mathcal{N}|\mathcal{H}_{\rm GR}, \mathrm{I})} \\
&= \frac{\alpha}{2^{N_T} - 1} \sum_{k=1}^{N_T} \sum_{i_1<i_2<\ldots<i_k} \frac{P(d_1, d_2, \ldots, d_\mathcal{N}|H_{i_1 i_2 \ldots i_k}, \mathrm{I})}{P(d_1, d_2, \ldots, d_\mathcal{N}|\mathcal{H}_{\rm GR}, \mathrm{I})}
\end{aligned}
$$

- Detections are *independent*:

$$
\begin{aligned}
P(d_1, d_2, \ldots, d_\mathcal{N}|H_{i_1 i_2 \ldots i_k}, \mathrm{I}) &= \prod_{A=1}^{\mathcal{N}} P(d_A|H_{i_1 i_2 \ldots i_k}, \mathrm{I}) \\
P(d_1, d_2, \ldots, d_\mathcal{N}|\mathcal{H}_{\rm GR}, \mathrm{I}) &= \prod_{A=1}^{\mathcal{N}} P(d_A|\mathcal{H}_{\rm GR}, \mathrm{I})
\end{aligned}
$$

Testing the strong-field dynamics with compact binaries

- Final expression for odds ratio with *catalog of detections*:

$$\boxed{\mathcal{O}^{\mathrm{modGR}}_{\mathrm{GR}} = \frac{\alpha}{2^{N_T}-1} \sum_{k=1}^{N_T} \sum_{i_1<i_2<\ldots<i_k} \prod_{A=1}^{\mathcal{N}} \frac{P(d_A|H_{i_1 i_2 \ldots i_k}, \mathrm{I})}{P(d_A|\mathcal{H}_{\mathrm{GR}}, \mathrm{I})}}$$

- Recall that hypothesis $H_{i_1 i_2 \ldots i_k}$ is tested with waveforms that have free parameters $\{\bar{\theta}, \psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_k}\}$

- Convenient to write

$$\psi_i = \psi_i^{\mathrm{GR}}(m_1, m_2, \vec{S}_1, \vec{S}_2)\;[1 + \delta\chi_i]$$

- $\delta\chi_i$ allowed to vary from source to source: if deviation from GR, would expect it to depend on masses, spins, additional coupling constants and charges, ...

Simulations

- Simulated sources:
  - Binary neutron star inspirals
  - $m_1, m_2 \in [1, 2]\ M_{sun}$
  - Distributed uniformly in sky position
  - Random orientations of inspiral plane
  - Distances $\in$ [100, 400] Mpc
  - Inject in noise consistent with sensitivities of Advanced LIGO/Virgo

- Simulated *catalogs* of sources: 15 per catalog

- Use 3 testing parameters: ψ_1 , ψ_2 , ψ_3

  $\rightarrow \quad \mathcal{H}_{\mathrm{modGR}} = H_1 \vee H_2 \vee H_3 \vee H_{12} \vee H_{13} \vee H_{23} \vee H_{123}$

- Odds ratio for individual source:

$$^{(3)}O^{\mathrm{modGR}}_{\mathrm{GR}} = \frac{\alpha}{7}\left[\frac{P(d|H_1, \mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}}, \mathrm{I})} + \ldots + \frac{P(d|H_{123}, \mathrm{I})}{P(d|\mathcal{H}_{\mathrm{GR}}, \mathrm{I})}\right]$$

- Odds ratio for catalog of 15 sources:

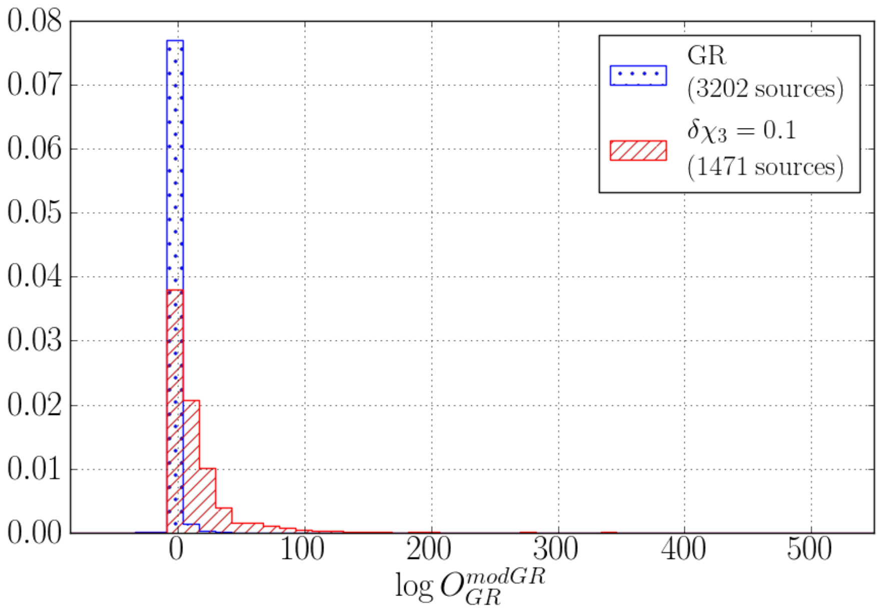
$$^{(3)}\mathcal{O}^{\mathrm{modGR}}_{\mathrm{GR}} = \frac{\alpha}{7}\left[\prod_{A=1}^{15}\frac{P(d_A|H_1, \mathrm{I})}{P(d_A|\mathcal{H}_{\mathrm{GR}}, \mathrm{I})} + \ldots + \prod_{A=1}^{15}\frac{P(d_A|H_{123}, \mathrm{I})}{P(d_A|\mathcal{H}_{\mathrm{GR}}, \mathrm{I})}\right]$$

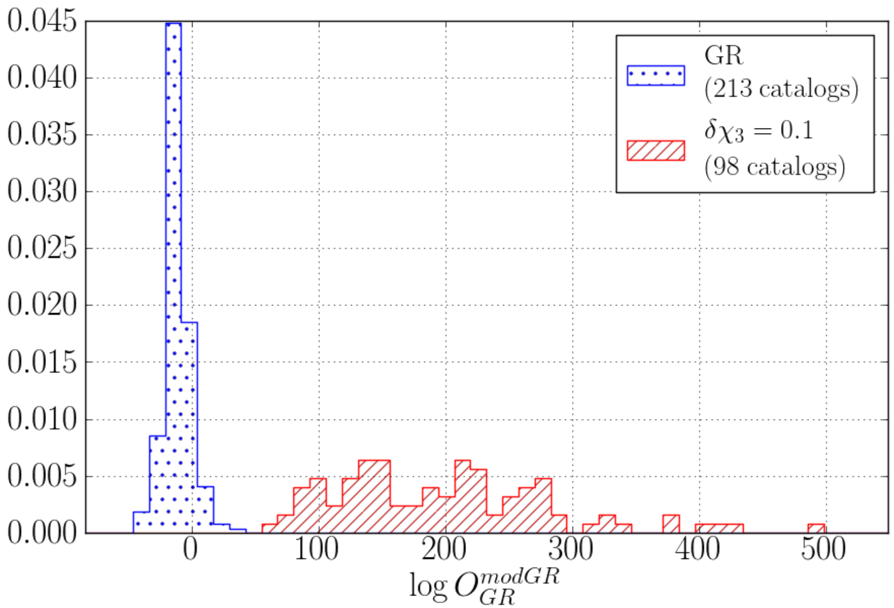
Example 1: Constant 10% shift at (v/c)^3

- Recall that phase coefficients parameterized as

$$\psi_i = \psi_i^{\rm GR}(m_1, m_2, \vec{S}_1, \vec{S}_2)\ [1 + \delta\chi_i]$$

- For the signals, choose constant $\delta\chi_3 = 0.1$





Individual sources

Catalogs of 15 sources each

Example 2: Signals with extra $(v/c)^{2.5}$ term in the phase

$$\boxed{\Phi(v(t)) = \left(\frac{v}{c}\right)^{-5} \sum_{n=0}^{7} \left[\psi_n + \psi_n^{(l)} \ln\left(\frac{v}{c}\right)\right] \left(\frac{v}{c}\right)^{n}}$$

- In GR, lower-order contributions to the phase look like $v^{-5}\psi_n v^n$ with integer n
- In the signals, introduce an anomalous *extra contribution* $v^{-5}\delta\chi_A v^{2.5}$
- But, auxiliary hypotheses used to analyze the data still the same as before:

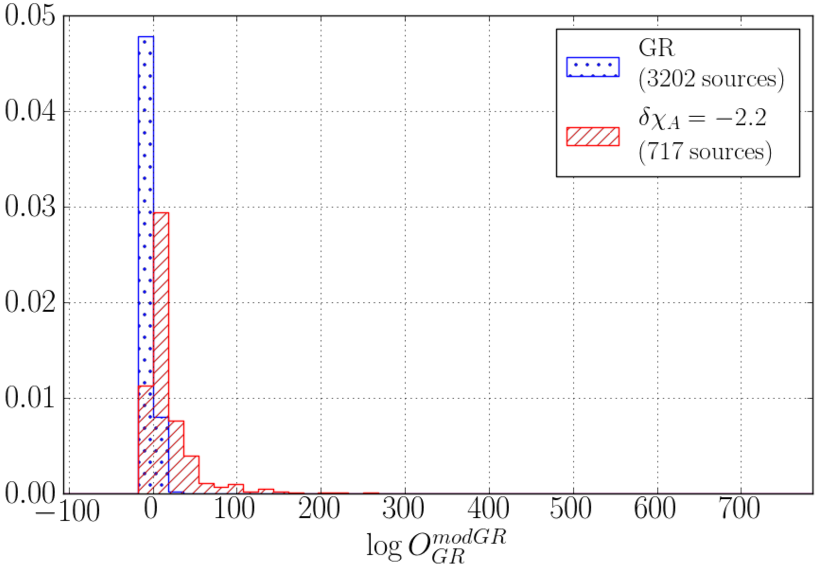
$$\{\bar{\theta}, \psi_1\}, \{\bar{\theta}, \psi_2\}, \{\bar{\theta}, \psi_3\}, \{\bar{\theta}, \psi_1, \psi_2\}, \{\bar{\theta}, \psi_1, \psi_3\}, \{\bar{\theta}, \psi_2, \psi_3\}, \{\bar{\theta}, \psi_1, \psi_2, \psi_3\}$$

corresponding to arbitrary shifts in coefficients of $(v/c)^{-5+1}$, $(v/c)^{-5+2}$, $(v/c)^{-5+3}$

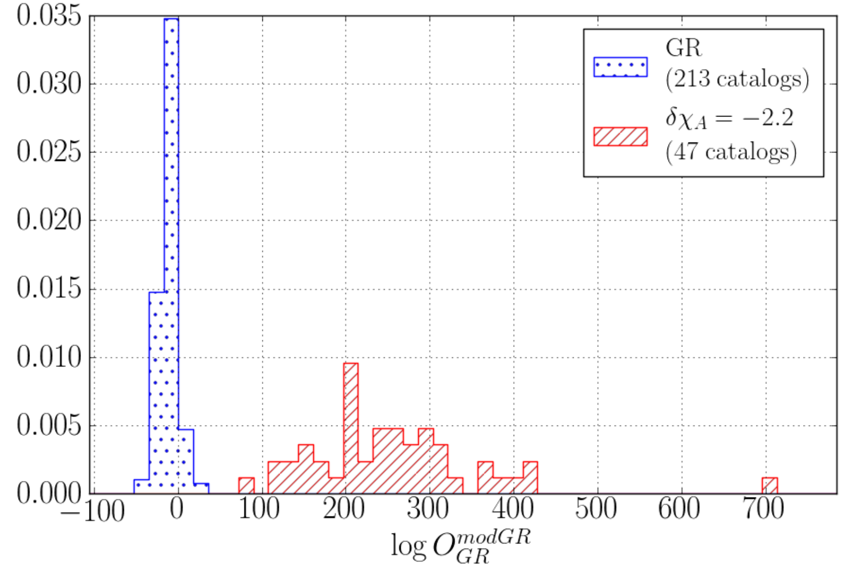
Expectation:

- Waveforms with extra degrees of freedom will be able to fit the signal better than GR
- Hence, GR violations can be picked up even if they are different from our auxiliary hypotheses

Example 2: Signals with extra $(v/c)^{2.5}$-term in the phase



Individual sources



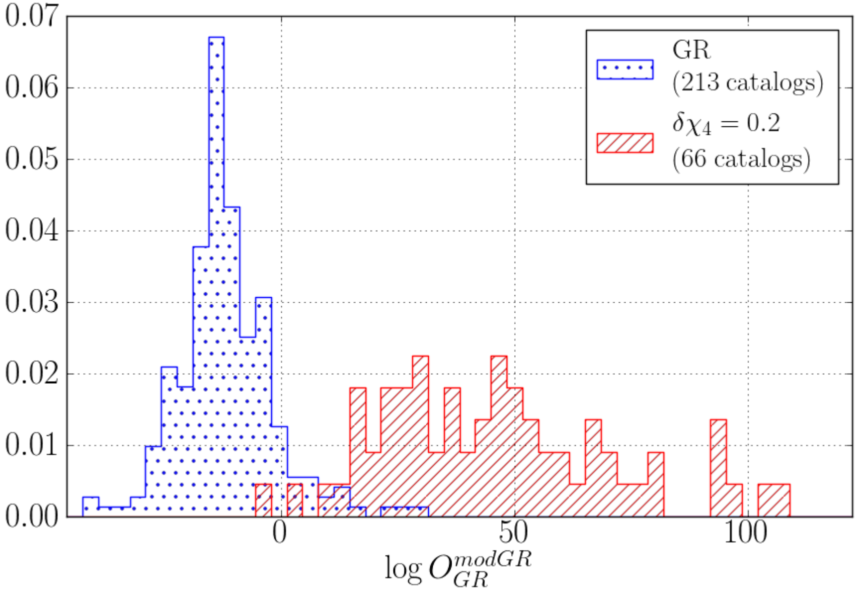
Catalogs of 15 sources each

- Anomalous behavior in the phase is indeed picked
- Reason: Waveforms will use available freedom in shifting phase coefficients to best match the signal

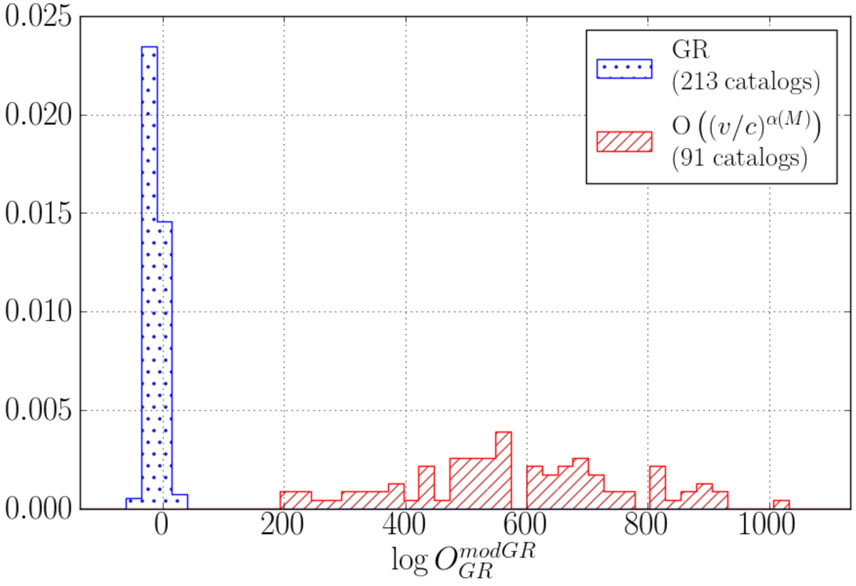
  → Modification of GR preferred over GR hypothesis

Further examples

$(v/c)^4$



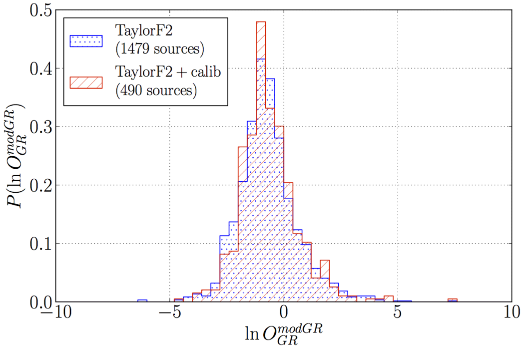
$(v/c)^{\alpha(M)}$



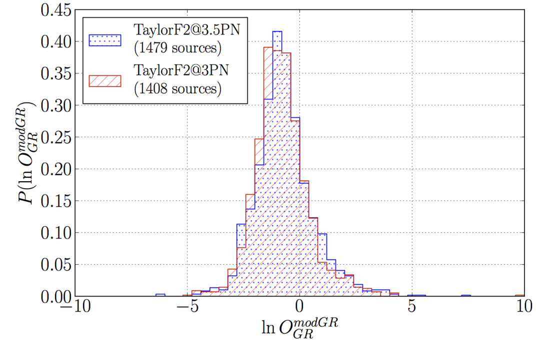
Method to find generic deviations from GR

Robustness against nuisance effects

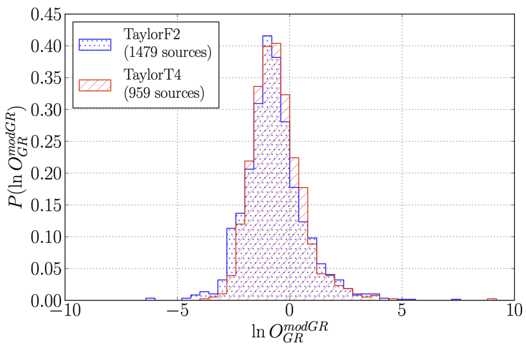
Instrumental calibration errors



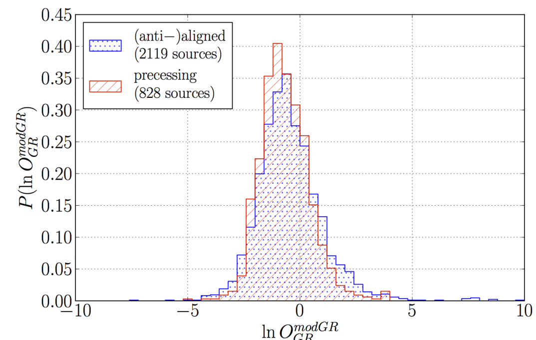
Finite number of known phase contributions



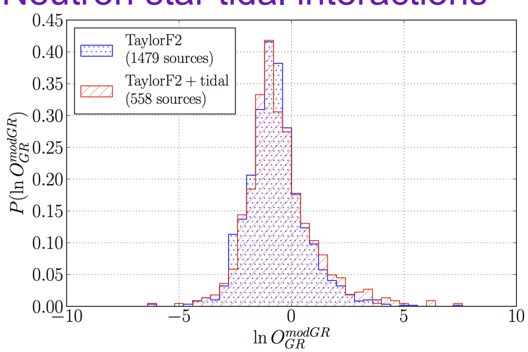
Different waveform approximations



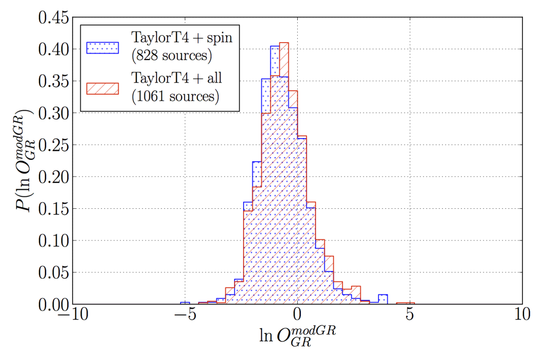
Neutron star spins



Neutron star tidal interactions

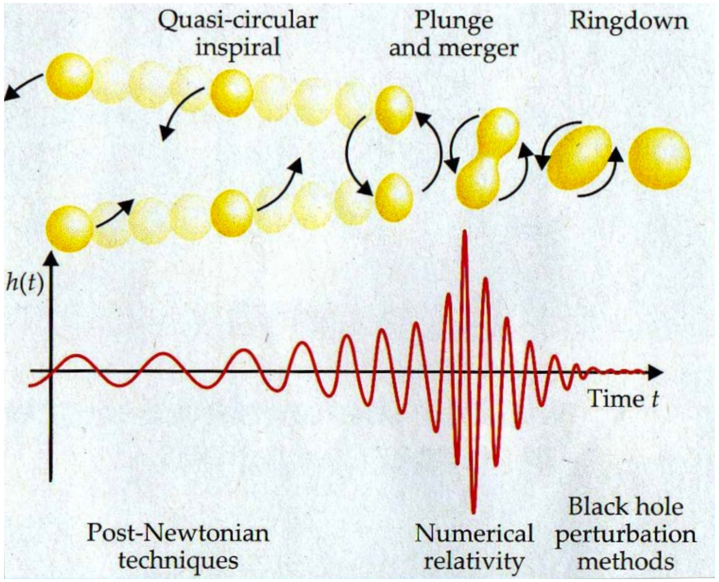


All effects together



What about binary black holes?

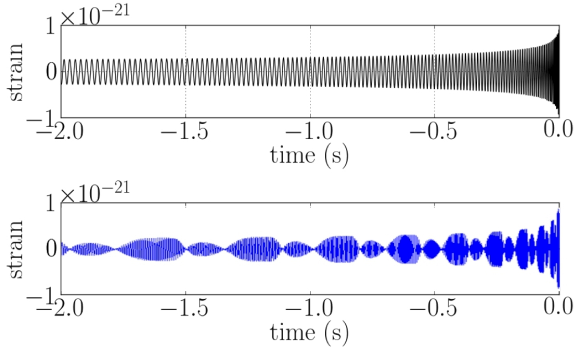
- For binary neutron stars a viable data analysis “pipeline” is at hand
  - Only inspiral part of the waveform in detectors' sensitive band
  - Small spins



- Binary black holes:
  - Inspiral, merger, ringdown
  - Large spins

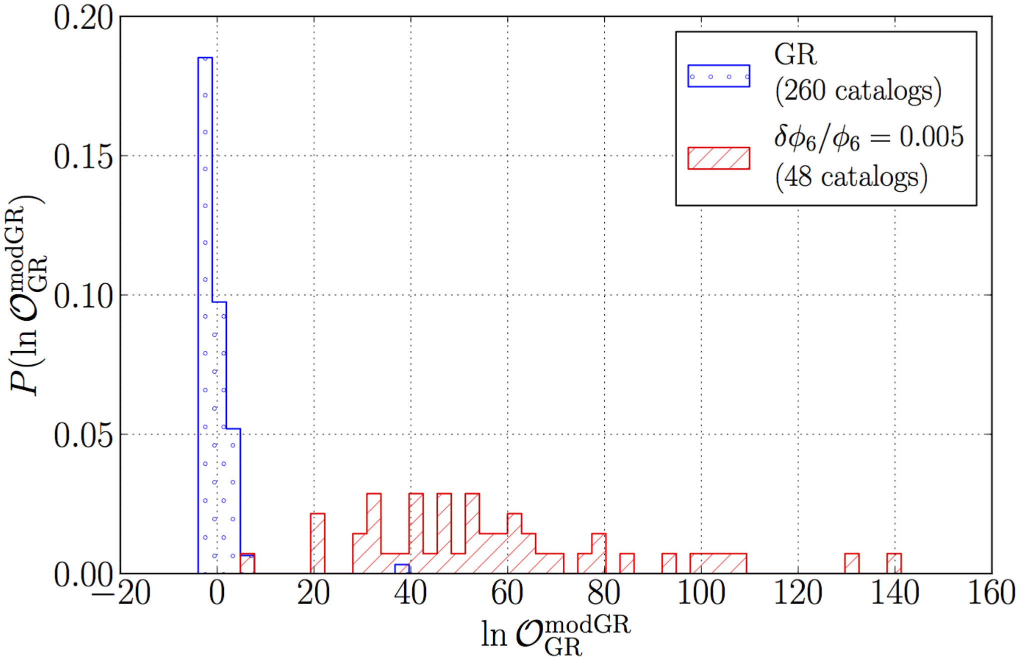
Dynamically richer, but...

  - Good waveform models becoming available only since recently
  - Analysis problem much harder



What about binary black holes?

- Exploratory work:
  - Reasonable waveform model with inspiral, merger, ringdown
  - Ignore spins



0.5% violation at $(v/c)^6$ beyond leading order can be seen

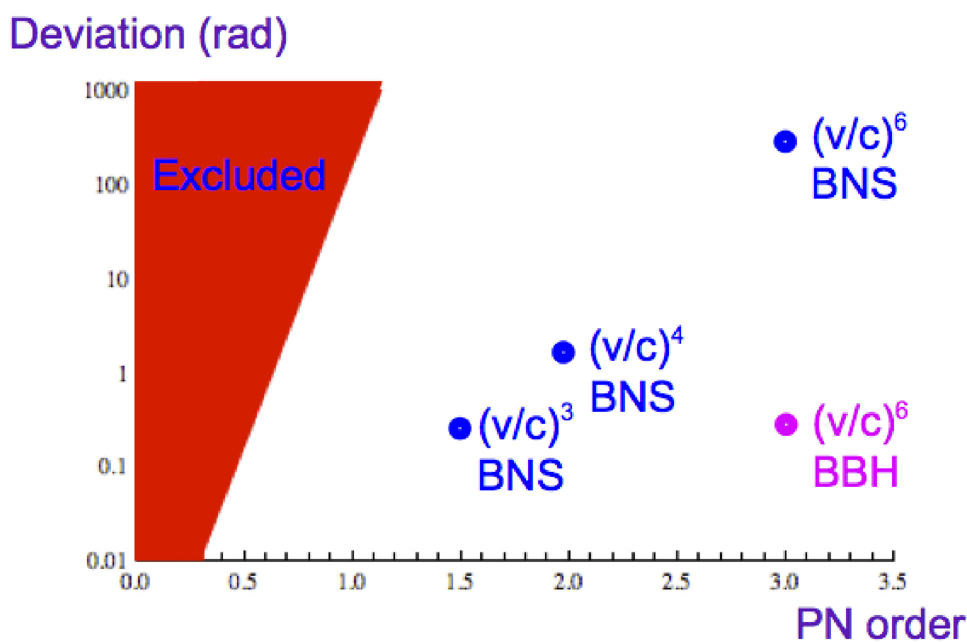
- More to be done... work in progress

Comparison with existing bounds?

- We will explore regime where (v/c) and GM/c^2R both $O(1)$
  - Compare binary pulsar studies from EM observations:

    $(v/c) \sim 2 \times 10^{-3}$ and $GM/c^2R \sim 4.4 \times 10^{-6}$

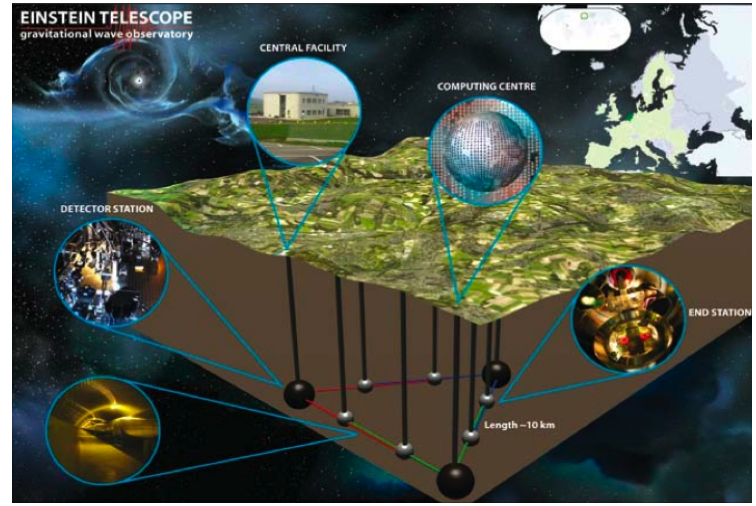
- Simple comparison: deviations from GR as $\Psi(v) \longrightarrow \Psi(v) + \beta(v/c)^b$
  - Red: already excluded by binary pulsar obs.
  - GW observations of BNS will already probe uncharted territory
  - Binary black holes (BBH) even more promising

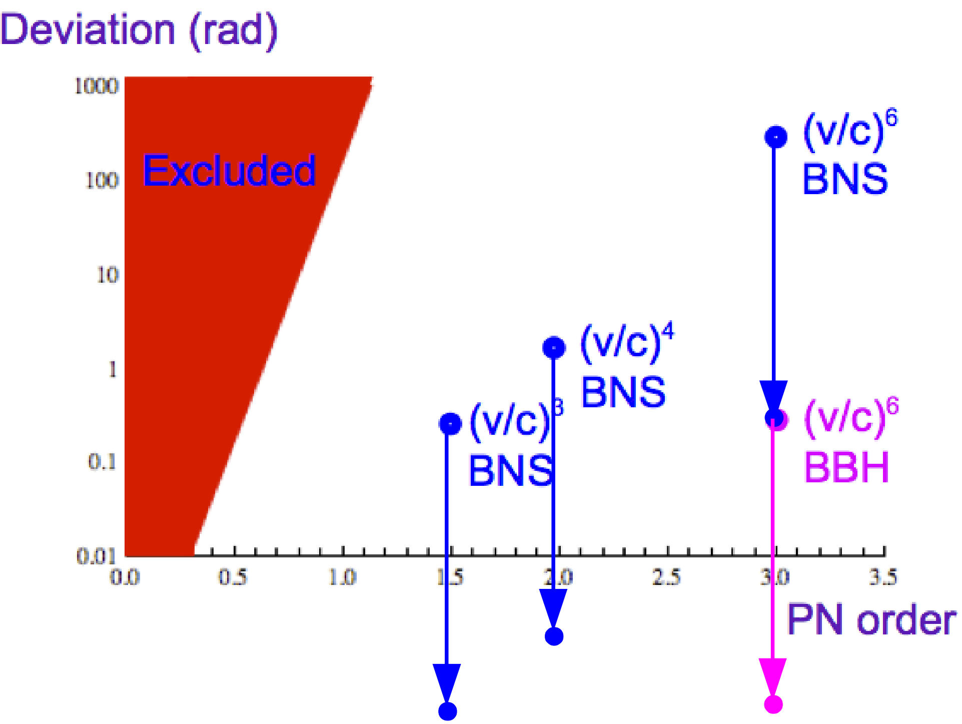


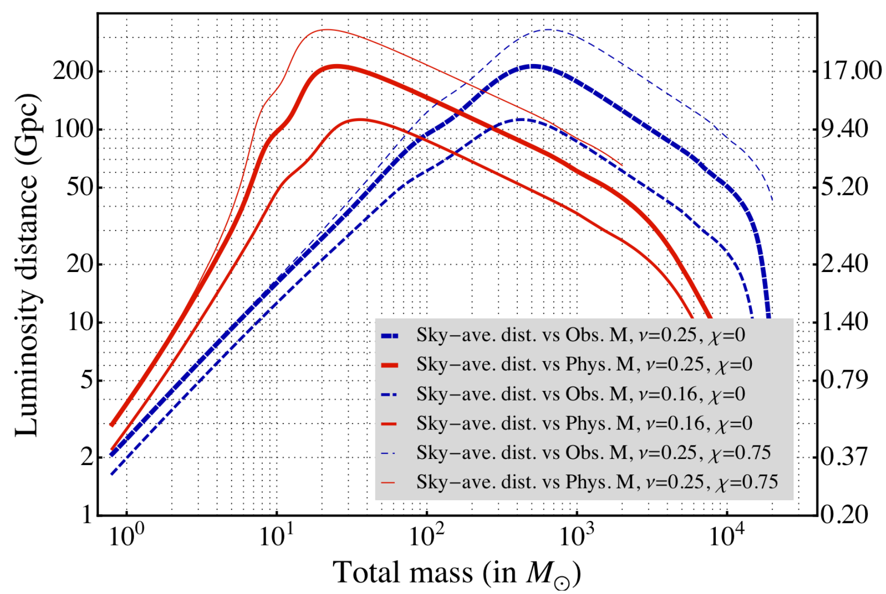
Einstein Telescope

- Same sources as in 2nd generation detectors will be seen with ~10 times higher SNR
- May see > 10^4 (weaker) sources out to redshift > 5

$\Rightarrow$ Gain of ~10^3

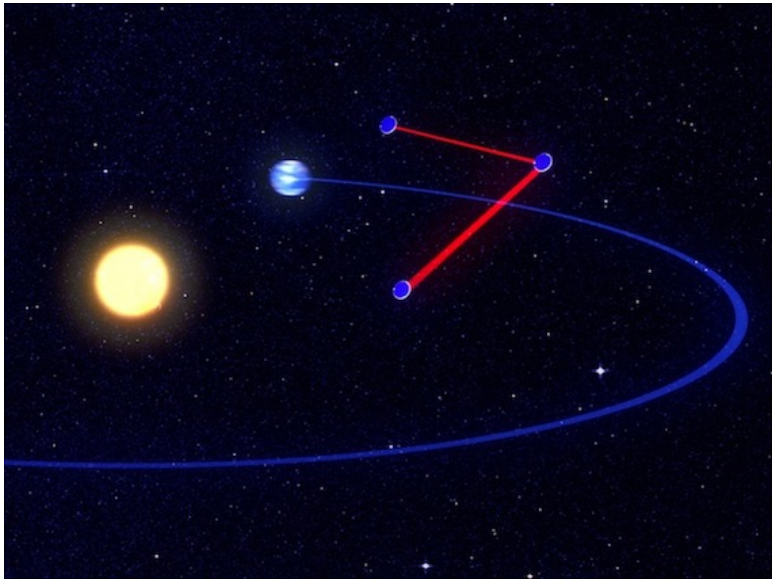


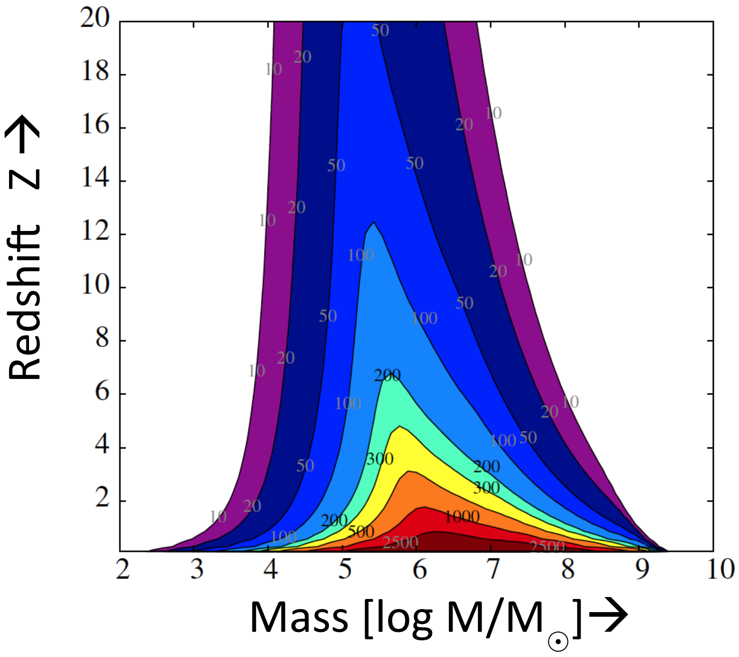


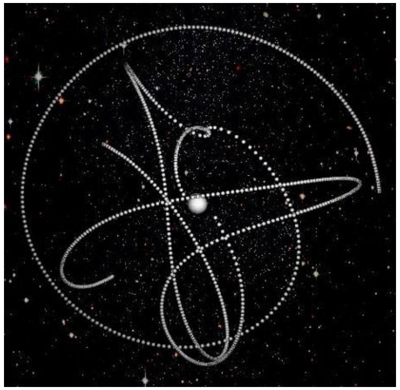


eLISA

- Supermassive binary black hole mergers
  - Here to, precision probing of inspiral, merger, ringdown
- Also tests of black hole no hair theorem with extreme mass ratio inspirals
  - Spacetime near astrophysical black holes only depends on mass M, spin J

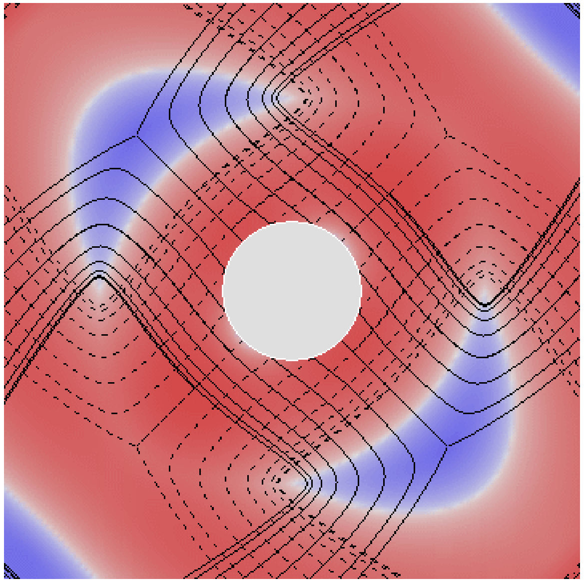






Testing the no hair theorem with black hole ringdowns

- Einstein Telescope and eLISA will allow us to clearly see ringdown signals of massive black hole binaries



- Superposition of modes with
  - Frequencies ω_{nlm}
  - Damping times τ_{nlm}
- Einstein equations force all of these

to depend only on mass M, spin J of final black hole:

$\omega_{nlm} = \omega_{nlm}(M, J), \qquad \tau_{nlm} = \tau_{nlm}(M, J)$

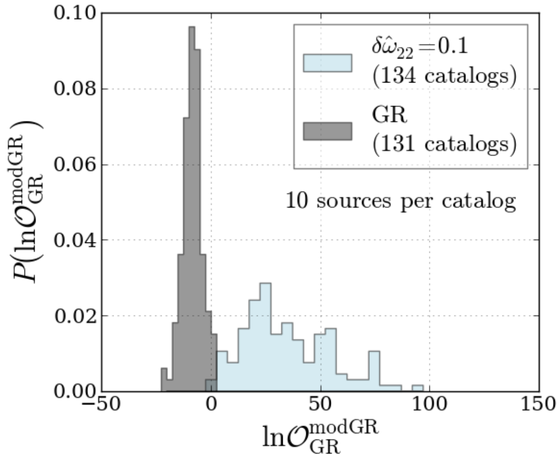
… hence only two of them are independent!

Again test of the no hair theorem

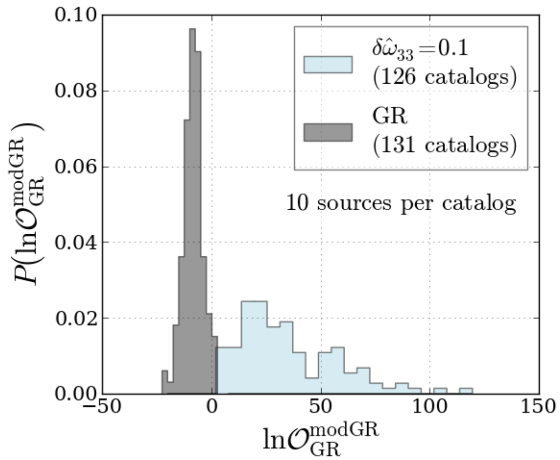
Testing the no hair theorem with Einstein Telescope

Use same model selection as before, but now for ω_{nlm}, τ_{nlm}

- 10% deviation in frequency of 22 mode:



- 10% deviation in frequency of 33 mode:



Outlook

- Direct gravitational wave detection will give us empirical access to the strong-field dynamics of spacetime
  - Rich physics
    - Observe dynamical self-interaction of spacetime itself
  - Variety of ways in which alternative theories of gravity can manifest themselves
- Already the 2nd generation detectors will take us into unexplored regime
  - Robust data analysis pipeline in place for BNS
  - NSBH/BBH more challenging, but great rewards
- Einstein Telescope and eLISA will herald ultra high precision tests
  - New kinds of tests possible, e.g. testing the no hair theorem