Bayesian model selection
and
parameter estimation

Bayesian inference: summary

- Parameter estimation: compute *posterior density function*

$$p(\vec{\theta}|d,H,I) \propto p(d|\vec{\theta},H,I)\,p(\vec{\theta}|H,I)$$

Posterior density function for one parameter, e.g. θ_1:

$$p(\theta_1|d,H,I) = \int_{\theta_2^{\min}}^{\theta_2^{\max}} \cdots \int_{\theta_N^{\min}}^{\theta_N^{\max}} p(\theta_1,\ldots,\theta_N|d,H,I)\,d\theta_2\ldots d\theta_N$$

- Model selection: compute *odds ratio*

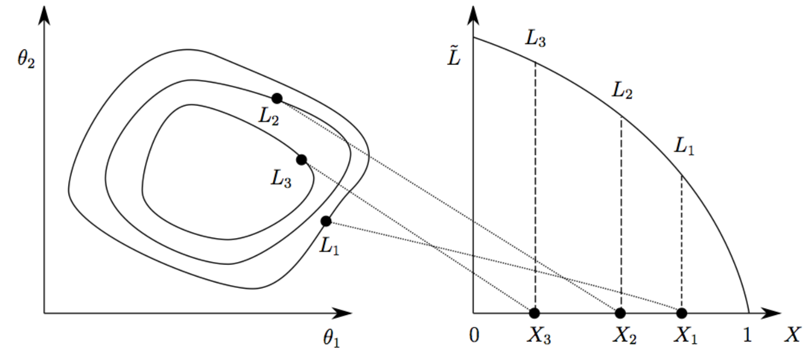
$$O_{H_2}^{H_1} \equiv \frac{p(H_1|d,I)}{p(H_2|d,I)} = \frac{p(d|H_1,I)}{p(d|H_2,I)}\frac{p(H_1|I)}{p(H_2|I)}$$

$$p(d|H,I) = \int p(d|\vec{\theta},H,I)\,p(\vec{\theta}|H,I)\,d\vec{\theta}$$

- Compute using e.g. *nested sampling*

$$p(d|H,I) = Z = \int \tilde{L}(X)\,dX \approx \sum_k L_k \Delta X_k$$

$$p(\vec{\theta}|d,H,I) = \tilde{P}(X) = \frac{\tilde{L}(X)}{Z}\Delta X$$



Gravitational-wave parameter estimation

- Parameter space is 15-dimensional:

$$\vec{\theta} = \{m_1, m_2, \vec{S}_1, \vec{S}_2, \alpha, \delta, \iota, \psi, d_{\mathrm{L}}, t_c, \varphi_c\}$$

- Different detectors *D* have different response to signals:

$$\tilde{h}^{(D)}(f) = \left[F_+^{(D)}\tilde{h}_+(f) + F_\times^{(D)}\tilde{h}_\times(f)\right] e^{-2\pi i f \Delta t^{(D)}}$$

where $F_+^{(D)}(\alpha, \delta, \psi, t_0)$ and $F_\times^{(D)}(\alpha, \delta, \psi, t_0)$ antenna pattern functions at geocentric arrival time t_0 while $\Delta t^{(D)}(\alpha, \delta, t_0)$ differences between arrival times at geocenter and at detectors

- Different noise realizations in different detectors:

$$\tilde{d}^{(D)}(f) = \tilde{h}^{(D)}(f) + \tilde{n}^{(D)}(f)$$

- Different noise power spectral densities:

$$\langle \tilde{n}^{(D)}(f)\, \tilde{n}^{(D')^*}(f') \rangle = \frac{1}{2}\delta(f - f')\delta_{DD'} S^{(D)}(f)$$

Gravitational-wave parameter estimation

- Examples of generative hypotheses:
  - Data is pure noise, $\mathcal{H}_N$
  - Data contains signal (with waveform from a certain family), $\mathcal{H}_S$

- Probability for noise realization $n^{(D)} = n_0$ is given by
  $p(n^{(D)} = n_0) \propto e^{-(n_0|n_0)/2}$
- Likelihood for $\mathcal{H}_N$:
  $p(d^{(D)}|\vec{\theta}, \mathcal{H}_N, I) \propto e^{-(d^{(D)}|d^{(D)})/2}$
- Likelihood for $\mathcal{H}_S$:
  $p(d^{(D)}|\vec{\theta}, \mathcal{H}_S, I) \propto e^{-(d^{(D)}-h^{(D)}|d^{(D)}-h^{(D)})/2}$
  (Recall $\tilde{d}^{(D)}(f) = \tilde{h}^{(D)}(f) + \tilde{n}^{(D)}(f)$)

- Data from different detectors $\vec{d} = \{d^{(H)}, d^{(L)}, ...\}$

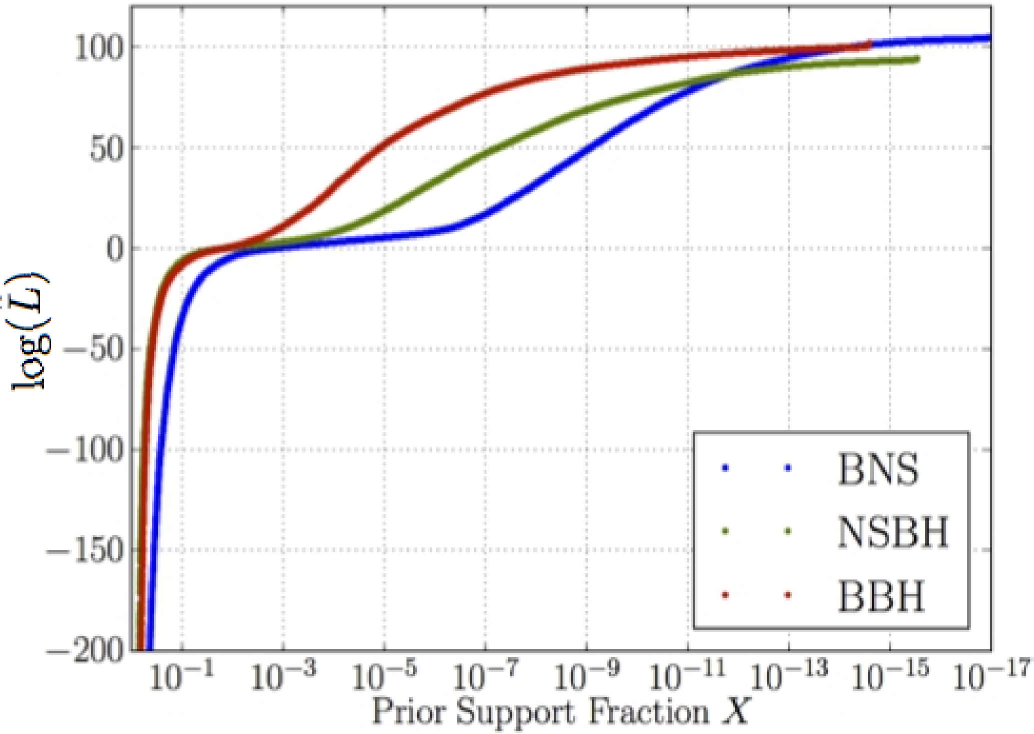
- Joint likelihood: $p(\vec{d}|\vec{\theta}, \mathcal{H}, I) = \prod_{(D)} p(\vec{d}^{(D)}|\vec{\theta}, \mathcal{H}, I)$

Gravitational-wave parameter estimation

- Assume that the search pipelines have found a compact binary coalescence signal
- Compute evidence for signal hypothesis $\mathcal{H}_S$:

$$p(d|\mathcal{H}_S, I) = Z = \int \tilde{L}(X)\, dX \approx \sum_k L_k \Delta X_k$$

- Typical growth of $\tilde{L}(X)$: usually convenient to consider logarithm



Gravitational-wave parameter estimation

- Posterior densities for parameters:

$$p(\vec{\theta}|d,\mathcal{H}_S,I) = \tilde{P}(X) = \frac{\tilde{L}(X)}{Z}\Delta X$$

- Marginalize to get posterior density for one particular parameter:

$$p(\theta_1|d,\mathcal{H}_S,I) = \int_{\theta_2^{\min}}^{\theta_2^{\max}} \cdots \int_{\theta_N^{\min}}^{\theta_N^{\max}} p(\theta_1,\ldots,\theta_N|d,\mathcal{H}_S,I)$$

… or get posterior density for two parameters jointly:

$$p(\theta_1,\theta_2|d,\mathcal{H}_S,I) = \int_{\theta_3^{\min}}^{\theta_3^{\max}} \cdots \int_{\theta_N^{\min}}^{\theta_N^{\max}} p(\theta_1,\ldots,\theta_N|d,\mathcal{H}_S,I)$$

- Often useful to change to a different set of parameters:

$$\begin{aligned} p(\vec{\theta}|d,\mathcal{H}_S,I)\,d\vec{\theta} &= p(\vec{\lambda}|d,\mathcal{H}_S,I)\,d\vec{\lambda} \\ &= p(\vec{\lambda}|d,\mathcal{H}_S,I)\,\det\frac{\partial(\lambda_1,\ldots,\lambda_N)}{\partial(\theta_1,\ldots,\theta_N)}\,d\vec{\theta} \end{aligned}$$

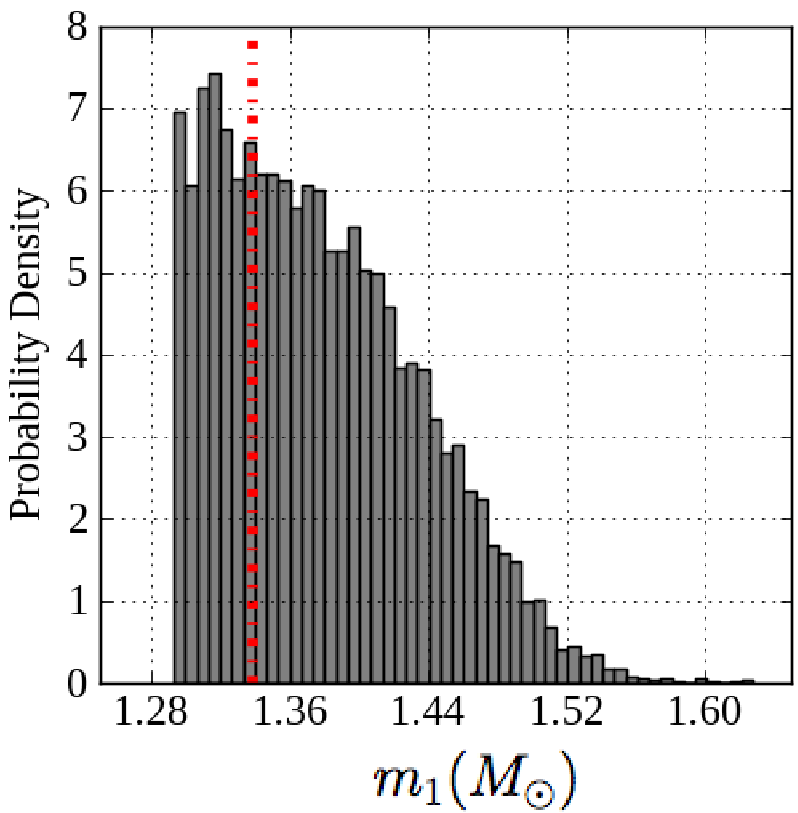
hence

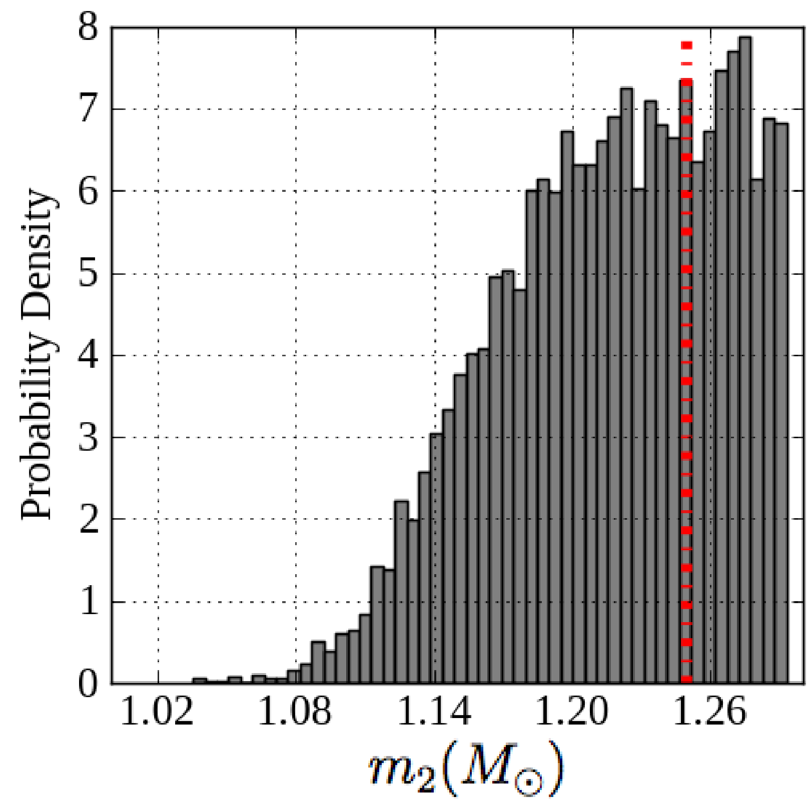
$$p(\vec{\theta}|d,\mathcal{H}_S,I) = p(\vec{\lambda}|d,\mathcal{H}_S,I)\,\det\frac{\partial(\lambda_1,\ldots,\lambda_N)}{\partial(\theta_1,\ldots,\theta_N)}$$

Gravitational-wave parameter estimation

Example: binary neutron star coalescence, $(m_1,m_2)=(1.34,1.25)M_{sun}$ at distance of 156.5 Mpc, simulated Gaussian noise for Advanced LIGO/Virgo

- Component masses:

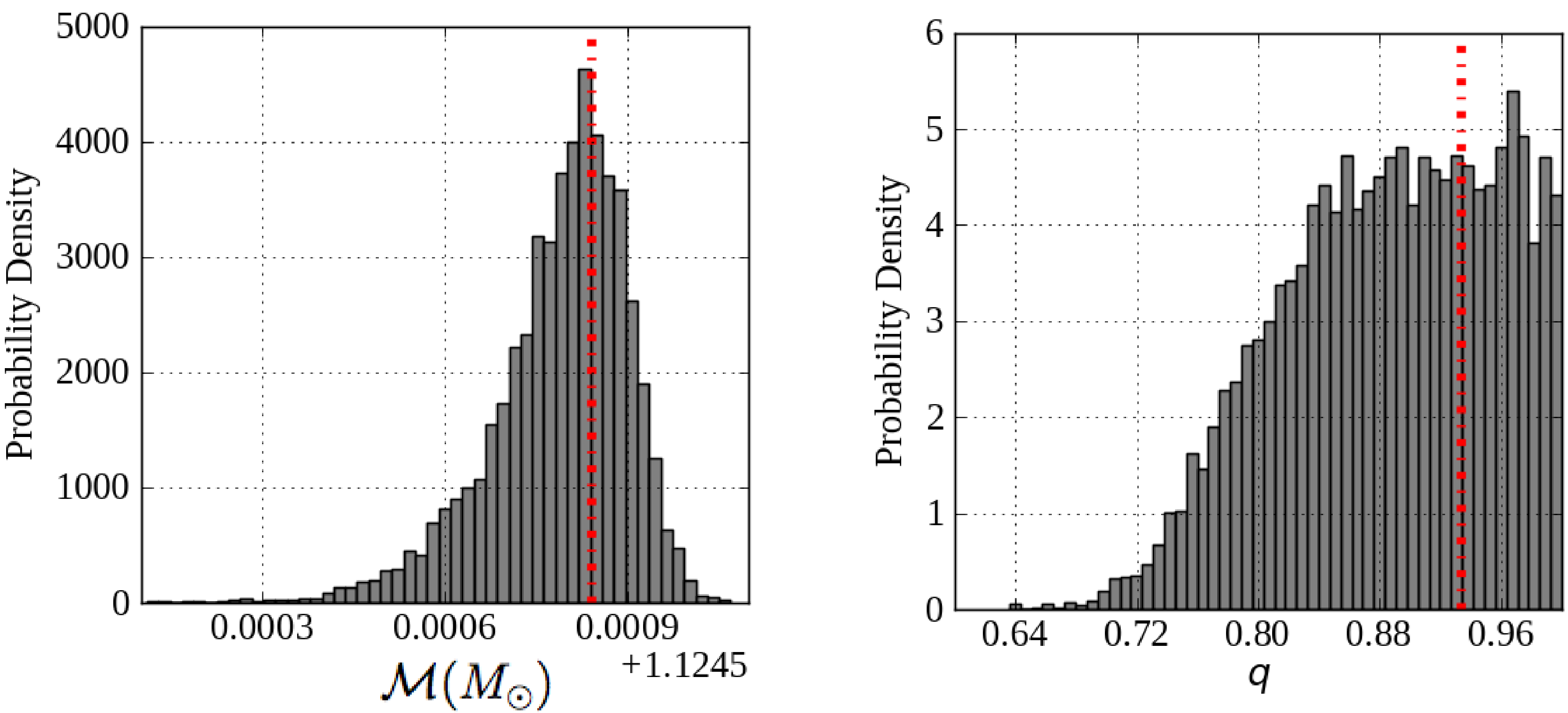




Gravitational-wave parameter estimation

Example: binary neutron star coalescence, $(m_1,m_2)=(1.34,1.25)M_{sun}$ at distance of 156.5 Mpc, simulated Gaussian noise for Advanced LIGO/Virgo

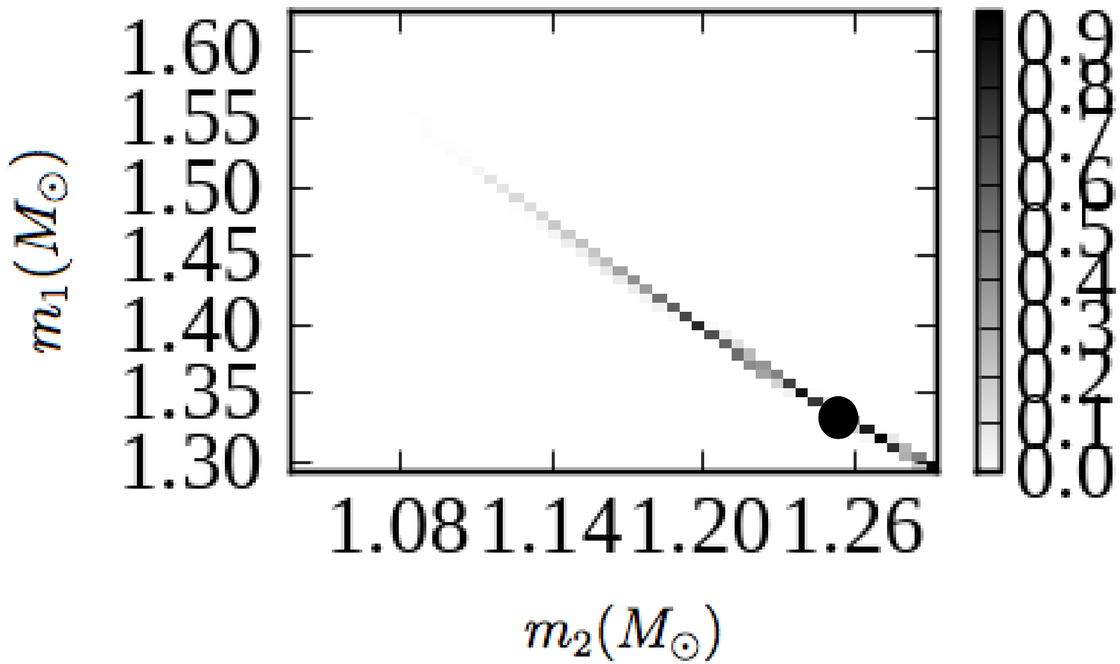
- Chirp mass $\mathcal{M}=(m_1+m_2)^{-1/5}(m_1 m_2)^{3/5}$ and mass ratio $q=m_2/m_1$:



Gravitational-wave parameter estimation

Example: binary neutron star coalescence, $(m_1,m_2)=(1.34,1.25)M_{sun}$ at distance of 156.5 Mpc, simulated Gaussian noise for Advanced LIGO/Virgo

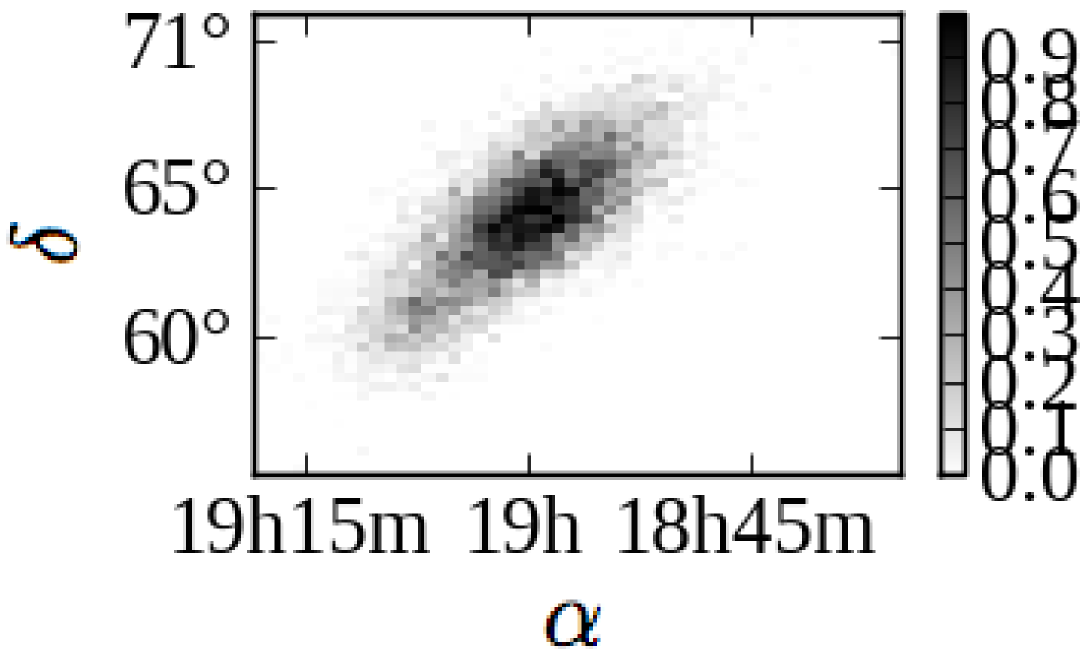
- Joint posterior density distribution for (m_1,m_2):



Gravitational-wave parameter estimation

Example: binary neutron star coalescence, $(m_1,m_2)=(1.34,1.25)M_{sun}$ at distance of 156.5 Mpc, simulated Gaussian noise for Advanced LIGO/Virgo

- Position on the sky:



Combining information from multiple sources

- Masses, sky position, distance, … are incidental
(though it will be of great interest to see how they are distributed!)

- Will sometimes want to check a *functional dependence*
E.g.:
- Dependence of post-Newtonian parameters on masses, spins
- Neutron star equation of state $P(\rho)$

- Two ways of doing this:
- Hypothesis testing: compare different possible functional dependences
- Measure parameters that determine functional dependence

- Assuming the functional dependence is universal, should be able to *combine information from multiple sources*

Example: the equation of state of neutron stars

- As binary neutron stars spiral towards each other, they start feeling each other's tidal effects

- Quadrupole deformation induced in one star by tidal tensor of the other:

  $Q_{ij} = -\lambda(m)\,\mathcal{E}_{ij}$

  Tidal deformability $\lambda(m)$ depends on neutron star equation of state $P(\rho)$

- Neutron star deformations affect surrounding spacetime curvature

  $\rightarrow$ Effect on orbital motion, e.g. angular motion $\Phi(t)$

  $\rightarrow$ Imprinted onto the phase of the gravitational wave signal $2\Phi(t)$

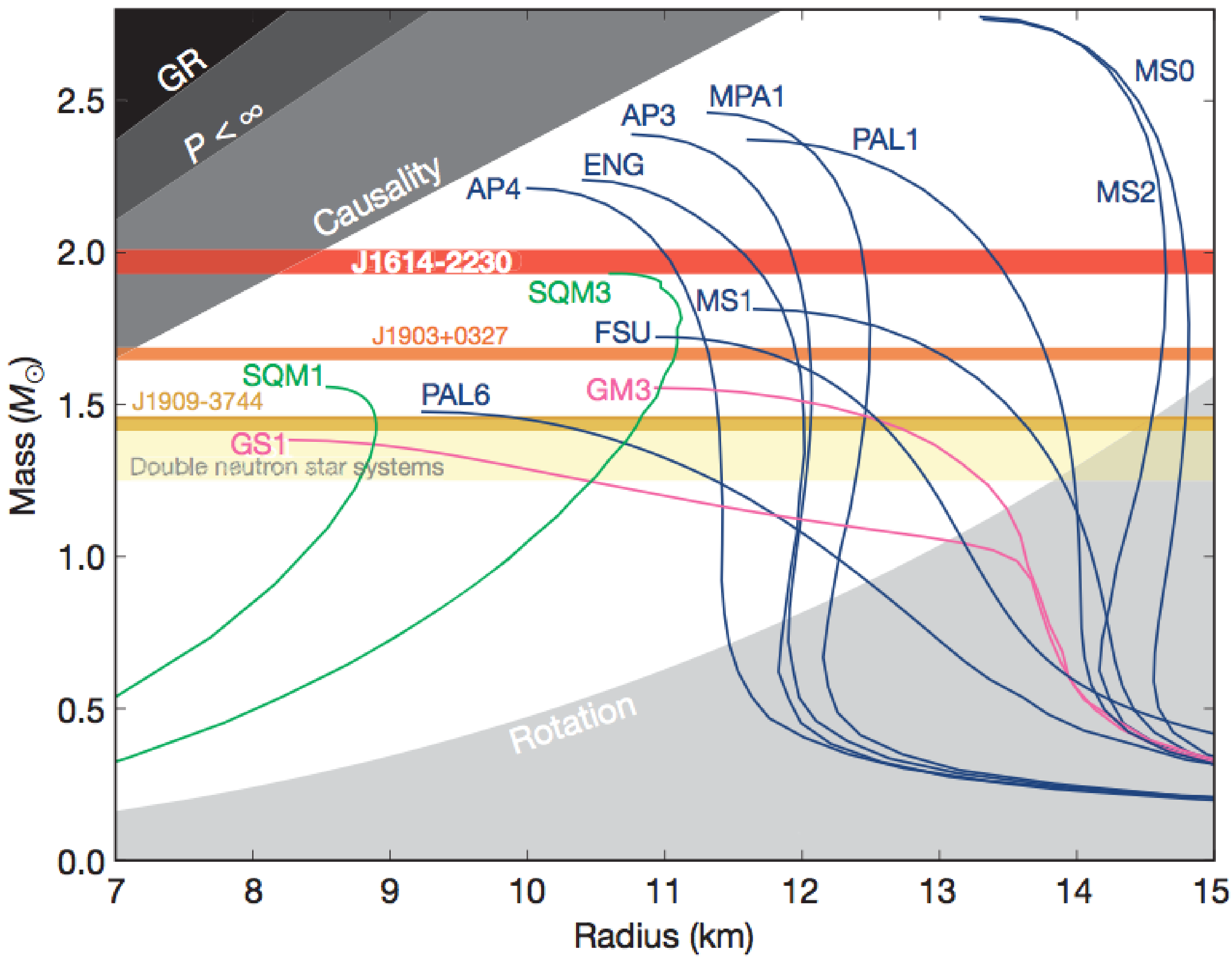
- Contributions to the phase don't show up until 5PN order in phase *but* with large prefactor:

  $\lambda(m)/M^5 \;\propto\; (R/M)^5 \;\sim\; 10^2 - 10^5$

Can we constrain neutron star equation of state with gravitational wave measurements?

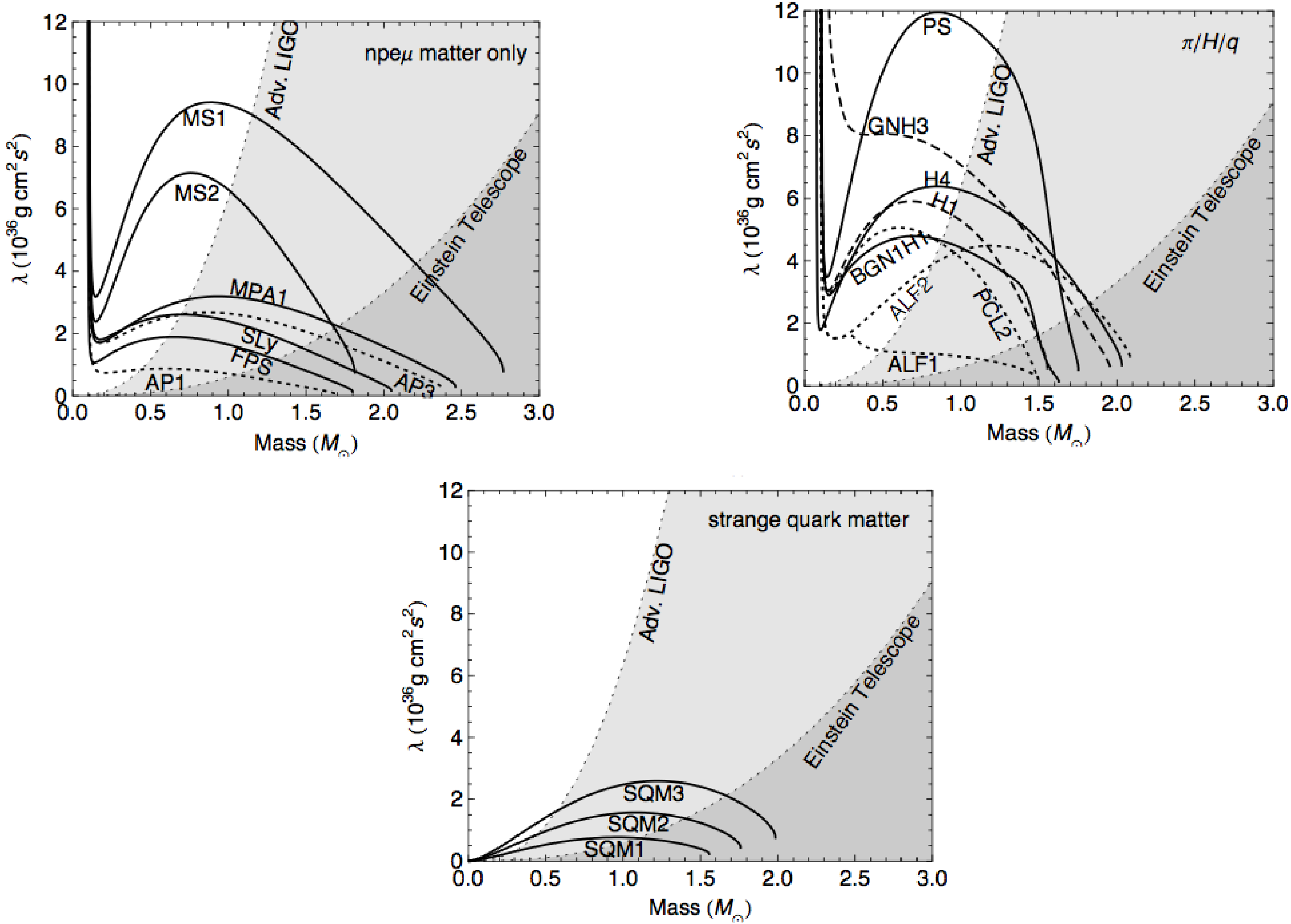
The equation of state of neutron stars

- Equation of state $P(\rho)$ maps to relation between radius and mass:



The equation of state of neutron stars

- Equation of state $P(\rho)$ maps to $\lambda(m)$:



Constraining the equation of state of neutron stars: Hypothesis ranking

- *Hypothesis ranking*:
  Take a set of (finitely many) EOS models $\{M_1, M_2, \ldots, M_K\}$
- Correspondingly, set of hypotheses $\{H_i; i = 1, \ldots, K\}$ where H_i states that M_i is the correct EOS
- Each EOS model comes with particular dependence $\lambda^{(i)}(m)$
- Let $\tilde{h}^{(i)}(\vec{\theta}; f)$ be the waveform model whose EOS contributions are determined by this $\lambda^{(i)}(m)$
- Then the likelihood function for the hypothesis H_i is given by

$$p(d|H_i, \vec{\theta}, I)$$
$$= \mathcal{N} \exp\left[-2 \int_{f_{\rm low}}^{f_{\rm cut}} \mathrm{d}f \frac{|\tilde{d}(f) - \tilde{h}^{(i)}(\vec{\theta}; f)|^2}{S_n(f)}\right]$$

- The evidence for H_i is

$$P(d|H_i, I) = \int \mathrm{d}\vec{\theta}\, p(\vec{\theta}|I)\, p(d|H_i, \vec{\theta}, I)$$

Constraining the equation of state of neutron stars: Hypothesis ranking

- *Odds ratio* in comparing any two hypotheses:

$$O^i_j \equiv \frac{P(H_i|d,I)}{P(H_j|d,I)} = \frac{P(H_i|I)}{P(H_j|I)}\frac{P(d|H_i,I)}{P(d|H_j,I)}$$

- Can this be extended to an odds ratio that combines information from multiple binary neutron star detections (stronger result)?

$$\begin{aligned} {}^{(N)}O^i_j &\equiv \frac{P(H_i|d_1,d_2,\ldots,d_N,I)}{P(H_j|d_1,d_2,\ldots,d_N,I)} \\ &= \frac{P(H_i|I)}{P(H_j|I)}\frac{P(d_1,d_2,\ldots,d_N|H_i,I)}{P(d_1,d_2,\ldots,d_N|H_j,I)} \end{aligned}$$

- Since different detections are independent,

$$P(d_1,d_2,\ldots,d_N|H_i,I) = \prod_{n=1}^{N} p(d_n|H_i,I)$$

Hence

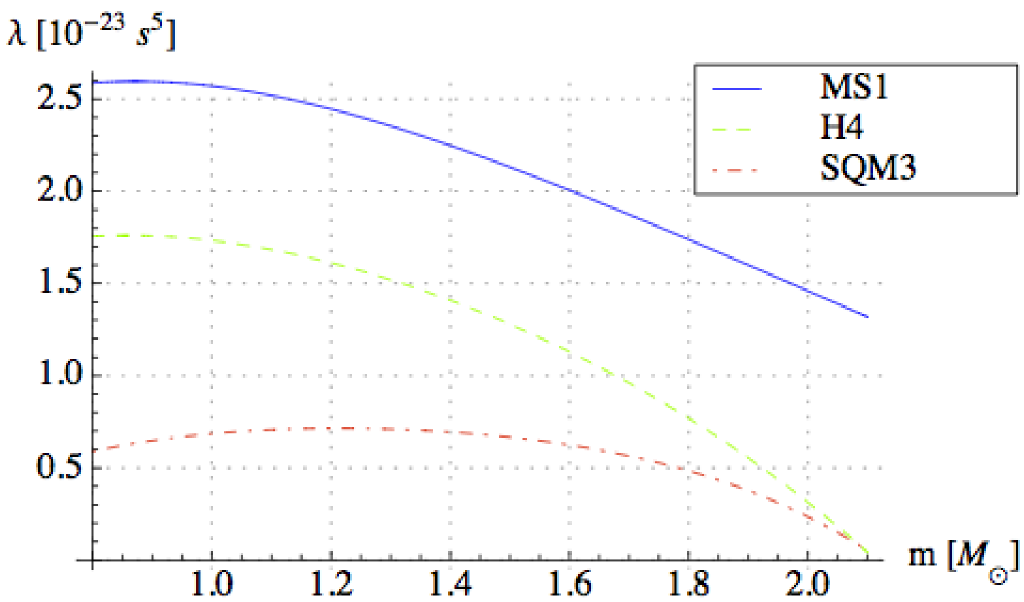
$$\boxed{{}^{(N)}O^i_j = \frac{P(H_i|I)}{P(H_j|I)}\prod_{n=1}^{N}\frac{P(d_n|H_i,I)}{P(d_n|H_j,I)}}$$

Constraining the equation of state of neutron stars: Hypothesis ranking

- Odds ratio after N binary neutron star detections:

$$ {}^{(N)}O^i_j = \frac{P(H_i|I)}{P(H_j|I)} \prod_{n=1}^{N} \frac{P(d_n|H_i,I)}{P(d_n|H_j,I)} $$

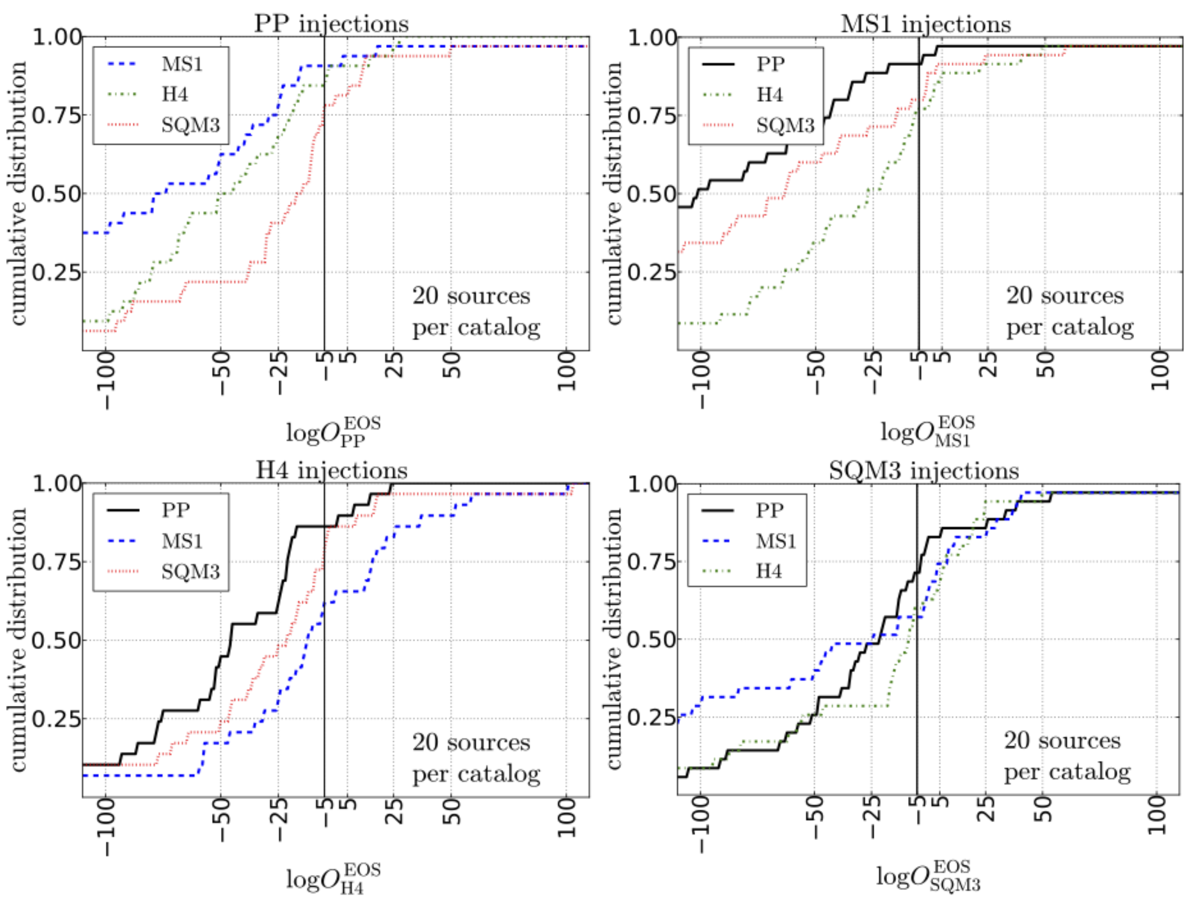
- As an example, consider three EOS models for both simulated signals and hypotheses:



- Will also consider "point particle" hypothesis, $\lambda(m) \equiv 0$

Constraining the equation of state of neutron stars: Hypothesis ranking

- Simulate large number of "catalogs" of 30 BNS detections each
- Cumulative histograms of log odds ratios:



Constraining the equation of state of neutron stars: Hypothesis ranking

Note:

- In reality, the true EOS will probably not be in the finite list of EOS used in the hypotheses

  - But, can expect the top-ranked EOS model to be “close” to the true EOS

- Even if the true EOS is in the list of hypotheses, it will not necessarily get ranked the highest!

  - Noise effects can interfere with the measurement

- Internal ranking of hypotheses largely as expected

  - E.g. if true EOS is stiff, then PP model most deprecated, next moderate EOS, next soft EOS

Constraining the equation of state of neutron stars: Parameter estimation

- Can measure $\lambda(m_1)$ and $\lambda(m_2)$ separately for each source

- Better idea: identify parameters to characterize the *function* $\lambda(m)$
  - Will be the same for each source
  - Should be possible to combine information on these parameters using all available detections

- Different choices in parameterizing $\lambda(m)$:
  - Model $P(\rho)$ by "piecewise polytropes":

    $p(\rho) = K_i \rho^{\Gamma_i}$ in density intervals $\rho_{i-1} < \rho < \rho_i$

    and compute $\lambda(m)$ from resulting expression
  - Write $\lambda(m)$ as a Taylor expansion around some reference mass:

$$\lambda(m) = \sum_{j=0}^{j_{\max}} \frac{1}{j!} c_j \left(\frac{m - m_0}{M_\odot} \right)^j$$

Constraining the equation of state of neutron stars: Parameter estimation

- Taylor expansion of $\lambda(m)$:

$$\lambda(m) = \sum_{j=0}^{j_{\max}} \frac{1}{j!} c_j \left(\frac{m - m_0}{M_\odot} \right)^j$$

If taken to sufficiently high order, coefficients c_j can be considered the same for all sources

- Construct waveform model $\tilde{h}(\vec{\theta}, \{c_j\}; f)$ where $\vec{\theta}$ are the parameters of the point particle waveform (masses, spins, ...) and the c_j enter through above expression for $\lambda(m)$

- Likelihood for a single source:

$$p(d|\vec{\theta}, \{c_j\}, I) = \mathcal{N} \exp \left[-2 \int_{f_{\rm low}}^{f_{\rm cut}} \mathrm{d}f \, \frac{|\tilde{d}(f) - \tilde{h}(\vec{\theta}, \{c_j\}; f)|^2}{S_n(f)} \right]$$

Constraining the equation of state of neutron stars: Parameter estimation

- Likelihood for a single source:

$$p(d|\vec{\theta},\{c_j\},I) = \mathcal{N}\, \exp\left[-2\int_{f_{\rm low}}^{f_{\rm cut}} \mathrm{d}f\, \frac{|\tilde{d}(f) - \tilde{h}(\vec{\theta},\{c_j\};f)|^2}{S_n(f)}\right]$$

- Joint posterior density function for all the parameters:

$$p(\vec{\theta},\{c_j\}|d,I) = \frac{p(d|\vec{\theta},\{c_j\},I)\, p(\vec{\theta},\{c_j\}|I)}{p(d|I)}$$

- Marginalize to get posterior density function for individual parameters, e.g.

$$p(c_0|d,I) = \int \mathrm{d}\vec{\theta}\,\mathrm{d}c_1 \ldots \mathrm{d}c_{j_{\rm max}}\, p(\vec{\theta},\{c_j\}|d,I)$$

- How to combine information from multiple sources?

Constraining the equation of state of neutron stars: Parameter estimation

- Posterior density function for individual parameters, e.g.

$$p(c_0|d,I) = \int \mathrm{d}\vec{\theta}\,\mathrm{d}c_1 \ldots \mathrm{d}c_{j_{\max}}\, p(\vec{\theta},\{c_j\}|d,I)$$

- Given detections $d_1, d_2, \ldots, d_N$:

$$\begin{aligned}
p(c_0|d_1,d_2,\ldots,d_N,I) &= \frac{p(d_1,d_2,\ldots,d_N|c_0,I)\,p(c_0|I)}{p(d_1,d_2,\ldots,d_N|I)} \\
&= p(c_0|I)\prod_{n=1}^{N}\frac{p(d_n|c_0,I)}{p(d_n|I)} \\
&= p(c_0|I)\prod_{n=1}^{N}\frac{p(c_0|d_n,I)\,p(d_n|I)}{p(c_0|I)\,p(d_n|I)} \\
&= p(c_0|I)^{1-N}\prod_{i=1}^{N}p(c_0|d_n,I)
\end{aligned}$$

- Can be viewed as posterior of *n*-th measurement becoming the prior for the *(n+1)*-th measurement! (Exercise)

Constraining the equation of state of neutron stars: Parameter estimation

- Posterior density function with multiple detections:

$$p(c_0|d_1, d_2, \ldots, d_N, I) = p(c_0|I)^{1-N} \prod_{i=1}^{N} p(c_0|d_n, I)$$

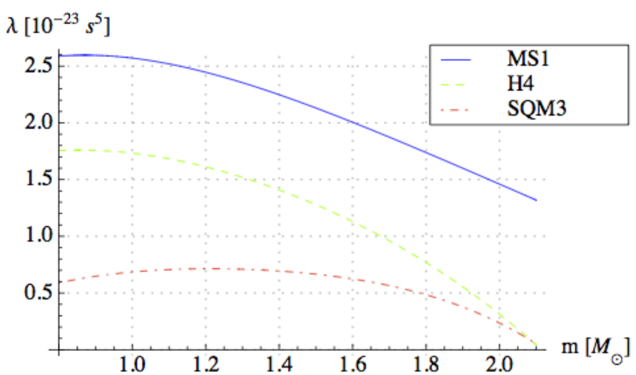
- Recall

$$\lambda(m) = \sum_{j=0}^{j_{\max}} \frac{1}{j!}\, c_j \left(\frac{m - m_0}{M_\odot}\right)^j$$

- For definiteness, restrict to quadratic order:

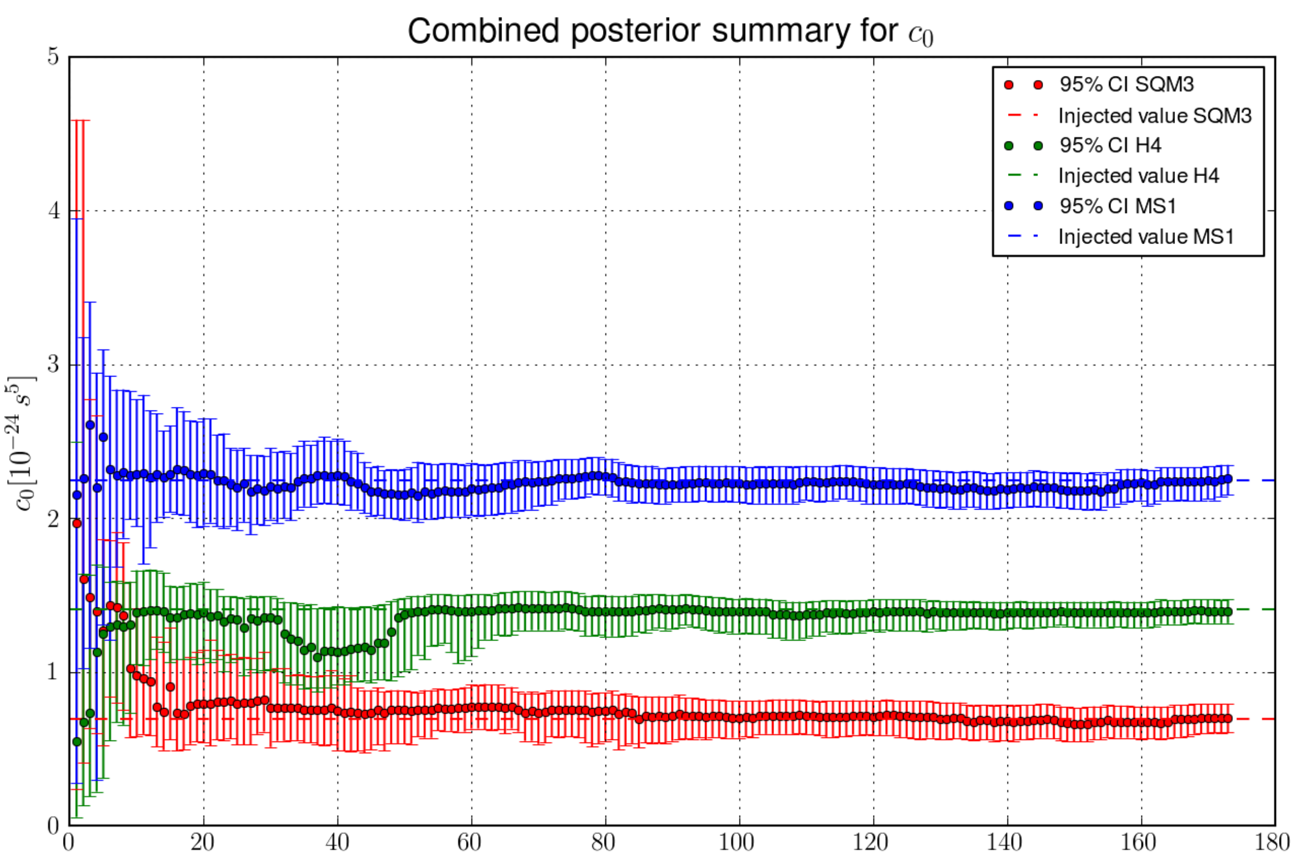
$$\lambda(m) \simeq c_0 + c_1 \left(\frac{m - m_0}{M_\odot}\right) + \frac{1}{2} c_2 \left(\frac{m - m_0}{M_\odot}\right)^2$$

- Again consider simulated signals with stiff, moderate, or soft EOS



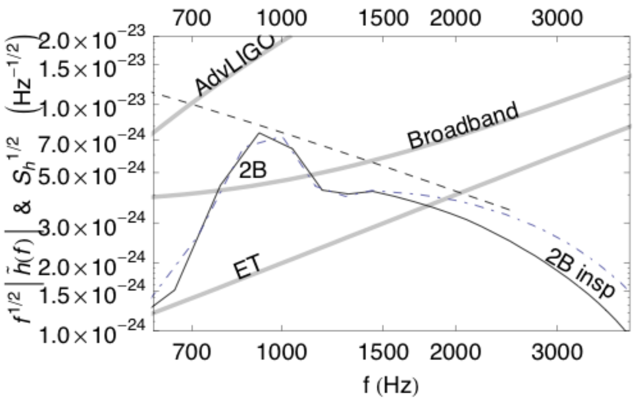
Constraining the equation of state of neutron stars: Parameter estimation

- Medians and 95% confidence intervals as more and more detections are made:

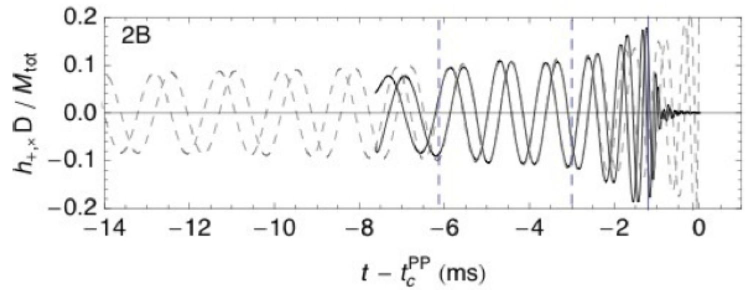


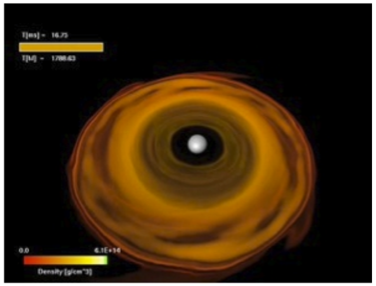
Constraining the equation of state of neutron stars: Merger effects

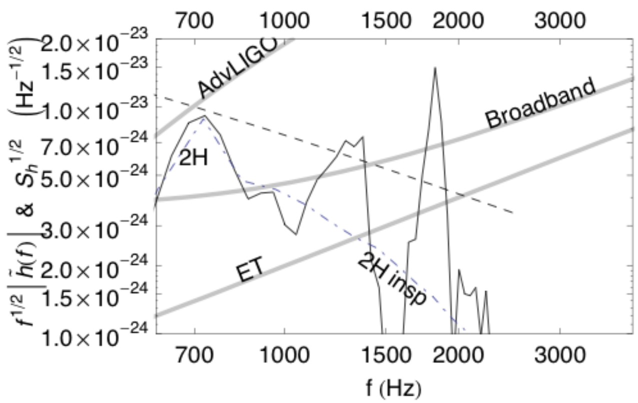
- Additional information to be gained from post-merger oscillations



- "Soft" EOS: prompt collapse to a black hole







- "Hard" EOS: unstable bar mode, eventually black hole