Tutorials 4 and 5: Nested sampling

I. COMPUTING INTEGRALS WITH NESTED SAMPLING

As a warm-up, let us first use nested sampling to perform a simple integral. Consider the following function of two variables, where $x, y \in [-0.5, 0.5]$:

$$F(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right] \tag{1.1}$$

To begin with, pick $\sigma_x = 0.1$ and $\sigma_y = 0.5$.

1. When calculating the integral

$$\int_{-0.5}^{0.5} dx \int_{-0.5}^{0.5} dy\, F(x, y) \tag{1.2}$$

by means of nested sampling, what plays the role of the likelihood function, the prior, and the evidence?

2. Write a nested sampling routine that computes (1.2). In doing so,

- Use print statements to monitor how the smallest likelihood for the set of live points evolves.
- Similarly, keep track of how the highest prior mass evolves.
- Keep track of how the evidence evolves.

You will need to choose a termination condition. Which condition appears to best suit the purpose? Also play with the number of live points used, to see how many points or needed for the end result to not be much affected any more.

3. To check that to what extent your calculation of the integral (1.2) is correct, compute it in some other way (e.g. using Mathematica or Matlab).

4. Pick different values for σ_x, σ_y from the ones above. How is the nested sampling result affected depending on whether the integrand $F(x, y)$ is more peaked or less peaked in one or both variables x, y?

5. As explained in the lectures, the nested sampling process gives you posterior samples for free; that is, you get a set of values $p(x, y)/Z$, where $p(x, y)$ is the likelihood function. Find a practical way to construct from these the joint posterior density distribution for (x, y). Plot this distribution. Had you expected the result?

6. In the language of Bayesian analysis, so far we have effectively been using a flat prior. Now introduce the following priors on x and y:

$$\pi_x(u) \propto -(u - 0.5), \tag{1.3}$$
$$\pi_y(v) \propto (v + 0.5). \tag{1.4}$$

Normalize the priors. Next, find a way to correctly draw points from the resulting prior distributions.

7. How are the steps in the nested sampling process affected by introducing the priors (1.3) and (1.4)? Implement this in your code.

8. Again compute the joint posterior density distribution for (x, y) and plot it. Find a way to marginalize over either x or y.

II. A DATA ANALYSIS PROBLEM

The output of a certain apparatus consists of a series of points $d(t_i)$, $i = 1, \ldots, N$ where $t_i = -1 + i\Delta t$, with $\Delta t = 2/N$. (For definiteness, you can pick $N = 1000$.) We are told that the output takes the following form:

$$d(t_i) = \frac{1}{\sqrt{2\pi}\bar{\sigma}} \exp\left[-\frac{(t_i - \bar{\mu})^2}{2\bar{\sigma}^2}\right], \tag{2.1}$$

where $\bar{\mu} \in [-0.5, 0.5]$ and $\sigma \in [0, 1]$, but we are not given the precise values of μ and σ. The likelihood takes the following form:

$$p(d|\mu, \sigma) \propto \exp\left[-\sum_{i=0}^{N}(d(t_i) - g(\mu, \sigma, t_i))^2\right], \tag{2.2}$$

where the functions $g(\mu, \sigma, t)$ play the role of waveform templates, which are of the same general form as (2.1):

$$g(\mu, \sigma, t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(t - \mu)^2}{2\sigma^2}\right]. \tag{2.3}$$

9. Pick values for $\bar{\mu}$ and $\bar{\sigma}$ (to be kept secret!), and produce files containing data of the above kind, to be given to your fellow students.

10. Write nested sampling code to produce posterior density functions for the parameters $\bar{\mu}$ and $\bar{\sigma}$ for data files you receive from your fellow students. Compute medians and 95% confidence intervals. How do your "measurements" compare with reality?

11. Add various levels of synthetic noise to the data. How do the results change?