Tutorials on GW data analysis

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I. DETECTION OF GRAVITATIONAL-WAVE "CHIRPS"

A. Waveforms

Consider a system of two compact stars (black holes or neutron stars) with masses m_1 and m_2 in a circular orbit. Such a system will lose energy by gravitational-wave (GW) emission and the stars will spiral inwards such that the orbital frequency increases with time following Kepler's third law. The emitted gravitational waveform is a "chirp" (similar to the chirping of birds) with both amplitude and frequency increasing with time. When the stars are widely separated, the problem can be treated perturbatively. In the leading order post-Newtonian approximation, the observed GW signal, which is a linear combination of the two polarizations $h_+(t)$ and $h_\times(t)$, can be computed as:

$$h(t) = A(t) \cos\varphi(t). \tag{1.1}$$

The amplitude $A(t)$ depends on a particular combination of the masses, called the *chirp mass* $\mathcal{M}$, the instantaneous frequency $F(t)$ of GWs, the luminosity distance D to the source, and a geometric factor C that depends on the location of the source in the sky and its orientation with respect to the detector.

$$A(t) = C\,\frac{4\mathcal{M}^{5/3}\pi^{2/3}F(t)^{2/3}}{D}. \tag{1.2}$$

For simplicity, we shall assume $C = 1$ which implies that the binary is conveniently oriented giving circular polarization and the source is located along the direction where the detector shows maximum directional sensitivity. The chirp mass can be expressed in terms of the total mass $M \equiv m_1 + m_2$ and reduced mass $\mu \equiv m_1 m_2/M$ as $\mathcal{M} = \mu^{3/5}M^{2/5}$. The frequency evolution $F(t)$ is given by

$$F(t) = \frac{(\mathcal{M}F_0^9)^{1/8}}{\left[(\mathcal{M}F_0)^{1/3} - 256\,F_0^3\mathcal{M}^2\pi^{8/3}\,t/5\right]^{3/8}} \tag{1.3}$$

where F_0 is the starting frequency of the signal: $F_0 := F(t = 0)$. It can be seen that the frequency sweeps from lower to higher frequencies, until the approximation breaks down at $t = t_c$. The *coalescence time* t_c can be computed as

$$t_c = \frac{5}{256\,(\pi F_0)^{8/3}\mathcal{M}^{5/3}}. \tag{1.4}$$

Finally, the phase $\varphi(t)$ of the GW signal can be expressed as

$$\varphi(t) = \varphi_0 - 2\left(\frac{1}{256\,(\pi\mathcal{M}F_0)^{8/3}} - \frac{t}{5\mathcal{M}}\right)^{5/8}, \tag{1.5}$$

where φ_0 is the phase at $t = 0$. Waveforms computed in this lowest order approximation are sometime called "Newtonian chirps".

Units: All expressions in this Section are written in geometrized units, in which $G = c = 1$. Mass and distance have units of seconds. Physical units can be obtained by replacing a mass $\mathcal{M}$ by $G\mathcal{M}/c^3$, and a distance D by cD. In geometrical units, $1M_\odot = 4.92549095 \times 10^{-6}$ s and 1pc $= 1.0292712503 \times 10^8$ s. A sample Newtonian chirp waveform is shown in Fig. 1.

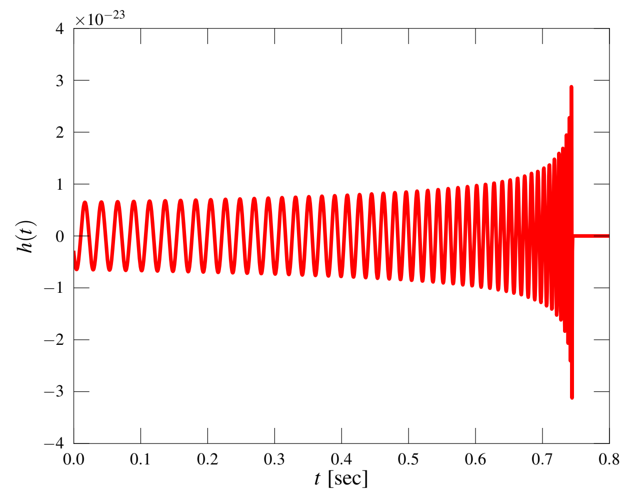

FIG. 1: An example of a "Newtonian chirp", with chirp mass $\mathcal{M} = 10M_\odot$, distance $D = 100$ Mpc, initial phase $\varphi_0 = 0$ and start frequency $F_0 = 40$ Hz.

B. Matched filter

In the case a known signal $h(t)$ buried in stationary Gaussian, white noise, the optimal technique for signal extraction is the *matched filtering*, which involves cross-correlating the data with a *template* of the signal.

The correlation function between two time series $x(t)$ and $\hat{h}(t)$ for a time shift τ is defined as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\,\hat{h}^\star(t-\tau)\,dt. \tag{1.6}$$

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Above, * denotes complex conjugation, and $\hat{h(t)} := h(t)/||h||$, where the norm $||h||$ of the template is defined by

$$||h||^2 = \int_0^{t_c} |h(t)|^2/\sigma^2\, dt,$$

where σ^2 is the variance of the noise. The optimal signal-to-noise ratio (SNR) is obtained when the template exactly matches with the signal.

$$\mathrm{SNR} = ||h|| \tag{1.7}$$

If the SNR is greater than a predetermined threshold (which corresponds to an acceptably small false alarm probability), a detection can be claimed. Note that the actual detector data is neither white and is only approximately Gaussian, which makes actual GW detection a siginificantly more complex exercise than mentioned above!

C. Problems

1. Write a code to generate the Newtonian chirp waveform $h(t)$ for arbitrary values of $\mathcal{M}, D, \varphi_0$.
2. A data set containing a Newtonian GW signal with $D = 100$ Mpc, $\varphi_0 = 0$, $F_0 = 40$Hz, but unknown $\mathcal{M}$ can be downloaded from https://home.icts.res.in/~archis/share/exercises/GWSchool_2015/gw_ex_data.dat.gz. The data $d(t)$ is comprised of the signal $h(t)$ and Gaussian white noise $n(t)$ of standard deviation $\sigma = 10^{-21}$. That is, $d(t) = h(t) + n(t)$. Write a code to detect the signal using the simple matched filtering method mentioned in the previous section. Since you don't know the chirp mass of the signal, choose a grid of chirp masses in the interval $\mathcal{M} \in (8, 12)M_\odot$ with some appropriate grid spacing. This is your template "bank"!
3. A sample of LIGO S5 L1 strain data can be downloaded from http://www.ligo.org/science/GRB051103/index.php (download the file L1-STRAIN_4096Hz-815045078-256.txt.gz). This contains 256 seconds of LIGO data from 2005, sampled at a rate of 4096 Hz. Compute the power spectral density of the data.