Gravitational wave data analysis: Detection

Gravitational waves

- Linearized general relativity: Einstein equations become a *wave equation*

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Gravitational radiation is primarily generated by *time-dependent mass quadrupole moment*:

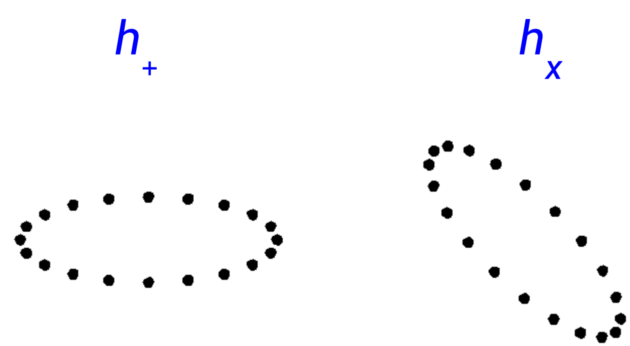
$$\left[h_{ij}^{\mathrm{TT}}(t,\mathbf{x})\right]_{\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^2}\Lambda_{ij,kl}(\hat{\mathbf{n}})\,\ddot{M}^{kl}(t-r/c)$$

- Nearby point particles in free fall, small separation ζ^μ

  $\rightarrow$ *Tidal effect:*

$$\ddot{\zeta}^i = \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}\zeta^j$$

"plus" and *"cross" polarizations:*

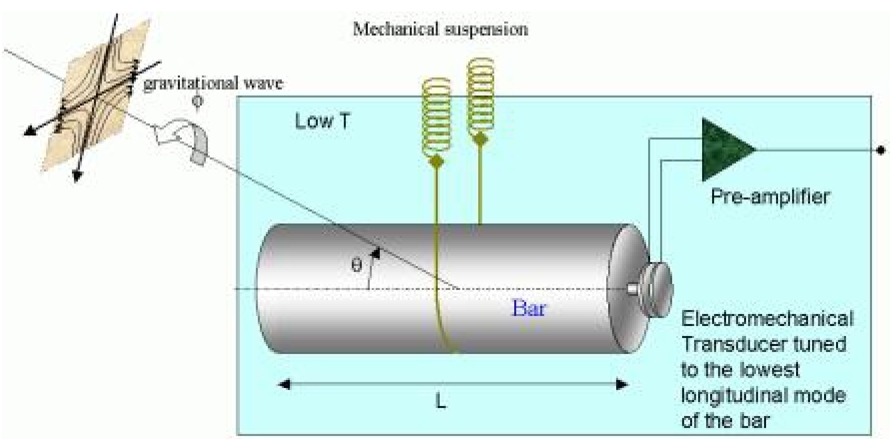


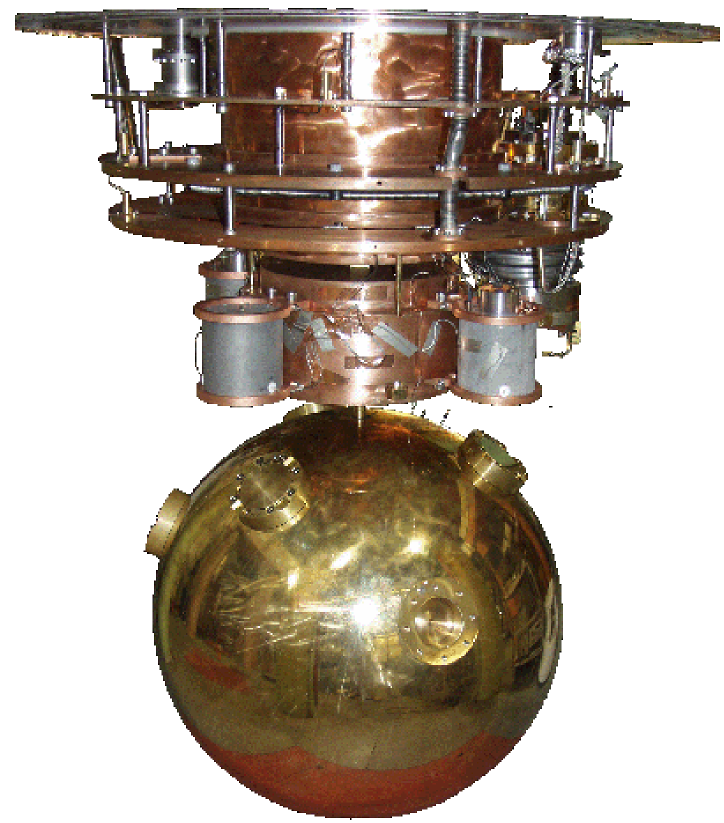
Detection of gravitational waves

$$\ddot{\zeta}^i = \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}\zeta^j$$

Exploit tidal effect on matter

- Resonant detectors
  - Metal bar: resonant frequency
  - Measure tiny length changes
  - Also spherical resonant detectors: Equal sensitivity in all directions (miniGRAIL, Leiden)



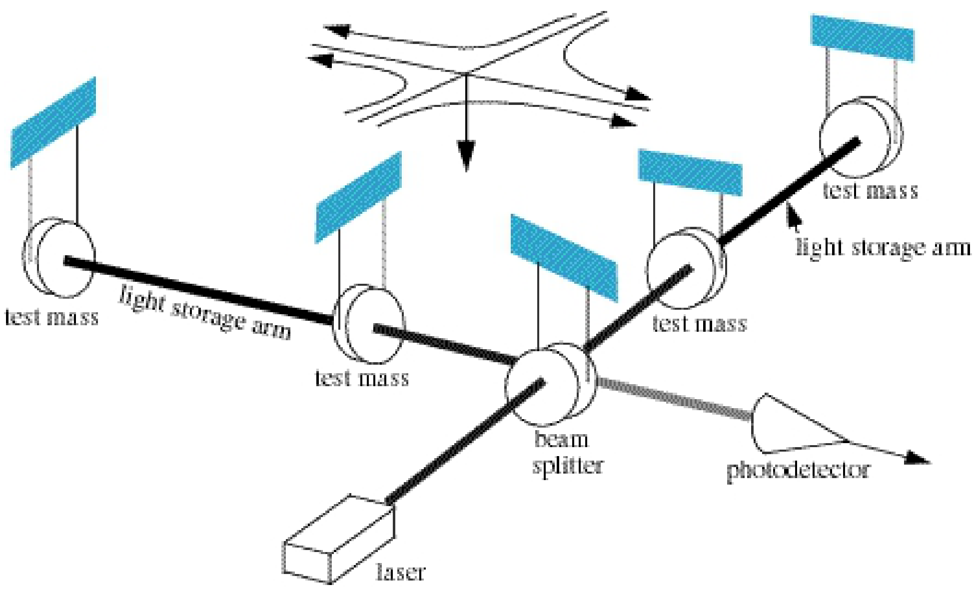


Interferometric detectors

$$\ddot{\zeta}^i = \frac{1}{2}\ddot{h}_{ij}^{\rm TT}\zeta^j$$

- Interferometric detectors
  - Laser beam through beam splitter
  - Power built up in long (several km) cavities
  - At output: destructive interference

  ... unless gravitational wave present



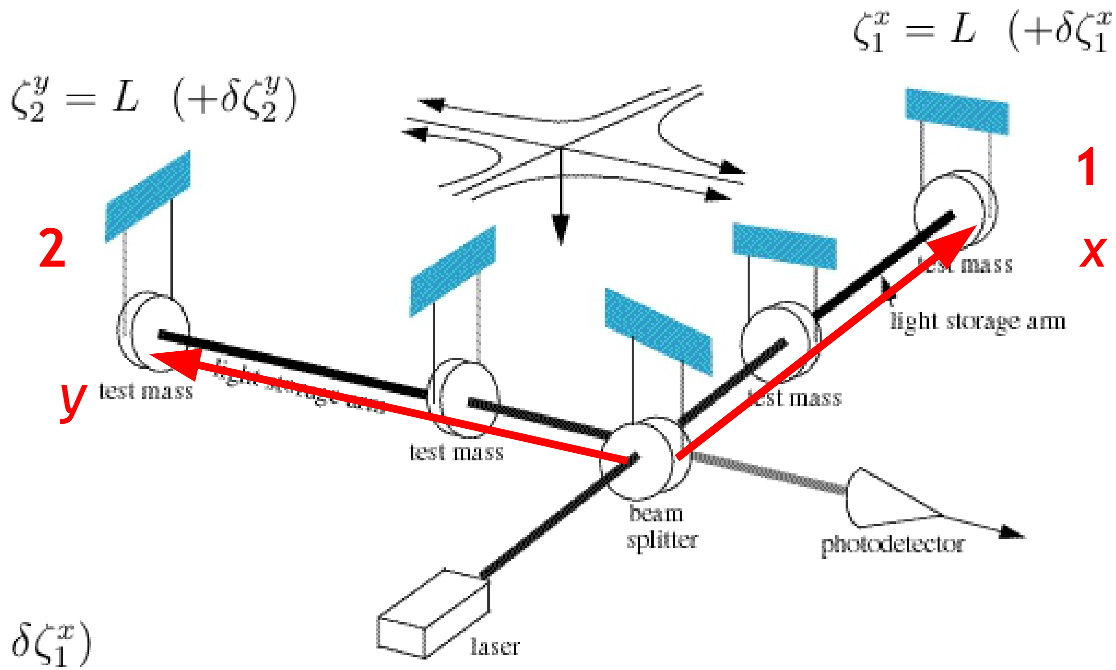
Interferometric detectors

$$\ddot{\zeta}^i = \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}}\zeta^j$$

- Motion of the end mirrors

$$\delta\ddot{\zeta}_1^x = \frac{1}{2}\ddot{h}_{xx}(L + \cancel{\delta\zeta_1^x})$$

$$\delta\ddot{\zeta}_2^y = \frac{1}{2}\ddot{h}_{yy}(L + \cancel{\delta\zeta_2^y})$$



... hence

$$\delta\zeta_1^x = \frac{1}{2}h_{xx}\,L$$

$$\delta\zeta_2^y = \frac{1}{2}h_{yy}\,L$$

- Measured *strain*:

$$h(t) \equiv \frac{\Delta L}{L} = \frac{(L + \delta\zeta_2^y) - (L + \delta\zeta_1^x)}{L}$$

$$h(t) = \frac{1}{2}(h_{xx} - h_{yy})$$

Detection of gravitational waves

- For L-shaped detectors: $$h(t) = \frac{1}{2}(h_{xx} - h_{yy})$$
- More generally:

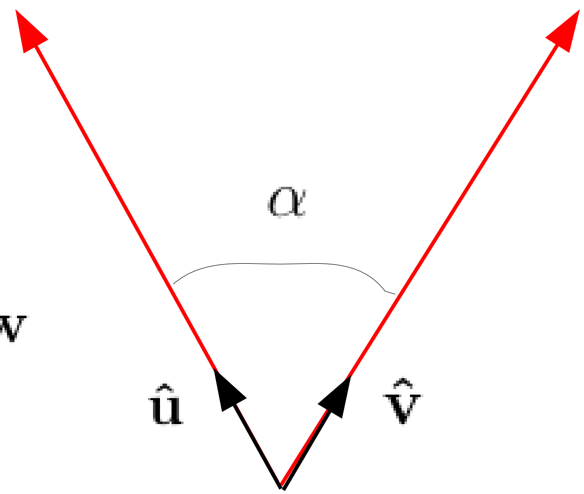
$$h(t) = \frac{1}{2}(h_{uu} - h_{vv})$$

- Detector tensor:

$$D^{ij} = \frac{1}{2}(u^i u^j - v^i v^j)$$

$$h(t) = D^{ij} h_{ij}$$

Top view



Detection of gravitational waves

$$h(t) = \frac{1}{2}(h_{xx} - h_{yy})$$

- Signal from arbitrary direction

  In transverse-traceless frame:

$$h_{ij}^{\hat{\mathbf{n}}=\hat{\mathbf{z}}'} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

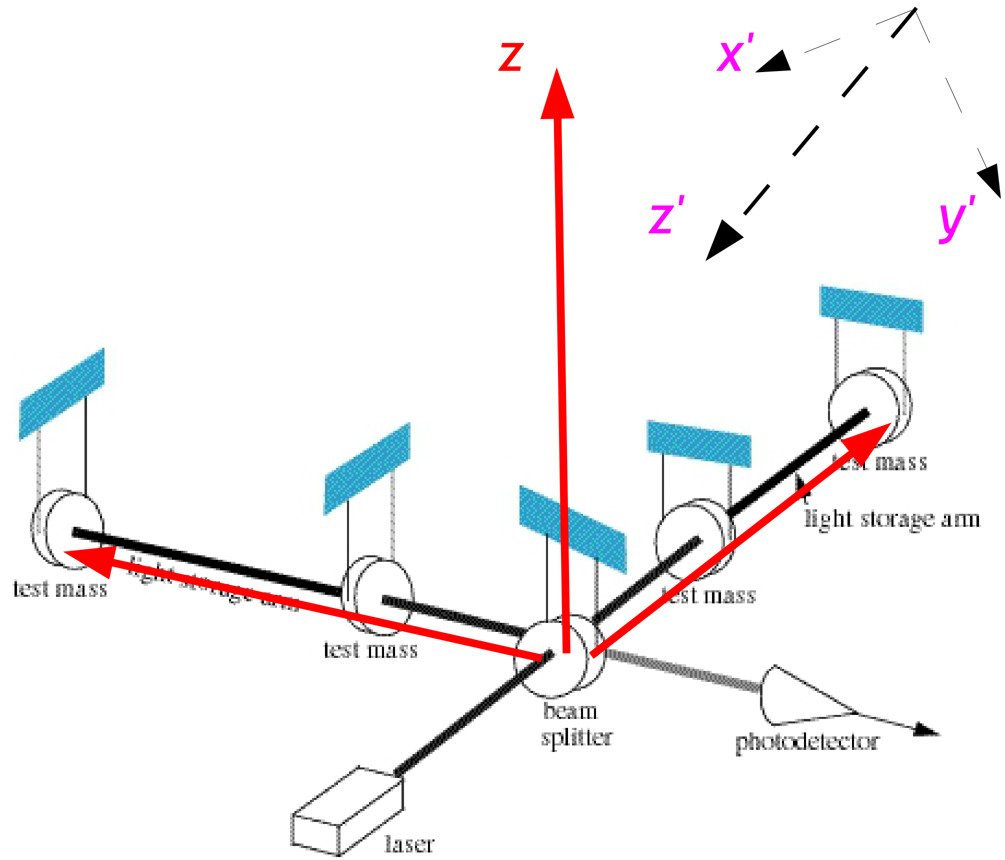
- If x′ = x, y′ = y, z′ = z then

$$h(t) = h_+(t)$$

- In general, need to apply linear transformation

$$h_{ij}^{\mathbf{det}} = \mathcal{R}_{ik}\mathcal{R}_{jl}h_{kl}^{\hat{\mathbf{n}}=\hat{\mathbf{z}}'}$$

- Projection onto detector also linear, hence



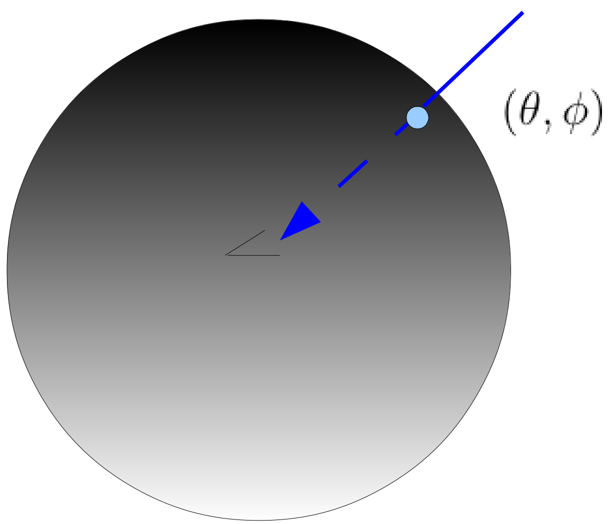
$$h(t) = D^{ij}h_{ij}^{\mathbf{det}} = D^{ij}\mathcal{R}_{ik}\mathcal{R}_{jl}h_{kl}^{\hat{\mathbf{n}}=\hat{\mathbf{z}}'} = F_+\, h_+(t) + F_\times\, h_\times(t)$$

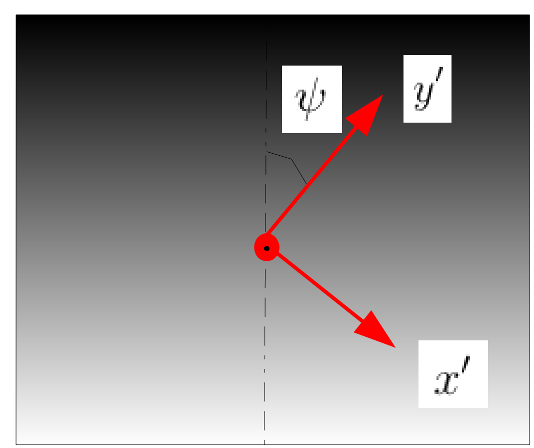
Detection of gravitational waves

$$h(t) = F_+ \, h_+(t) + F_\times \, h_\times(t)$$

- For an L-shaped interferometer:

$$F_+ \;=\; \frac{1}{2}(1+\cos^2(\theta))\;\cos(2\phi)\;\cos(2\psi) - \cos(\theta)\;\sin(2\phi)\;\sin(2\psi)$$
$$F_\times \;=\; \frac{1}{2}(1+\cos^2(\theta))\;\cos(2\phi)\;\sin(2\psi) + \cos(\theta)\;\sin(2\phi)\;\cos(2\psi)$$





- (θ, ϕ) sky position; ψ polarization angle

Digging a signal out of noise

- If there is a signal: measured strain is *sum of noise and signal*:

$$s(t) = n(t) + h(t)$$

- If shape of signal approximately known: integrate against the output

$$\frac{1}{T}\int_0^T s(t)\,h(t)\,dt = \frac{1}{T}\int_0^T n(t)\,h(t)\,dt + \frac{1}{T}\int_0^T h(t)^2 dt$$

oscillatory — positive definite

$$\sim \left(\frac{\tau_0}{T}\right)^{1/2} n_0 h_0 \qquad \sim h_0^2$$

τ_0 period of the GW signal,

h_0 characteristic signal amplitude

n_0 characteristic amplitude of the noise

- *To detect the signal, don't need* $h_0 > n_0$ *but only* $h_0 > (\tau_0/T)^{1/2} n_0$
- Binary objects: $\tau_0 \sim 10^{-2}\,\mathrm{s}$, $T \sim 100\,\mathrm{s} \;\rightarrow\; (\tau_0/T)^{1/2} \sim 10^{-2}$
- Millisecond pulsars: $\tau_0 = 1\,\mathrm{ms}$, $T \sim 1\,\mathrm{yr} \;\rightarrow\; (\tau_0/T)^{1/2} \sim 10^{-5}$

Characterizing the noise

- Detector data comes in as time series
- If only noise:

$$(n(t_0), n(t_1), \ldots, n(t_N)) \text{ where } t_{i+1} = t_i + \Delta t$$

- Often convenient to take a (discrete) Fourier transform:

$$(\tilde{n}(f_0), \tilde{n}(f_1), \ldots, \tilde{n}(f_N)) \text{ where } f_{i+1} = f_i + \Delta f$$

Notation: $\tilde{n}(f_i) = \tilde{n}_i$

- Some noise realizations are more probable than others
Probability distribution in each frequency bin: $p(\tilde{n}_i)$
- We will assume that the noise is *stationary* and *Gaussian*:

$$p(\tilde{n}_i) \propto e^{-\frac{|\tilde{n}_i|^2}{2\sigma_i^2}}$$

Stationarity and Gaussianity:

$$\langle \tilde{n}_i \rangle = \int \tilde{n}_i \, p(\tilde{n}_i) \, d\tilde{n}_i = 0 \qquad \langle |\tilde{n}_i|^2 \rangle = \int |\tilde{n}_i|^2 \, p(\tilde{n}_i) \, d\tilde{n}_i = \sigma_i^2$$

- Probability density for noise realization *as a whole*:

$$p[n] = p(\tilde{n}_0, \tilde{n}_1, \ldots, \tilde{n}_N) = \prod_{i=1}^{N} p(\tilde{n}_i)$$

Characterizing the noise

- Probability density for noise realization *as a whole*:

$$p[n] = p(\tilde{n}_0, \tilde{n}_1, \ldots, \tilde{n}_N) = \prod_{i=1}^{N} p(\tilde{n}_i)$$

- For stationary, Gaussian noise:

$$p[n] = \prod_{i=1}^{N} p(\tilde{n}_i) = \mathcal{N}\, e^{-\frac{1}{2}\sum_{i=1}^{N} \frac{|\tilde{n}_i|^2}{\sigma_i^2}}$$

- For purpose of this lecture, convenient to take continuum limit:

$$\begin{aligned} p[n] &= \mathcal{N}\, e^{-\frac{1}{2}\sum_{i=1}^{N} \frac{|\tilde{n}_i|^2}{\sigma_i^2}} \\ &= \mathcal{N}\, e^{-\frac{1}{2}\sum_{i=1}^{N} \frac{|\tilde{n}_i|^2}{\sigma_i^2 \Delta f}\Delta f} \\ &\to \mathcal{N}\, e^{-\frac{1}{2}\int_{-\infty}^{\infty} \frac{|\tilde{n}(f)|^2}{\frac{1}{2}S_n(f)}\, df} \end{aligned}$$

Variance:

$$\sigma_i^2 \Delta f \to \frac{1}{2} S_n(f) \quad \text{or} \quad \sigma_i^2 \to \underbrace{\delta(f - f')}_{\sim \frac{1}{\Delta f}} \frac{1}{2} S_n(f)$$

$$\langle \tilde{n}^*(f)\, \tilde{n}(f') \rangle = \delta(f - f')\, \frac{1}{2}\, S_n(f)$$

Characterizing the noise

- Probability density and variance for noise realizations:

$$p[n] = \mathcal{N}\, e^{-\frac{1}{2}\int_{-\infty}^{\infty} \frac{|\tilde{n}(f)|^2}{\frac{1}{2}S_n(f)}\, df}$$

$$\langle \tilde{n}^*(f)\, \tilde{n}(f') \rangle = \delta(f - f')\, \frac{1}{2}\, S_n(f)$$

- Could also have worked in time domain

- Stationarity: $\langle n(t) \rangle = 0$

- Gaussianity: completely determined by $R(\tau) \equiv \langle n(t+\tau)\, n(t) \rangle$

where again $\langle \ldots \rangle$ denotes average over noise realizations

Defining *noise power spectral density* as

$$\frac{1}{2} S_n(f) \equiv \int_{-\infty}^{\infty} d\tau\, R(\tau)\, e^{2\pi i f \tau}$$

one finds

$$\langle \tilde{n}^*(f)\, \tilde{n}(f') \rangle = \frac{1}{2} \delta(f - f')\, S_n(f)$$

Matched filtering

- Instead of integrating the data against waveforms, use more generic filter:

$$\hat{s} = \int_{-\infty}^{\infty} dt\, s(t)\, \underbrace{K(t)}_{filter}$$

- Define *S* to be the expected value if a signal *h(t)* is present, and *N* the root-mean-square value if no signal present:

$$S \;=\; \langle \hat{s} \rangle_h \qquad\qquad N = \left[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2 \right]^{1/2}$$

- Define *signal-to-noise ratio*:

$$\frac{S}{N} = \frac{\langle \hat{s} \rangle_h}{\left[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle_{h=0}^2 \right]^{1/2}}$$

Now find out which filter K(t) maximizes S/N

Matched filtering

$$\frac{S}{N} = \frac{\langle \hat{s} \rangle_h}{\left[\langle \hat{s}^2 \rangle_{h=0} - \langle \hat{s} \rangle^2_{h=0}\right]^{1/2}}$$

$$\hat{s} = \int_{-\infty}^{\infty} dt\, s(t)\, \underbrace{K(t)}_{filter}$$

Write S in the frequency domain:

$$\begin{aligned} S &= \langle \hat{s} \rangle_h \\ &= \int_{-\infty}^{\infty} dt\, \langle s(t) \rangle\, K(t) \\ &= \int_{-\infty}^{\infty} dt\, h(t)\, K(t) \\ &= \int_{-\infty}^{\infty} df\, \tilde{h}(f)\, \tilde{K}^*(f) \end{aligned}$$

... and N:

$$\begin{aligned} N &= \left[\langle \hat{s}^2 \rangle - \langle \hat{s} \rangle^2\right]^{1/2}_{h=0} \\ &= \left[\langle \hat{s}^2 \rangle\right]^{1/2}_{h=0} \\ &= \left[\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, \langle n(t)\, n(t') \rangle\, K(t)\, K(t')\right]^{1/2} \\ &= \left[\int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f)\, |\tilde{K}(f)|\right]^{1/2} \end{aligned}$$

Matched filtering

... to arrive at:

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df\, \tilde{h}(f)\, \tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f)\, |\tilde{K}(f)|^2\right]^{1/2}}$$

- Now define the *noise-weighted inner product*

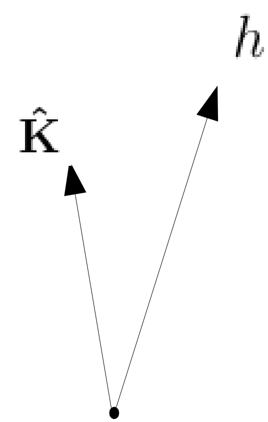
$$\begin{aligned} (A|B) &\equiv \mathrm{Re} \int_{-\infty}^{\infty} df\, \frac{\tilde{A}^*(f)\, \tilde{B}(f)}{\frac{1}{2} S_n(f)} \\ &= 4\,\mathrm{Re} \int_{0}^{\infty} \frac{\tilde{A}^*(f)\, \tilde{B}(f)}{S_n(f)} \end{aligned}$$

... and rewrite *S/N* as

$$\frac{S}{N} = \frac{(\mathbf{K}|h)}{(\mathbf{K}|\mathbf{K})^{1/2}} \qquad \mathbf{K} = \frac{1}{2} S_n(f)\, \tilde{K}(f)$$

... or

$$\frac{S}{N} = (\hat{\mathbf{K}}|h) \qquad \hat{\mathbf{K}} = \frac{\mathbf{K}}{(\mathbf{K}|\mathbf{K})^{1/2}}$$

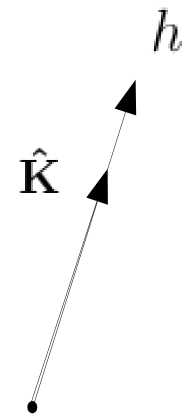


Maximizing S/N is equivalent to making $\hat{\mathbf{K}}$ *point in the same direction as* h *!*

Matched filtering

$$\frac{S}{N} = (\hat{\mathbf{K}}|h) \qquad \hat{\mathbf{K}} = \frac{\mathbf{K}}{(\mathbf{K}|\mathbf{K})^{1/2}} \qquad \mathbf{K} = \frac{1}{2} S_n(f)\,\tilde{K}(f)$$

- Need to make $\hat{\mathbf{K}}$ point in the same direction as h



$$\hat{\mathbf{K}} \propto h \quad \longrightarrow \quad \mathbf{K} \propto h$$

$$\longrightarrow \quad \boxed{\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_n(f)}}$$

"Wiener filter"

- Substituting back into the expression for S/N gives the highest signal-to-noise ratio one can attain:

$$\boxed{\begin{aligned} \frac{S}{N} &= \frac{(h|\langle s\rangle)}{\sqrt{(h|h)}} \\ &= \left[4\int_0^\infty df\, \frac{|\tilde{h}(f)|^2}{S_n(f)}\right]^{1/2} \end{aligned}}$$

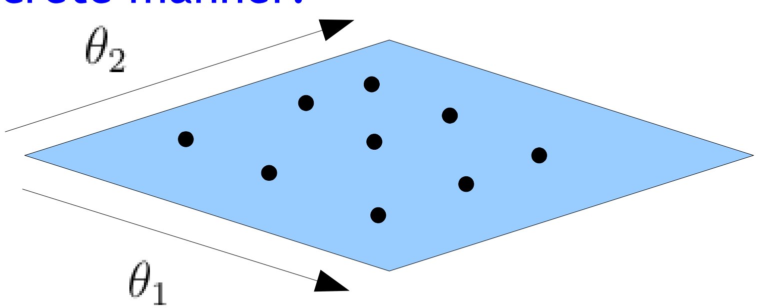
Matched filtering

$$\frac{S}{N} = \frac{(h|\langle s\rangle)}{\sqrt{(h|h)}}$$
$$= \left[4\int_0^\infty df\,\frac{|\tilde{h}(f)|^2}{S_n(f)}\right]^{1/2}$$

- In practice:
  - No access to *expected* s(t), only the actual detector output s(t)
  - Waveforms are characterized by source parameters $\bar{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$
  - Use optimal filter for many choices of parameters and find largest S/N:

$$\frac{S}{N} = \max_{\bar{\theta}} \frac{(h(\bar{\theta})|s)}{\sqrt{(h(\bar{\theta})|h(\bar{\theta}))}}$$

  - Parameter space can only be sampled in a discrete manner: "*template banks*"



  - Noise not perfectly Gaussian or stationary

Matched filtering in practice: compact binary coalescence

- Simplest compact binary coalescence waveforms:
  - Inspiral only
  - Amplitude to leading PN order only

$$h_+ = h_0 \cos\left(2\varphi(t)\right)$$

$$h_\times = h_{\frac{\pi}{2}} \sin\left(2\varphi(t)\right)$$

- Detector response:

$$h(t) = F_+ h_+ + F_\times h_\times$$

- Can be written as:

$$h(t) = A(t)\cos(2\varphi(t))\cos(\Phi_0) + A(t)\sin(2\varphi(t))\sin(\Phi_0)$$

where

$$A(t) = \left[F_+^2 h_0^2 + F_\times^2 h_{\frac{\pi}{2}}^2\right]^{1/2}$$

$$\cos\Phi_0 = \frac{F_+ h_0}{A},$$

$$\sin\Phi_0 = \frac{F_\times h_{\frac{\pi}{2}}}{A}.$$

Define

$$h_c = A(t)\cos 2\varphi(t)$$

$$h_s = A(t)\sin 2\varphi(t)$$

Matched filtering in practice: compact binary coalescence

$$h(t) = A(t)\cos(2\varphi(t))\cos(\Phi_0) + A(t)\sin(2\varphi(t))\sin(\Phi_0)$$

$$h_c = A(t)\cos 2\varphi(t)$$

$$h_s = A(t)\sin 2\varphi(t)$$

- Normalize h_c, h_s:

$$\hat{h}_c = \frac{h_c}{(h_c|h_c)} \qquad \hat{h}_s = \frac{h_s}{(h_s|h_s)}$$

- Matched filtering against detector output *s*:

$$\begin{aligned}(s|\hat{h}) &= (s|\hat{h}_c)\cos(\Phi_0) + (s|\hat{h}_s)\sin(\Phi_0) \\ &= \left[(s|\hat{h}_c)^2 + (s|\hat{h}_s)\right]^{1/2}\cos(\Phi_0 - \alpha)\end{aligned}$$

where $\alpha = \tan^{-1}\frac{(s|\hat{h}_s)}{(s|\hat{h}_c)}$

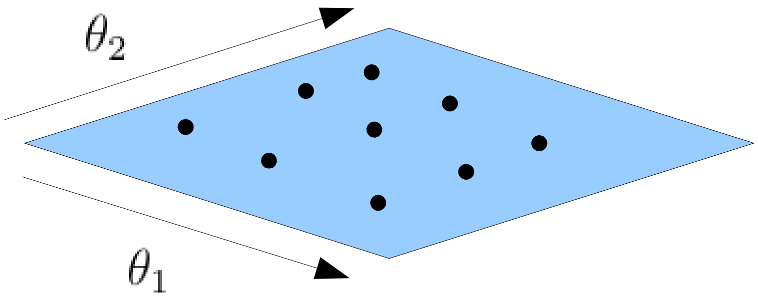
and the S/N ratio is maximized when $\Phi_0 = \alpha$

- Matched filtering reduces to:

$$(s|\hat{h}) = \left[(s|\hat{h}_c)^2 + (s|\hat{h}_s)\right]^{1/2}$$

No dependence of templates on extrinsic parameters!
Only dependence on masses and spins

Placement of template banks



- Match between nearby templates:

$$M = (\hat{h}(\bar{\theta})|\hat{h}(\bar{\theta}+\Delta\bar{\theta}))$$

- Expand in small quantities:

$$\begin{aligned} M &\simeq (\hat{h}(\bar{\theta})|\hat{h}(\bar{\theta})) + \frac{\partial M}{\partial \theta^i}\Delta\theta^i + \frac{1}{2}\frac{\partial^2 M}{\partial\theta^i \partial\theta^j}\Delta\theta^i\Delta\theta^j \\ &= 1 + \frac{1}{2}\frac{\partial^2 M}{\partial\theta^i \partial\theta^j}\Delta\theta^i\Delta\theta^j \end{aligned}$$

- Define a *metric tensor* on parameter space:

$$g_{ij} \equiv -\frac{1}{2}\frac{\partial^2 M}{\partial\theta^i \partial\theta^j}$$

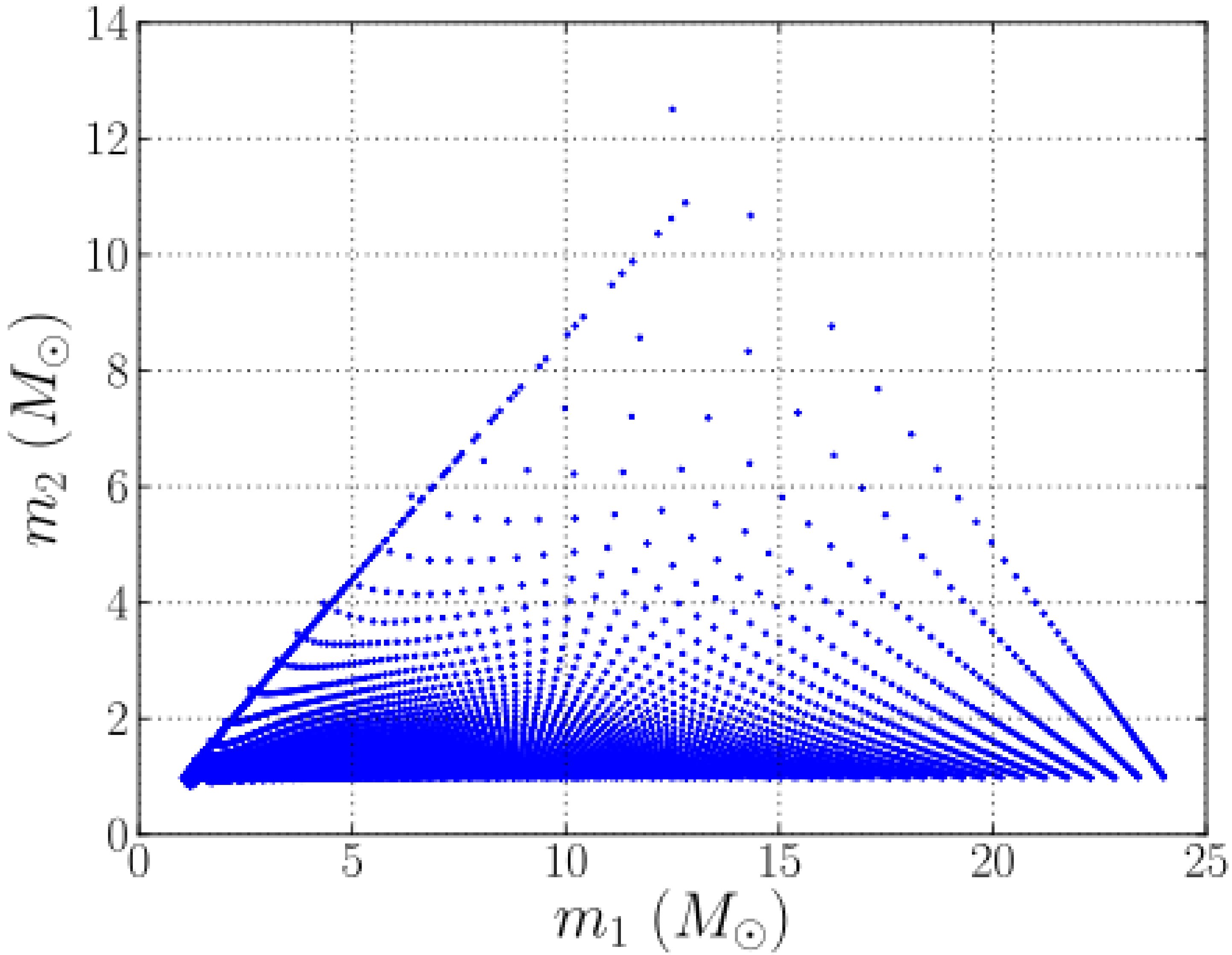
- *Mismatch* between nearby templates:

$$1 - M = g_{ij}\Delta\theta^i\Delta\theta^j$$

Place templates on parameter space such that metric distance never larger than a pre-specified mismatch, e.g. 0.03

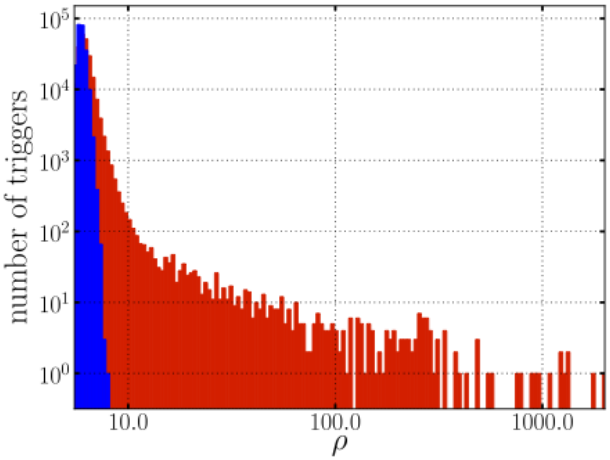
Placement of template banks

- Example where spins ignored so that templates only placed in (m_1,m_2):

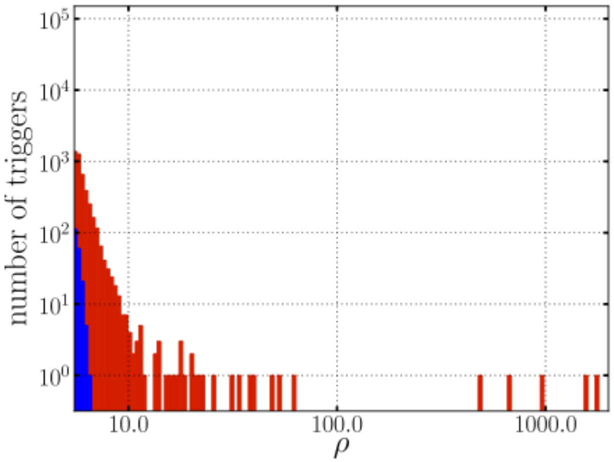


Handling of potential signals

- Glitches in the instruments can pose as gravitational wave signals!
- Distribution of SNRs for noise triggers in a single detector, in Gaussian noise (**blue**) and in real data (**red**)



- Demand that event seen in at least two detectors, with *same parameters* up to expected uncertainties (to be explained later)



Signal consistency tests

- Demand that build-up of SNR over frequency is as expected if a signal is really present

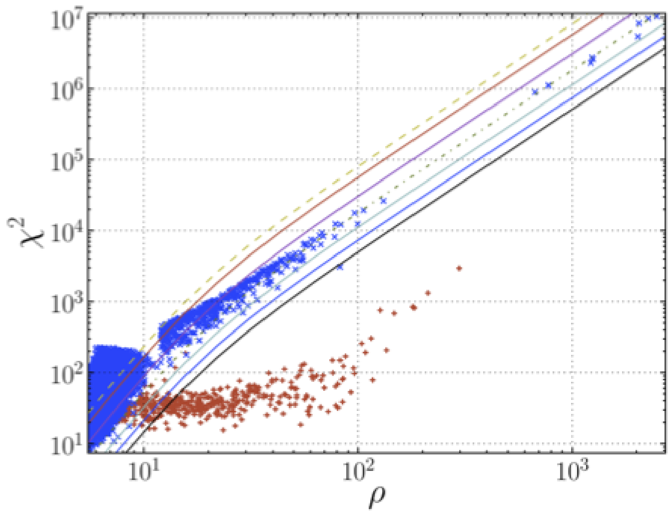
  - Divide frequency range in n bins that should contribute equally to SNR

$$(h_k|h_k) = \frac{(h|h)}{n}$$

  - Construct χ^2 *statistic*

$$\chi^2 = \sum_{k=1}^{n} \left[\left((s|h_{c,k}) - \frac{(h_c|h_c)}{n} \right)^2 + \left((s|h_{s,k}) - \frac{(h_c|h_c)}{n} \right)^2 \right]$$

Noise transients will typically have high χ^2



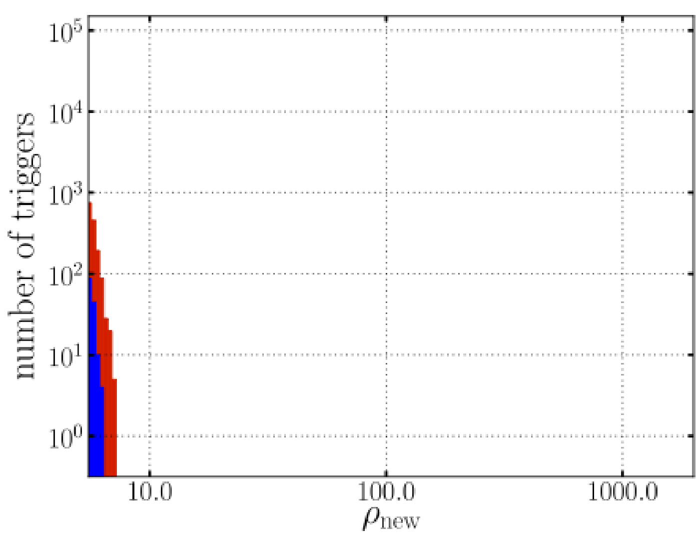
Compare distribution of noise triggers (**blue**) and simulated GW signals (**red**)

Signal consistency tests

- Construct a detection statistic that is large when SNR large and χ^2 small, e.g. "new SNR"

$$\rho_{\rm new} = \begin{cases} \rho & \text{for } \chi^2 \leq n_{\rm dof} \\ \rho \left[\frac{1}{2} \left(1 + \left(\frac{\chi^2}{n_{\rm dof}} \right)^3 \right) \right]^{-1/6} & \text{for } \chi^2 > n_{\rm dof} \end{cases}$$

- Distribution of new SNR for noise triggers in Gaussian noise (**blue**) and real data (**red**)

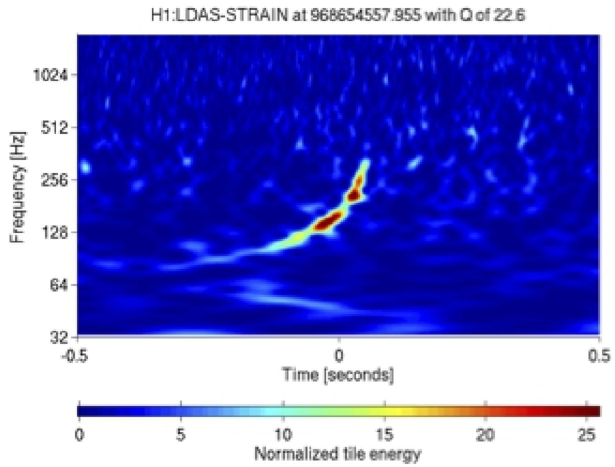


- Given data from multiple detectors, slide data stream in time w.r.t. each other
- Construct distribution of new SNR for multi-detector triggers (necessarily fake events!)
- Use this to set a *threshold* for a potential signal to overcome
- Given a candidate signal, assess its significance by comparing with this distribution

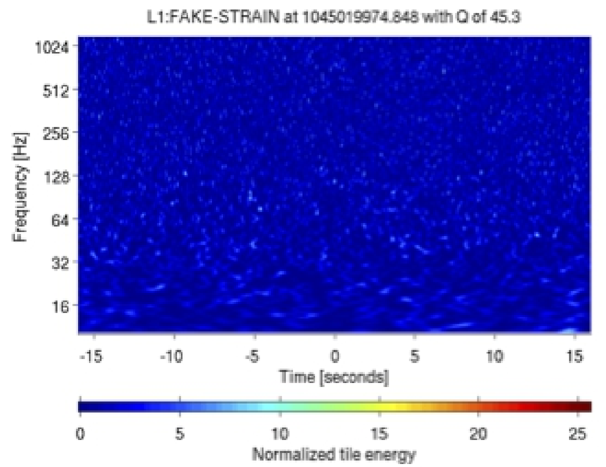
Other diagnostic tools

- Detector characterization and data quality studies
- Visual inspection of time-frequency spectrograms

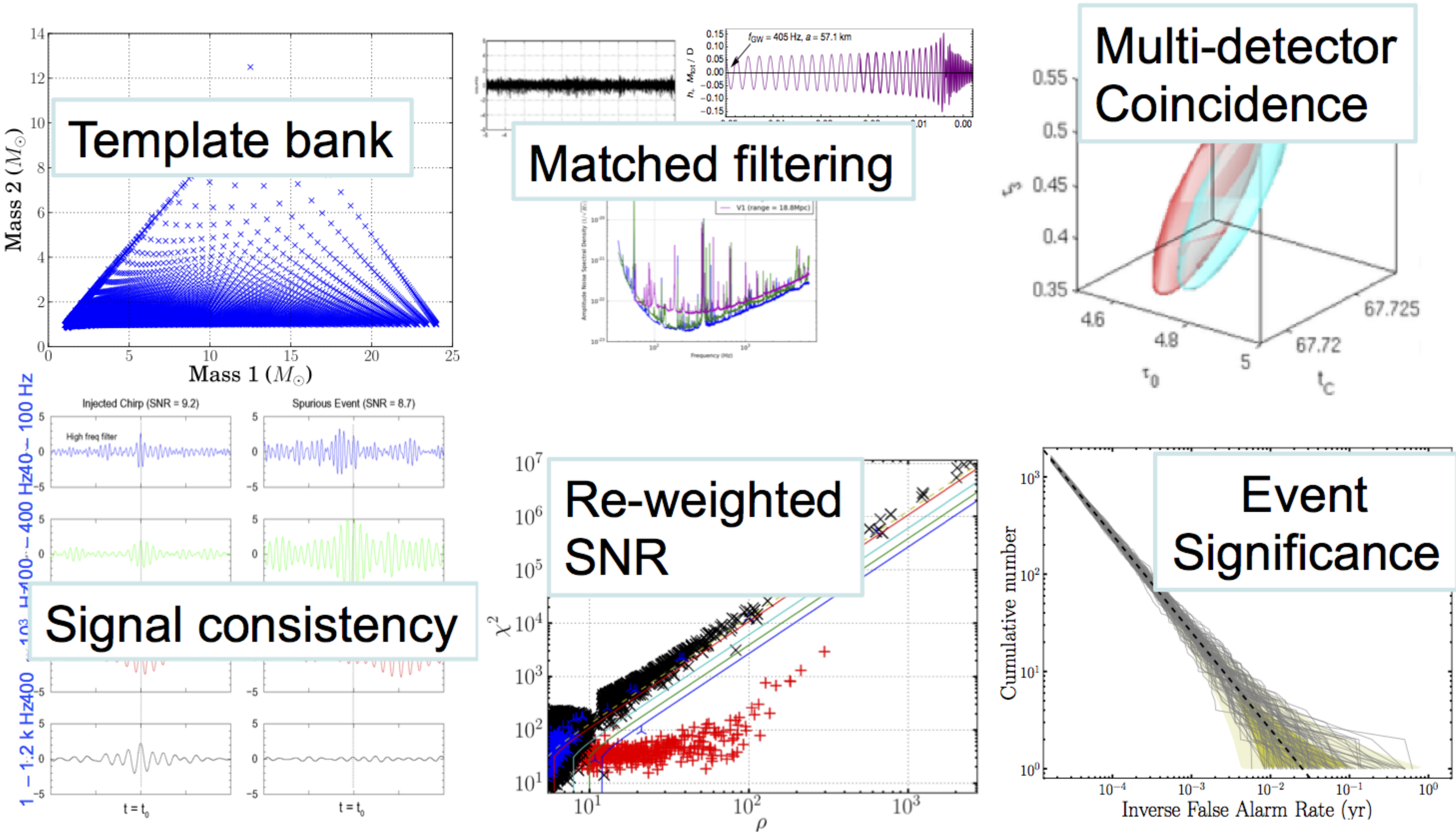
  - Neutron star-black hole coalescence:



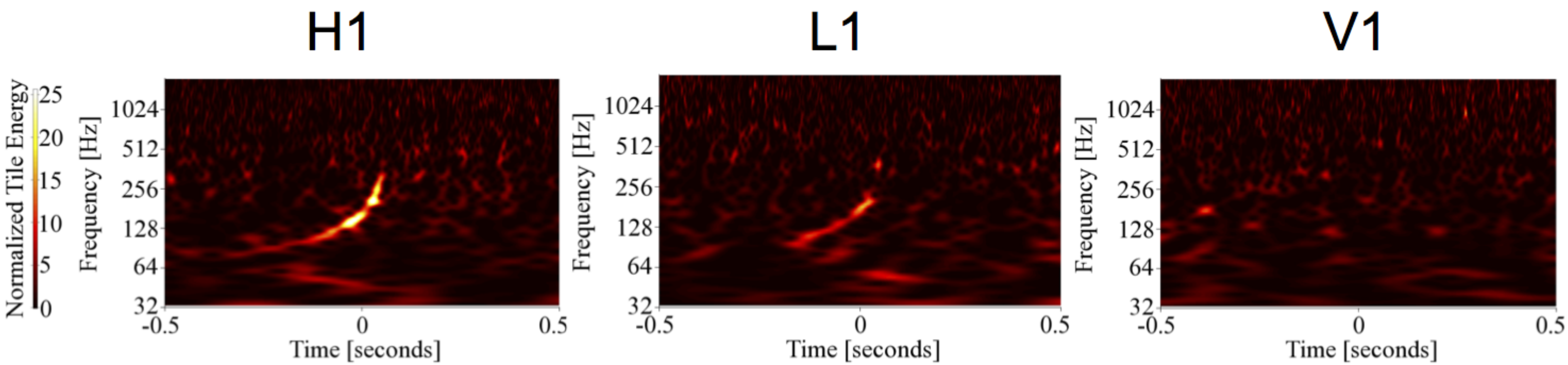
  - Binary neutron star coalescence:



Summary so far



The Big Dog



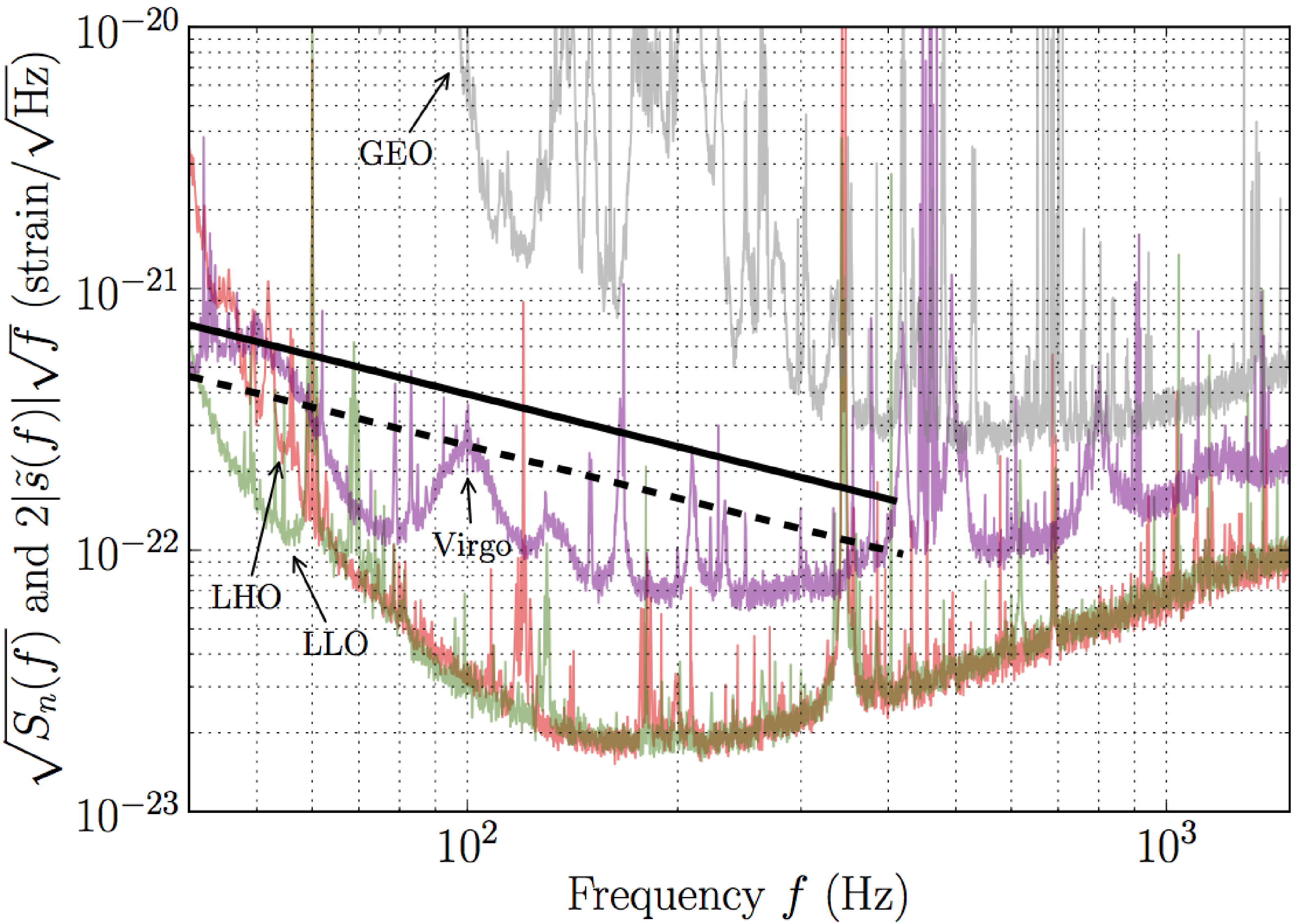
- SNR = 15
- Chirp Mass =4.7$M_\odot$

- SNR 10
- Chirp Mass = 4.4$M_\odot$

No trigger above threshold

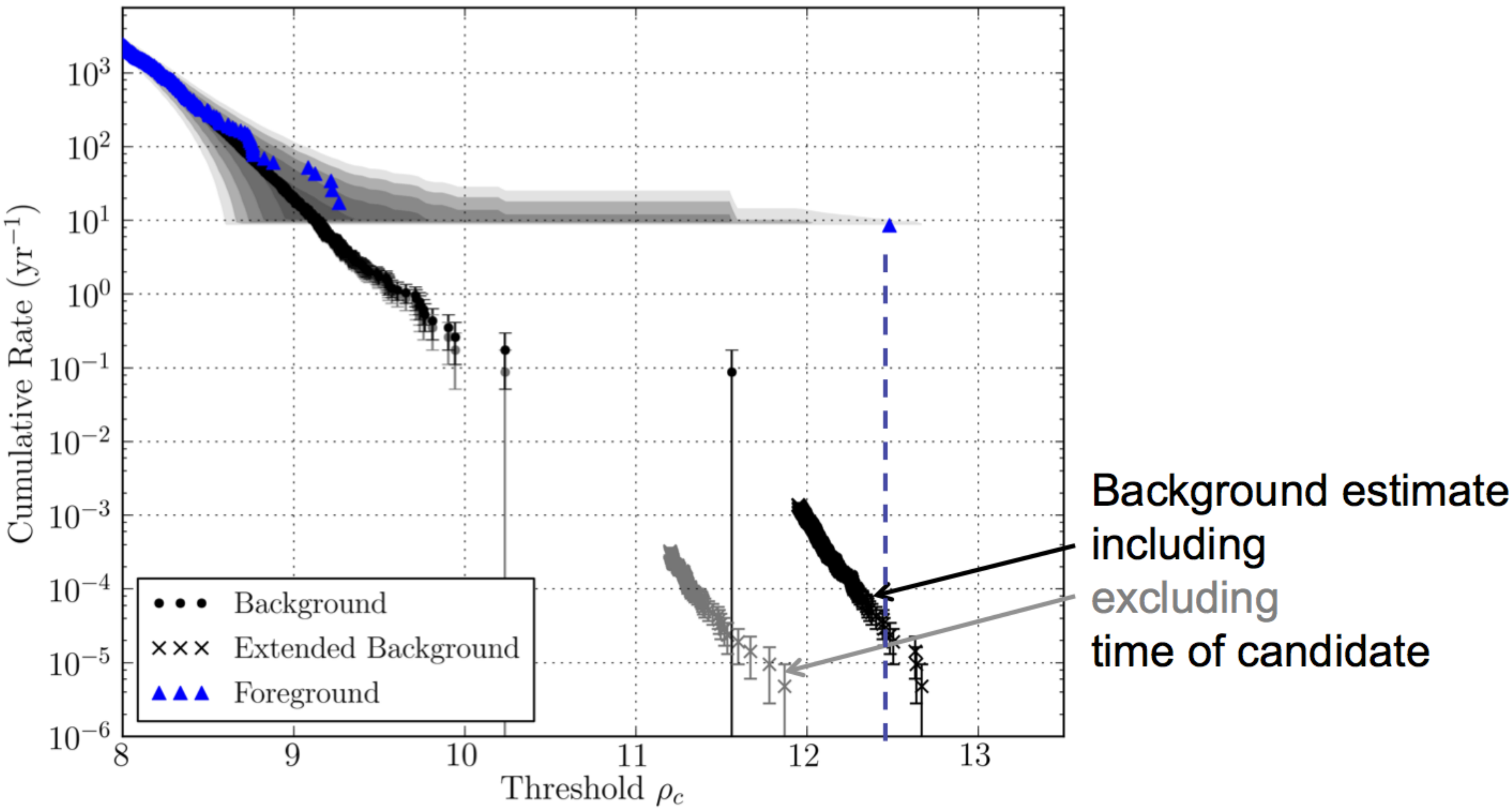
A coincident signal was observed by the two LIGO detectors at 2010 September 16, 06:42:23 UTC, with ρ_c= 12.5

The Big Dog



GEO
Virgo
LHO
LLO
10^{-20}
10^{-21}
10^{-22}
10^{-23}
10^{2}
10^{3}
Frequency f (Hz)
$\sqrt{S_n(f)}$ and $2|\tilde{s}(f)|\sqrt{f}$ (strain/$\sqrt{\text{Hz}}$)

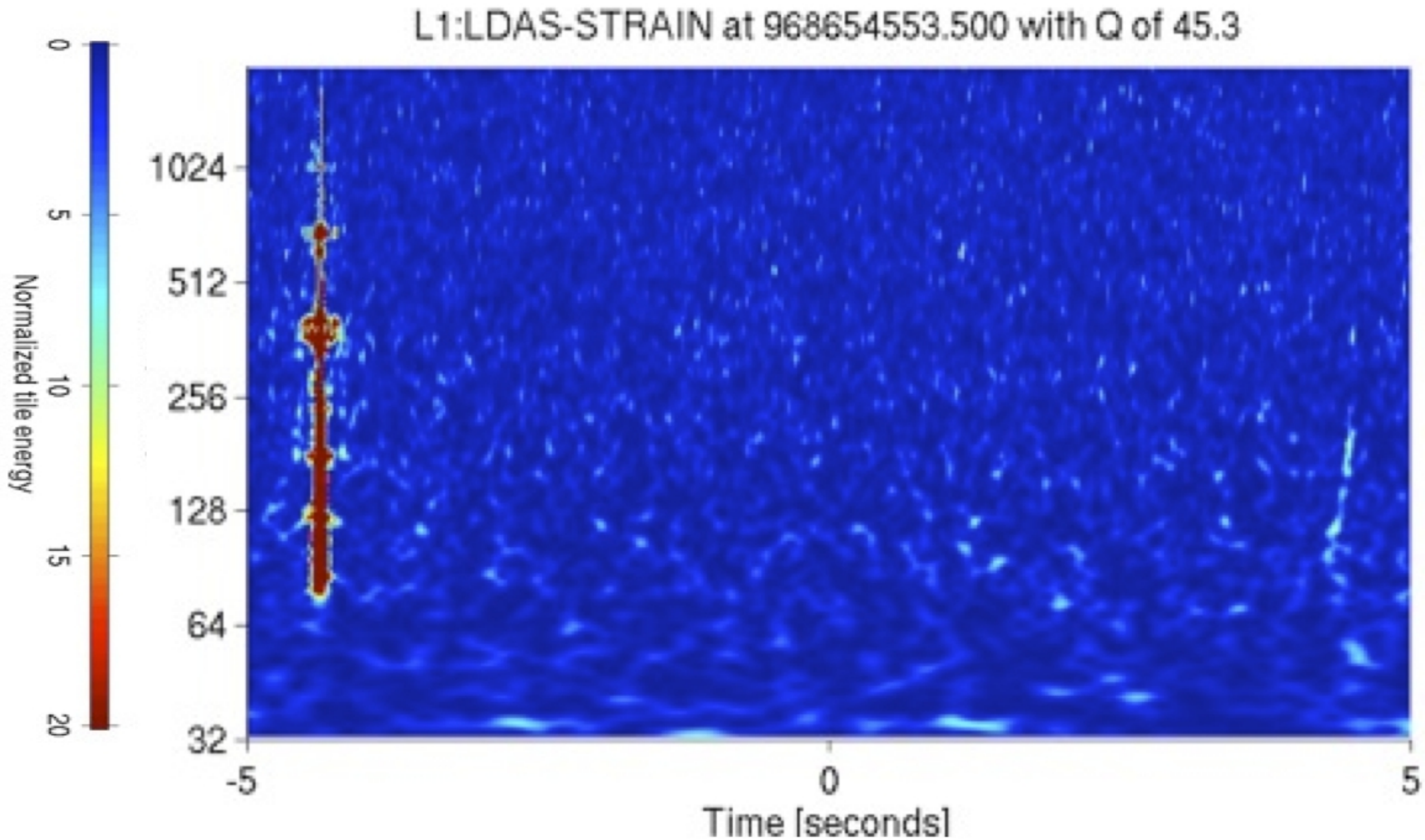
The Big Dog: significance of the event



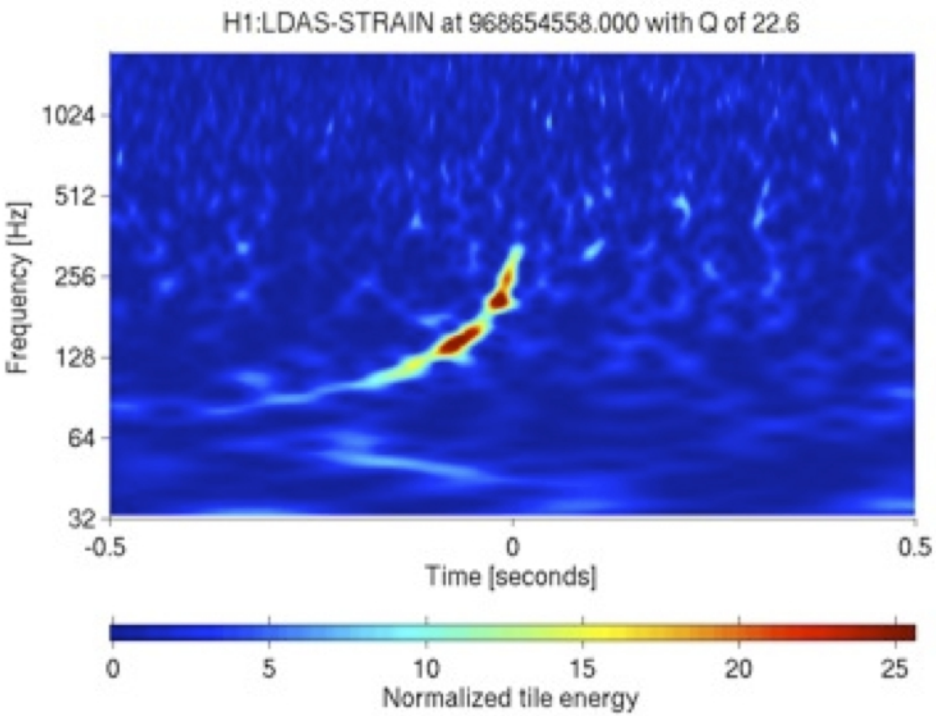
The event has a false alarm rate of less than 1 in 7000 y.

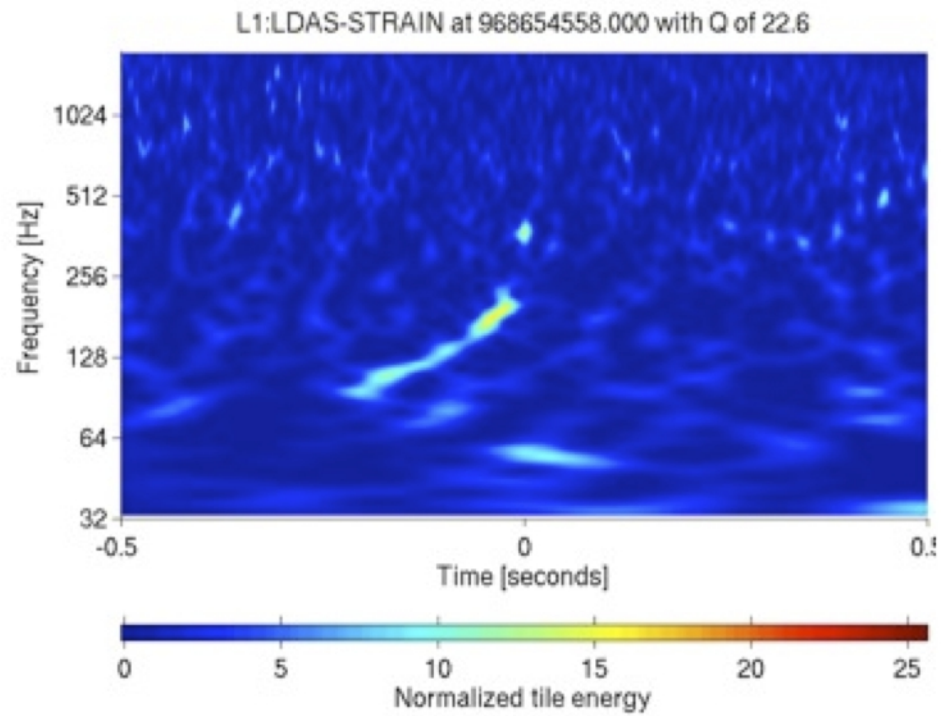
The Big Dog: instrumental validation

We find no evidence that the signal was of instrumental or environmental origin.

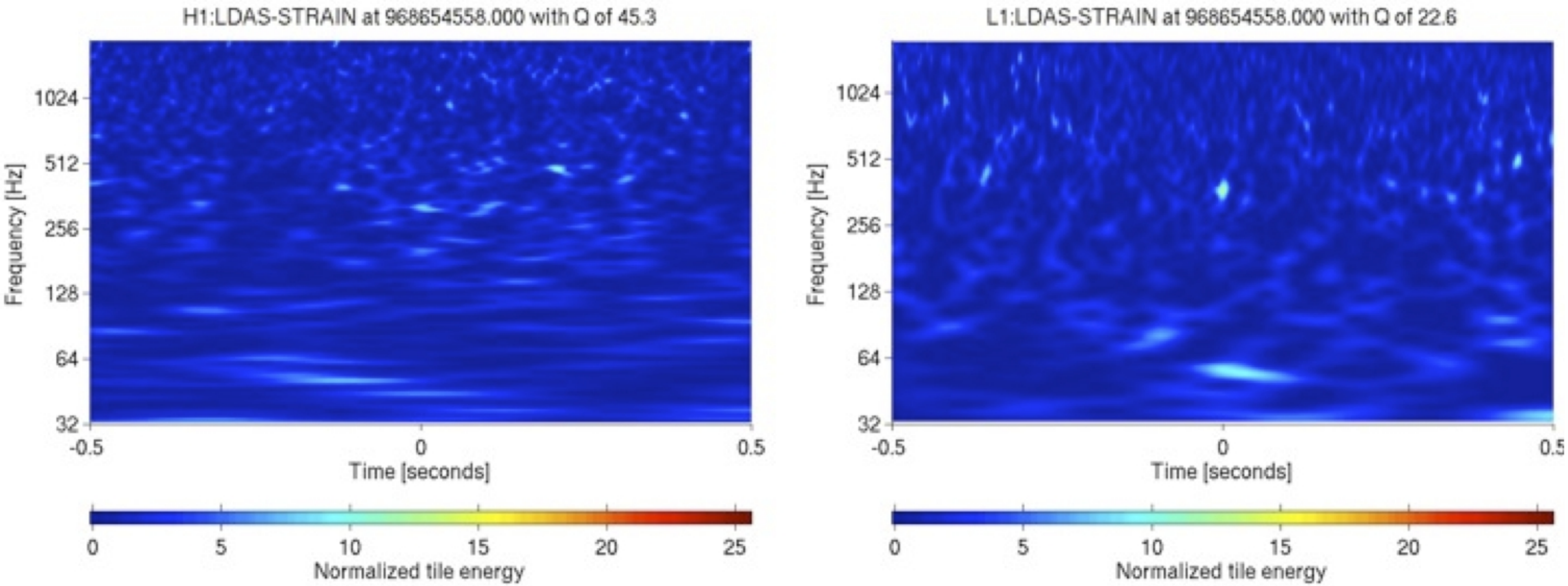


The Big Dog: waveform subtraction

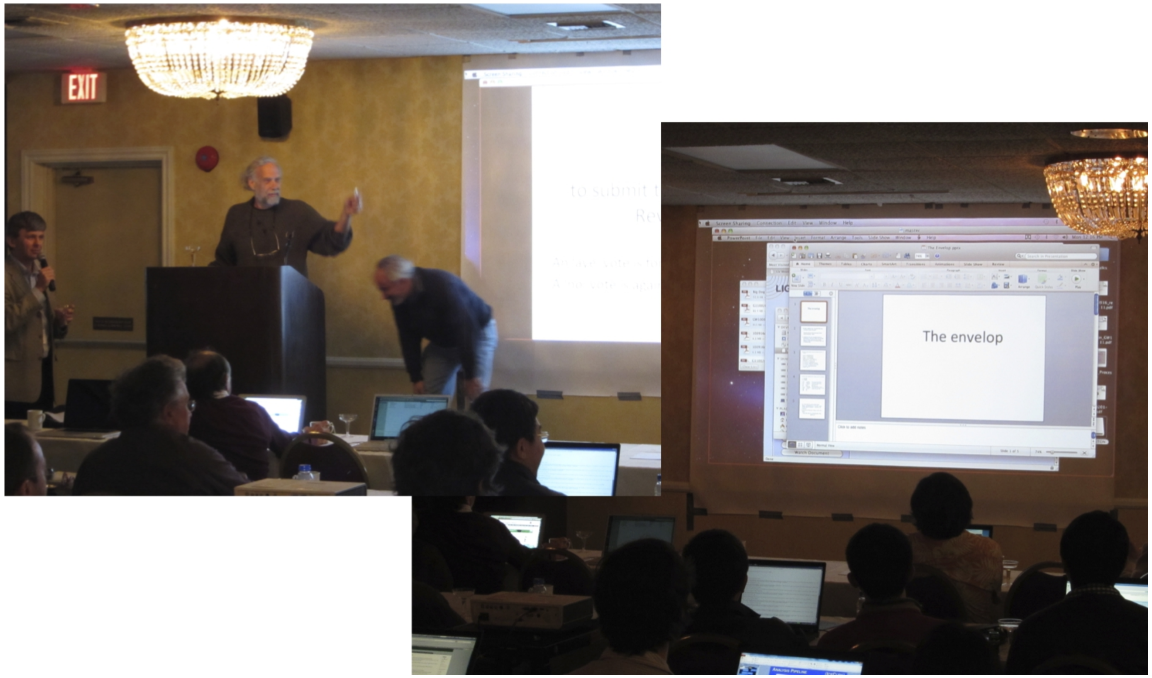




The Big Dog: waveform subtraction



But this one wasn't real...



Blind hardware injection:

- Mirrors are shaken so as to mimick what a real signal would do
- Nobody is told!
- Check if the data analysis algorithms are working properly
- Plausible deniability...

Better luck in the advanced detector era!