# Tutorials 2: Gravitational wave signals as seen in a detector

**1.** To leading post-Newtonian order, the evolution of the inspiral gravitational wave frequency looks like

$$f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{G\mathcal{M}}{c^3}\right)^{-5/8}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}, \tag{0.1}$$

where $\mathcal{M} = (m_1 + m_2)^{-1/5}(m_1 m_2)^{3/5}$ is the chirp mass, and $\tau = t_c - t$, with $t$ the time of coalescence. Invert this to obtain $\tau(f_{\rm gw})$.

**2.** Inspiral signals roughly end at the frequency of last stable orbit (LSO), defined by

$$f_{\rm LSO} = \frac{c^3}{6^{3/2}\pi G M}, \tag{0.2}$$

with $M$ the total mass of the binary. Calculate how long the signal will be in the sensitive frequency band of a detector, for the following cases:

- A typical binary neutron star inspiral with $(m_1, m_2) = (1.4, 1.4)\,M_\odot$, and detector lower frequency cut-offs of, respectively $f_{\rm low} = 40$ Hz, 20 Hz, and 3 Hz. (The latter could be appropriate for Einstein Telescope.)
- A typical binary black hole inspiral with $(m_1, m_2) = (10, 10)\,M_\odot$, and detector lower frequency cut-off of $f_{\rm low} = 40$ Hz, 20 Hz, and 3 Hz.
- A supermassive binary black hole inspiral with $(m_1, m_2) = (10^6, 10^6)\,M_\odot$ and LISA's lower frequency cut-off of $f_{\rm low} = 10^{-5}$ Hz.

**3.** The number of wave cycles is

$$N_{\rm cyc} = \int_{t_{\rm min}}^{t_{\rm max}} f_{\rm gw}\, dt, \tag{0.3}$$

where $t_{\rm min}$ is the time at which the signal enters the band, and $t_{\rm max}$ the time at which the signal terminates. Using Eq. (0.1), rewrite this as a simple expression in terms of chirp mass and lower frequency cut-off.

**4.** Using this expression, calculate the number of cycles in band for the same cases as in problem **2** above.

**5.** The detector response to an inspiral signal is:

$$h(t) = F_+(\theta, \phi, \psi)\, h_+(t) + F_\times(\theta, \phi, \psi)\, h_\times(t), \tag{0.4}$$

where to leading post-Newtonian order in amplitude we have

$$\begin{aligned} h_+(t) &= A(t)\,(1+\cos^2(\iota))\,\cos(\Phi(t)), \\ h_\times(t) &= A(t)\,2\,\cos(\iota)\,\sin(\Phi(t)), \end{aligned} \tag{0.5}$$

with

$$A(t) = \frac{4}{D}\left(\frac{G\mathcal{M}}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}. \tag{0.6}$$

Show that one can write

$$h(t) = A(t)\,\sqrt{F_+^2(1+\cos^2(i))^2 + F_\times^2 4\cos^2(\iota)}\,\cos(\Phi(t)+\varphi_0) \tag{0.7}$$

where

$$\varphi_0 = \arctan\left(\frac{-F_\times 2\cos(\iota)}{F_+(1+\cos^2(\iota))}\right) \tag{0.8}$$

**6.** For matched filtering we need the Fourier transform of the detector response:

$$\tilde{h}(f) = \int dt\,h(t)\,e^{2\pi i f t}. \tag{0.9}$$

The *stationary phase approximation* is a simple but useful approximation of $\tilde{h}(f)$, which we now derive in steps. First argue that

$$\int dt\,A(t)\,\cos(\Phi(t)-\varphi_0)\,e^{2\pi i f t} \simeq \frac{1}{2}e^{-i\varphi_0}\int dt\,A(t)\,e^{i(2\pi f t-\Phi(t))}. \tag{0.10}$$

Explain why the largest contribution to the integral comes from the time $t = t_s$ where $2\pi f = \dot{\Phi}(t_s(f))$. Now Taylor expand the exponent in the integrand around $t_s$:

$$2\pi f t - \Phi(t) \simeq 2\pi f t_s - \Phi(t_s) - \frac{1}{2}\ddot{\Phi}(t_s)\,(t-t_s)^2 + \ldots \tag{0.11}$$

Keeping only terms up to quadratic order, show that

$$\begin{aligned} &\int dt\,A(t)\,\cos(\Phi(t)-\varphi_0)\,e^{2\pi i f t} \\ &\simeq \frac{1}{2}A(t_s(f))\,e^{-i\varphi_0}\,e^{i(2\pi f t_s(f)-\Phi(t_s(f)))}\left(\frac{2}{\ddot{\Phi}(t_s(f))}\right)^{1/2}\int_{-\infty}^{\infty} dx\,e^{-ix^2}. \end{aligned} \tag{0.12}$$

It is known that

$$\int_{-\infty}^{\infty} dx\,e^{-ix^2} = \sqrt{\pi}e^{-i\pi/4}, \tag{0.13}$$

so that finally

$$\int dt\,A(t)\,\cos(\Phi(t)-\varphi_0)\,e^{2\pi i f t} \simeq \frac{\sqrt{\pi}}{2}A(t(f))\,e^{-i\varphi_0}\left(\frac{2}{\ddot{\Phi}(t_s(f))}\right)^{1/2} e^{i\Psi(f)}, \tag{0.14}$$

where

$$\Psi(f) = 2\pi f t(f) - \Phi(t(f)) - \pi/4. \tag{0.15}$$

**7.** Using Eq. (0.1), derive leading-order expressions for $\Phi(t)$ and $\ddot{\Phi}(t)$ as well as $t(f)$. With the results (0.14), (0.15), using (0.6), and reinstating the angle-dependent prefactor, show that

$$\tilde{h}(f) \simeq \sqrt{F_+^2(1+\cos^2(i))^2 + F_\times^2 4\cos^2(\iota)}\sqrt{\frac{5\pi}{96}}(\pi f)^{-7/6}\frac{c}{D}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/6} \times \exp\left[i\left(2\pi f t_c - \Phi_c + \frac{3}{4}\left(\frac{G\mathcal{M}f}{c^3}\right)^{-5/3} - \varphi_0 - \pi/4\right)\right]. \quad (0.16)$$

**8.** Show that in the stationary phase approximation, the optimal signal-to-noise ratio (SNR) takes the form

$$\rho = \left[(F_+^2(1+\cos^2(i))^2 + F_\times^2 4\cos^2(\iota))\frac{5\pi}{24}\pi^{-7/3}\frac{c^2}{D^2}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}\int_{f_{\min}}^{f_{\max}}\frac{f^{-7/3}}{S_n(f)}\,df\right]^{1/2}. \quad (0.17)$$

Using the expressions in the lecture slides for $F_+(\theta,\phi,\psi)$ and $F_\times(\theta,\phi,\psi)$, make plots to explore the dependence of $\rho$ on the sky position and orientation of the binary.

**9.** For a given distance, which sky position and orientation of the binary leads to the highest SNR, and what does the expression (0.17) reduce to in this case? Optional because laborious: show that if $\langle\cdot\rangle$ denotes the average over both the sky position $(\theta,\phi)$ and the orientation $(\iota,\phi)$,

$$\left\langle F_+^2\left(\frac{1+\cos^2(\iota)}{2}\right)^2 + F_\times^2\cos^2(\iota)\right\rangle = \frac{4}{25} \quad (0.18)$$

so that

$$\rho_{\text{averaged}} = \frac{2}{5}\rho_{\text{highest}}. \quad (0.19)$$

**10.** Show that given a minimum SNR $\rho_0$ needed for detection, the "angle-averaged" distance reach of a detector is:

$$D_{\text{averaged}} = \frac{2}{5}\left[\frac{5\pi}{6}\pi^{-7/3}c^2\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}\int_{f_{\min}}^{f_{\max}}\frac{f^{-7/3}}{S_n(f)}\,df\right]^{1/2}\rho_0^{-1} \quad (0.20)$$

**11.** You will have received a sensitivity curve for LIGO-India at design sensitivity. Write code to numerically evaluate the integral

$$\int_{f_{\min}}^{f_{\max}}\frac{f^{-7/3}}{S_n(f)}\,df. \quad (0.21)$$

Then, for a minimum SNR of $\rho_0 = 8$, make a 3D plot of $D_{\text{averaged}}$ as a function of component masses $m_1$ and $m_2$, in the mass regime of interest for binaries consisting of neutron stars and/or stellar mass black holes.