Beyond Linearized Approximation to GR for GW detection

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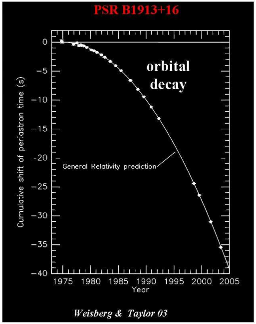
- Complication: For self-gravitating systems orders in velocity are related to orders in non-linearity..Virial theorem $\rightarrow \Phi = GM/R$ same order as v^2 ..Reaction terms of order $(v/c)^5$ from linear theory will be accompanied by terms $(v/c)^3 \, \Phi/c^2$, $(v/c)^1 \, (\Phi/c^2)^2$..
  Higher order PN calculation requires dealing dealing with higher order non-linearities of Einstein's Equations.

Gravitational Waves exist

1974, Binary Pulsar, Hulse and Taylor

High quality data ∼ Proof that GW exist

Nobel Prize (1993).



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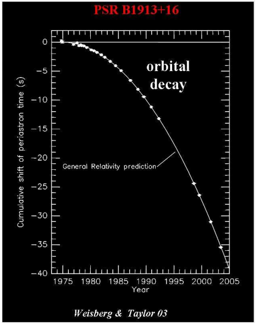

Prediction from general relativity

$$\dot{P} = -\frac{192\pi}{5c^5}\frac{\mu}{M}\left(\frac{2\pi G\,M}{P}\right)^{5/3}\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \approx -2.4\,10^{-12}\mathrm{s/s}$$

- Leading order (Newtonian) EOM [Peters & Mathews 1963] adequate to compute leading order GW Flux (Quadrupole formula) and infer *consequent* 2.5PN gravitational radiation reaction effect..Heuristic Balance arguments ..

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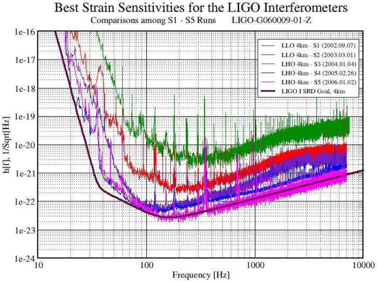
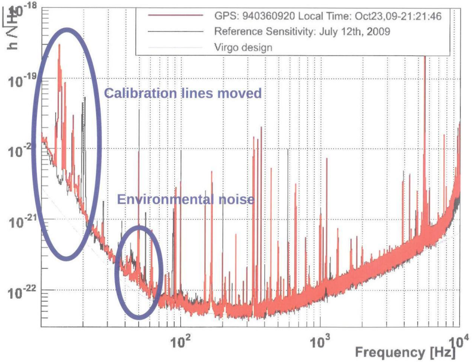
- Binary Psr population crucially underlie the estimated number of GW events for LIGO/Virgo. Existence of binary neutron star sources emitting GW for hundreds of million years before coalescing spectacularly in sensitivity BW of LIGO/Virgo.

Noise Curve of Detectors





Prototype Sources: Coalescing Compact Binaries

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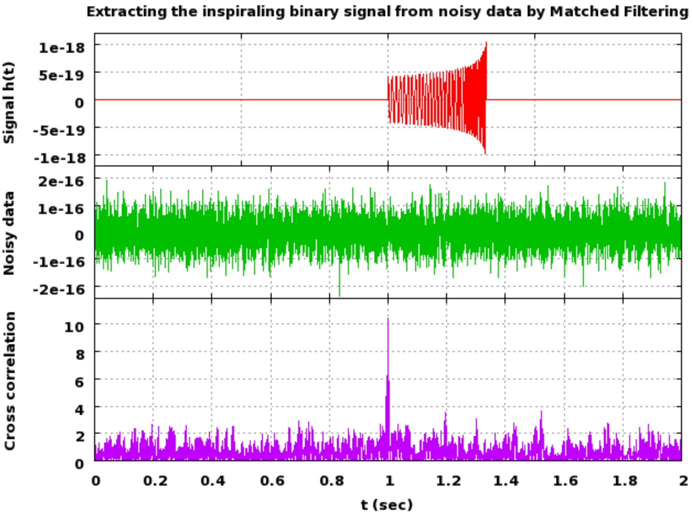
- Favours sources like CCB (NS-NS, BH-BH, NS-BH) over unmodelled sources like supernovae or GRB
- Spectacular Theoretical Progress in 2-body problem in GR complementing spectacular progress in GW detection endeavours

Chirp Signal, Matched Filtering



Courtesy Anand Sengupta (IUCAA)

Last three minutes.. Cutler et al 1993

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- (ii) Since matched filtering is sensitive to the phase it is more important to first control higher order phasing than higher order amplitudes - Newtonian Amplitude + Best available phasing: Restricted waveform
- (iii) The inspiral can be treated in the adiabatic approximation as sequence of circular orbits..This allows one to treat separately the **radiation reaction** effects and the **conservative** effects

GW from ICB - Three Modules

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- Radiation Reaction: Given the Conserved energy and Radiated Flux of Energy (and AM), *ASSUME* the Balance Eqns to Compute the effect of Radiation on the Orbit. Compute $F(t)$ and $\phi(t)$ (and $r(t)$)

Newtonian Phasing - Adiabatic Approximation

- Use E_N, the CM energy at Newtonian order and $\mathcal{L}_{\mathcal{N}}$, GW luminosity or energy flux $v^2 \equiv x = \left(\frac{Gm\omega_{\rm orb}}{c^3}\right)^{2/3}$ $\mu = \frac{m_1 m_2}{m_1+m_2}$, $\nu = \frac{\mu}{m_1+m_2}$

$$E_N = -\frac{1}{2}\mu c^2 x; \;\; \mathcal{L}_N = \frac{32}{5} \; \frac{c^5}{G} \; \nu^2 x^5; \;\; \frac{c^5}{G} \approx 3.63 \times 10^{52} \text{ W},$$

- And heuristic Energy Balance equation

$$\frac{dE_N}{dt} = -\mathcal{L}_N \; ; \to \frac{dt}{dv} = \frac{E_N'(v)}{\mathcal{L}_N(v)}$$

$$x_N(t) = \frac{1}{4}\tau_N^{-1/4}; \;\; \tau_N = \frac{c^3\nu}{5Gm}(t_c - t); \;\; \phi_N(t) = \int \omega dt = -\frac{5}{64\nu}\int x^{-7/2}dx$$

$$\phi(t) = \phi_c - \frac{x(t)^{-5/2}}{32\nu}; \;\; \mathcal{N} = \frac{\phi_0 - \phi}{\pi} = \frac{1}{32\pi}x^{-5/2} = \mathcal{O}(v/c)^{-5}$$

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Inverse RR order.

Requirement $\delta\mathcal{N} \sim 1$ implies PN corrections to Phase due to Inspiral will play crucial role to at least 2.5PN order

Higher order Phasing - Adiabatic Approx

At higher PN orders

$$\frac{dE_n}{dt} = -\mathcal{L}_n \, ; \, \rightarrow \frac{dt}{dv} = \frac{E_n'(v)}{\mathcal{L}_n(v)}$$

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- Higher order Phasing is equivalent to inclusion of higher order Gravitational Radiation Reaction (GRR). 3PN (v^6/c^6) Flux beyond leading quadrupole determines 3PN (v^6/c^6) RR relative to leading RR at 2.5PN (v^5/c^5).

$$\begin{aligned} a_i &= a_i^{\text{Newton}} + a_i^{1\text{PN}} + a_i^{2\text{PN}} + a_i^{2.5\text{PN}} + a_i^{3\text{PN}} + a_i^{3.5\text{PN}} \\ &+ a_i^{4\text{PN}} + a_i^{4.5\text{PN}} + a_i^{5\text{PN}} + a_i^{5.5\text{PN}} + a_i^{6\text{PN}} \end{aligned}$$

Are we there???

Contributions to the accumulated number $\mathcal{N} = \frac{1}{\pi}(\phi_{\rm ISCO} - \phi_{\rm seismic})$ of gravitational-wave cycles. Frequency entering the bandwidth is $f_{\rm seismic} = 10$ Hz; terminal frequency is assumed to be at the Schwarzschild innermost stable circular orbit $f_{\rm ISCO} = \frac{c^3}{6^{3/2}\pi G m}$. A$\equiv 2 \times 1.4 M_\odot$ B$\equiv 10 M_\odot + 1.4 M_\odot$ C$\equiv 2 \times 10 M_\odot$

RR Order	A	B	C
Newtonian	16031	3576	602
1PN	441	213	59
1.5PN	-211	-181	-51
2PN	9.9	9.8	4.1
2.5PN	-12.2	-20.4	-7.5
3PN	2.6	2.3	2.2
3.5PN	-1.0	-1.9	-0.9

Blanchet, Faye, BRI and Joguet

Blanchet, Damour, Esposito-Farese and BRI

Comparison of Detection Templates for Gravitational Waves from Inspiralling Compact Binaries

GWDA Problem

- In searching for GW from ICB one is faced with the following data analysis problem: We have some (unknown) exact GWF $h^X(t; \lambda_k)$ where λ_k, $k = 1, \ldots, n_\lambda$, are the parameters of the signal (e.g., masses m_1 and m_2). We have theoretical calculations of the motion and gravitational radiation from binary systems consisting of (NS) or (BH) giving the PN expansions of an energy function $E(x \equiv v^2)$, which is related to the total relativistic energy $E_{\rm tot}$ via $E_{\rm tot} = (m_1 + m_2)(1 + E)$, and a GW luminosity (or "flux") function $\mathcal{F}(v)$. The dimensionless argument $v \equiv x^{\frac{1}{2}}$ is an invariantly defined "velocity" related to the instantaneous GW frequency F ($=$ twice the *orbital* frequency) by $v \equiv (\pi m F)^{\frac{1}{3}}$.
- Given PN expansions of the motion of and grav radn from a binary system, one needs to compute the *"phasing formula"*, i.e. an accurate mathematical model for the evolution of the GW phase [within the "restricted" waveform approximation which keeps only the leading harmonic in the GW signal] $\phi^{\rm GW} = p[t; \lambda_i]$, involving the set of parameters $\{\lambda_i\}$ carrying information about the emitting binary system.

Phasing formula - Adiabatic Approx

- In the adiabatic approximation the phasing formula is easily derived from the energy and flux functions. Standard energy-balance equation $dE_{\rm tot}/dt = -\mathcal{F}$ gives the following parametric representation of the phasing formula:

$$t(v) = t_{\rm ref} + m \int_v^{v_{\rm ref}} dv \, \frac{E'(v)}{\mathcal{F}(v)}, \quad \phi(v) = \phi_{\rm ref} + 2 \int_v^{v_{\rm ref}} dv v^3 \, \frac{E'(v)}{\mathcal{F}(v)},$$

  $t_{\rm ref}$ and $\phi_{\rm ref}$ are integration constants and $v_{\rm ref}$ an arbitrary reference velocity.

- From the view point of computation more efficient to work with the following pair of coupled, non-linear, ordinary differential equations (ODE's) that are equivalent to the above parametric formulas:

$$\frac{d\phi}{dt} - \frac{2v^3}{m} = 0, \quad \frac{dv}{dt} + \frac{\mathcal{F}(v)}{mE'(v)} = 0.$$

- For massive systems, adiabatic approximation fails and one must replace the two ODE's by a more complicated ODE system.

T-APPROXIMANTS

- Denote by E_{T_n} and $\mathcal{F}_{T_n}$ the $n^{\rm th}$-order. "Taylor" approximants (as defined by the PN expansion) of the energy and flux functions. (Label n refers to an approximant accurate up toa $v^n = x^{(n/2)}$ included)

$$E_{T_{2n}}(x) \equiv E_N(x) \sum_{k=0}^{n} \hat{E}_k(\eta) x^k,$$

$$\mathcal{F}_{T_n}(x) \equiv \mathcal{F}_N(x) \left[\sum_{k=0}^{n} \hat{\mathcal{F}}_k(\eta) v^k + \sum_{k=6}^{n} \hat{L}_k(\eta) \log(v/v_0) v^k \right],$$

$$\text{where,} \quad E_N(x) = -\frac{1}{2}\eta x, \quad \mathcal{F}_N(x) = \frac{32}{5}\eta^2 x^5.$$

Subscript N denotes the "Newtonian value", $\eta \equiv m_1 m_2/m^2$ the symmetric mass ratio, and v_0 is a fiducial constant to be chosen below.

T-APPROXIMANTS

- In the test mass limit, i.e. $\eta \to 0$, $E(x)$ is known exactly, from which the Taylor expansion of $E_{T_n}(v, 0)$, can be computed to all orders. In the $\eta \to 0$ limit, the exact flux is known numerically and the Taylor expansion of flux is known up to order $n = 11$
(Poisson 93, Tanaka et al 96).
- For η finite, the above Taylor approximants are known up to seven-halves PN order, i.e. $n = 7$. (Damour, Jaranowski, Schäfer; Blanchet, Faye, Iyer, Joguet; Blanchet, Damour, Esposito-Farese, Iyer)
- Problem is to construct a sequence of approximate waveforms $h_n^A(t; \lambda_k)$, starting from the PN expansions of $E(v)$ and $\mathcal{F}(v)$. In formal terms, any such construction defines a *map* say T from the set of the Taylor coefficients of E and $\mathcal{F}$ into the (functional) space of waveforms

$$(E_{T_n}\,,\,\mathcal{F}_{T_n}) \xrightarrow{T} h_n^T(t, \lambda_k)\,,$$

obtained by inserting the successive Taylor approximants into the phasing formula. For brevity, refer to these "Taylor" approximants as "T-approximants".

T-APPROXIMANTS

- Beware: Even within this Taylor family of templates, there are at least three ways of proceeding further, leading to the following three *inequivalent* constructs:

Taylor $T1$ Approximant

- To compute $v(t)$ and $\phi(t)$;
  Std approach to Evoln of the GWF under RR:
  Adiabatic approximation;
  Cutler et al (1993)
- E and $\mathcal{F}$ given as PN Expansions in v
  Use E and $\mathcal{F}$ both to same *relative* PN order
- Phasing of GW given by following coupled ODE

$$\frac{d\varphi}{dt} = \frac{2v^3}{m}, \qquad \frac{dv}{dt} = -\frac{\mathcal{F}(v)}{mE'(v)},$$

  $E'(v) = dE(v)/dv$; $m = m_1 + m_2$
- Retain the rational polynomial $\mathcal{F}_{T_n}/E_{T_n}$ as it appears above
- Integrate the two ODE's numerically.
- Denote the phasing formula so obtained as $\phi^{(1)}_{T_n}(t)$:

Taylor $T2$-approximants

- Re-expand the rational function $\mathcal{F}_{T_n}/E_{T_n}$ appearing in the phasing formula and truncate it at order v^n,
- The integrals can be worked out analytically, to obtain a *parametric* representation of the phasing formula in terms of polynomial expressions in the auxiliary variable v

$$\phi^{(2)}_{T_n}(v) = \phi^{(2)}_{\mathrm{ref}} + \phi^v_N(v)\sum_{k=0}^{n}\hat{\phi}^v_k v^k, \quad t^{(2)}_{T_n}(v) = t^{(2)}_{\mathrm{ref}} + t^v_N(v)\sum_{k=0}^{n}\hat{t}^v_k v^k,$$

- The superscript on the coefficients (eg. ϕ^v_1) indicates that v is the expansion parameter
- The coefficient of ϕ^v_k include in some cases, a $\log v$ dependence

Taylor $T3$-approximants

- Alternatively, the second of the polynomials in Eq. (t as fn of v) can be inverted to obtain a polynomial for v in terms of t
- This can be substituted in $\phi^{(2)}(v)$ to arrive at an explicit time-domain phasing formula

$$\phi^{(3)}_{T_n}(t) = \phi^{(3)}_{\rm ref} + \phi^t_N \sum_{k=0}^{n} \hat{\phi}^t_k \theta^k, \qquad F^{(3)}_{T_n}(t) = F^t_N \sum_{k=0}^{n} \hat{F}^t_k \theta^k,$$

$\theta = [\eta(t_{\rm ref} - t)/(5m)]^{-1/8}$ and $F \equiv d\phi/2\pi dt = v^3/(\pi m)$ is the instantaneous GW frequency.

- The coefficients in these expansions are all listed in DIS

Frequency-domain phasing- Adiabatic Approx

- Frequency-domain phasing uses the usual stationary phase approximation for chirp signals. Consider a signal of the form,

$$h(t) = 2a(t)\cos\phi(t) = a(t)\left[e^{-i\phi(t)} + e^{i\phi(t)}\right],$$

  $\phi(t)$ is the implicit solution of one of the phasing formulas in the phasing eqn for some choice of functions E' and $\mathcal{F}$.
- Quantity $2\pi F(t) = d\phi(t)/dt$ defines the instantaneous GW frequency $F(t)$, and is assumed to be continuously increasing. (We assume $F(t) > 0$.) Fourier transform $\tilde{h}(f)$ of $h(t)$ is defined as

$$\tilde{h}(f) \equiv \int_{-\infty}^{\infty} dt e^{2\pi i f t} h(t) = \int_{-\infty}^{\infty} dt\, a(t)\left[e^{2\pi i f t - \phi(t)} + e^{2\pi i f t + \phi(t)}\right].$$

  Above transform can be computed in the SPA. For positive frequencies only the first term on the right contributes and yields the following *usual* SPA:

$$\tilde{h}^{\rm uspa}(f) = \frac{a(t_f)}{\sqrt{\dot{F}(t_f)}} e^{i[\psi_f(t_f) - \pi/4]}, \quad \psi_f(t) \equiv 2\pi f t - \phi(t),$$

Frequency-domain phasing- Adiabatic Approx

- t_f is the saddle point defined by solving for t, $d\psi_f(t)/dt = 0$, i.e. the time t_f when the GW frequency $F(t)$ becomes equal to the Fourier variable f. In the (adiabatic) approximation, the value of t_f is given by the following integral:

$$t_f = t_{\rm ref} + m \int_{v_f}^{v_{\rm ref}} \frac{E'(v)}{\mathcal{F}(v)} dv,$$

$v_f \equiv (\pi m f)^{1/3}$. Using t_f from the above equation and $\phi(t_f)$ one finds that

$$\psi_f(t_f) = 2\pi f t_{\rm ref} - \phi_{\rm ref} + 2 \int_{v_f}^{v_{\rm ref}} (v_f^3 - v^3) \frac{E'(v)}{\mathcal{F}(v)} dv.$$

- Big computational [with respect to its time-domain counterpart] advantage, is that, in the frequency domain, there are no equations to solve iteratively; the Fourier amplitudes are given as explicit functions of frequency.
- In the Fourier domain too there are many inequivalent ways in which the phasing ψ_f can be worked out. The most popular being

$F1$ - Approximant

- Substitute (without doing any re-expansion or re-summation) for the energy and flux functions their PN expansions or the P-approximants of energy and flux functions and solve the integral in Eq. numerically to obtain the T-approximant SPA or P-approximant SPA, respectively.

$F2$ - Approximant

- Use PN expansions of energy and flux but re-expand the ratio $E'(v)/\mathcal{F}(v)$ in Eq. in which case the integral can be solved explicitly. This leads to the following explicit, Taylor-like, Fourier domain phasing formula:

$$\psi_f(t_f) = 2\pi f t_{\rm ref} - \phi_{\rm ref} + \tau_N \sum_{k=0}^{5} \hat{\tau}_k (\pi m f)^{(k-5)/3}$$

  $\hat{\tau}_k$ are the chirp parameters listed in the paper. The latter is one of the standardly used frequency-domain phasing formulas. Therefore, one uses that as one of the models in our comparison of different inspiral model waveforms. Refer to it as "type-f2" frequency-domain phasing.

- Just as in the time-domain, the frequency-domain phasing is most efficiently computed by a pair of coupled, non-linear, ODE's:

$$\frac{d\psi}{df} - 2\pi t = 0, \qquad \frac{dt}{df} + \frac{\pi m^2}{3v^2}\frac{E'(f)}{\mathcal{F}(f)} = 0,$$

  rather than by numerically computing the integral in Eqs.

Approximation Schemes, regimes of validity

- Three main methods to deal with two body problem in GR:
  - Post Newtonian (PN) approach,
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MPM-PN

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- Perturbation theory suitable to describe motion and radiation of a small body moving around a large body. Expands EE around BH metric in mass ratio m_2/m_1. Can deal with **Extreme mass ratio** inspiral (EMRI); + **Quasi-Normal Mode (QNM) ringing**.

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- NR solves EE on computer for coalescence, merger and ringdown beyond inspiral. Can be used for any mass ratio, separation or velocity. Range of validity constrained by computational resources and accuracy requirements on the numerical solutions.
- Beyond PNA; Padè Resummation; `Effective one body approach`: EOB provides analytical description of both motion and radiation of CB from early inspiral, right thro plunge, merger and final ringdown.

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- (i) The general method (MPM expansion) applicable to extended or fluid sources with compact support, based on the mixed PM and multipole expansion (asymptotically) matched to some PN (slowly moving, weakly gravitating, small-retardation) source. IR divergences arising from the retardation expansion dealt by analytic continuation

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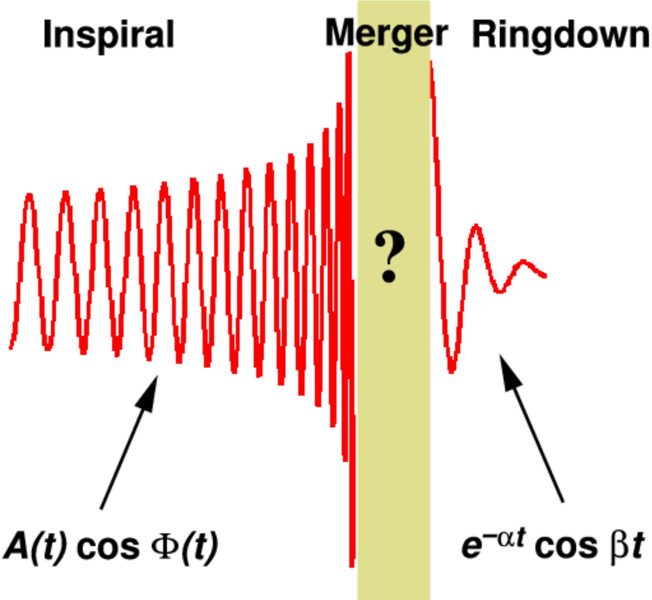
- (ii) The particular application to describe inspiralling compact binaries (ICB) by use of point particle models. Self-field regularisation to deal with UV divergences arising from use of Delta functions to model point particles - Riesz, Hadamard partie finie, Dimensional regularisation

Stages of Binary Coalescence



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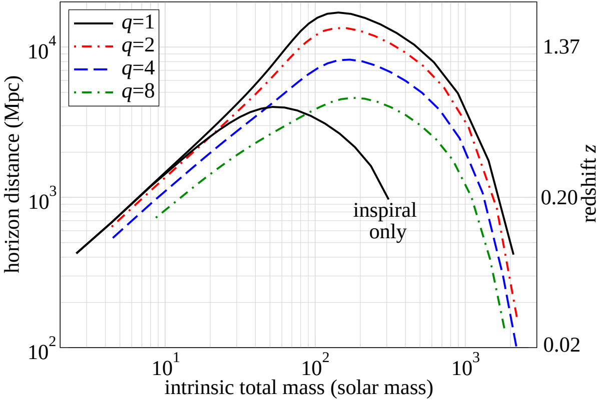

Credit: Buonanno, Sathyapakash

Implication for Ground Based GW detectors

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- Require Accurate Templates for **Detection** by matched filtering; More accurate templates for **parameter extraction** and eventual GW Astronomy. LIGO vs eLISA

Towards Detectors in FZ from Source in NZ

- Computation of Flux begins with computation of the Six Source moments (Mass I_L, Current J_L & 4 gauge (W_L, X_L, Y_L, Z_L) moments) from the non-linear source $\tau^{\mu\nu}$. The computation involves analytic continuation to deal with IR divergences, Dimensional Regularization to deal with the UV divergences and WL renormalization.

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- From the 6 source moments the two Canonical moments M_L and S_L can the algorithmically constructed. From the canonical moments the Radiative moments U_L and V_L can be constructed. The FZ fluxes and GW polarisations can be constructed from the Radiative moments.

ZONES

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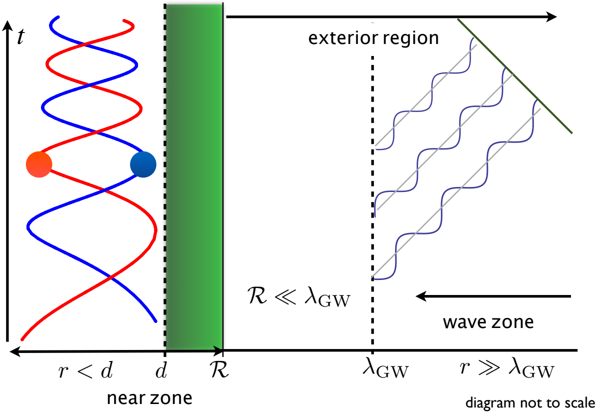


Credit: Buonanno, Sathyapakash

FZ flux - Radiative Multipoles

Following Thorne (1980), the expression for the 3PN accurate far zone energy flux in terms of symmetric trace-free (STF) radiative multipole moments read as

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\rm far-zone} = \frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij}\right.$$
$$+\frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk}+\frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm}+\frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right]$$
$$\left.+\frac{1}{c^6}\left[\frac{1}{594000}U^{(1)}_{ijkmn}U^{(1)}_{ijkmn}+\frac{4}{14175}V^{(1)}_{ijkm}V^{(1)}_{ijkm}\right]+\mathcal{O}(8)\right\}.$$

- For a given PN order only a finite number of Multipoles contribute
- At a given PN order the mass l-multipole is accompanied by the current $l-1$-multipole (Recall EM)
- To go to a higher PN order Flux requires new higher order l-multipoles and more importantly higher PN accuracy in the known multipoles.
- 3PN Energy flux requires 3PN accurate Mass Quadrupole, 2PN accurate Mass Octupole, 2PN accurate Current Quadrupole,........ N Mass 2^5-pole, Current 2^4-pole

State of the art restricted GW phasing for ICB

Damour, Jaranowski, Schäfer 2014 $x \equiv (GM\omega/c^3)^{2/3}$

$$
\begin{aligned}
E_4(x) &= -\frac{1}{2}\nu x \left[1 - \left(\frac{3}{4} + \frac{1}{12}\nu\right) x - \left(\frac{27}{8} - \frac{19}{8}\nu + \frac{1}{24}\nu^2\right) x^2 \right. \\
&- \left\{ \frac{675}{64} - \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu + \frac{155}{96}\nu^2 + \frac{35}{5184}\nu^3 \right\} x^3 \\
&+ \left(-\frac{3969}{128} + \left(\frac{9037\pi^2}{1536} - \frac{123671}{5760} + \frac{448}{15}(2\gamma_{\mathrm{E}} + \ln(16x))\right)\nu \right. \\
&+ \left. \left. \left(\frac{3157\pi^2}{576} - \frac{498449}{3456}\right)\nu^2 + \frac{301\nu^3}{1728} + \frac{77\nu^4}{31104}\right) x^4 + \mathcal{O}(x^5) \right],
\end{aligned}
$$

State of the art restricted GW phasing for ICB

Blanchet, Iyer, Damour, Esposito-Farese $x \equiv (GM\omega/c^3)^{2/3}$

$$
\begin{aligned}
\mathcal{L}_{3.5}(x) &= \frac{32c^5}{5G}x^5\nu^2\left\{1+\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x+4\pi x^{3/2}\right.\\
&+ \left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2+\left(-\frac{8191}{672}-\frac{535}{24}\nu\right)\pi x^{5/2}\\
&+ \left(\frac{6643739519}{69854400}+\frac{16\pi^2}{3}-\frac{1712}{105}C-\frac{856}{105}\ln(16x)\right.\\
&+ \left.\left[\frac{41\pi^2}{48}-\frac{134543}{7776}\right]\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right)x^3\\
&+ \left.\left(-\frac{16285}{504}+\frac{176419}{1512}\nu+\frac{19897}{378}\nu^2\right)\pi x^{7/2}+\mathcal{O}(x^4)\right\}
\end{aligned}
$$

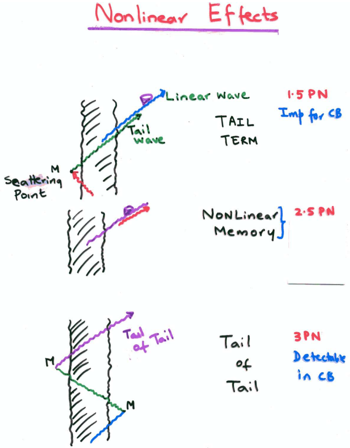
3PN GW polarisations: Blanchet, Faye, Iyer, Sinha, Favata

3.5PN GW polarisations h^{22}, h^{33}, h^{31}

Faye, Blanchet, Marsat BRI

3.5PN GW Flux requires and includes..

Control of all the non-linear couplings between multipole moments up to order 3.5PN for general matter sources; those couplings involve in addition to many 'instantaneous' terms, the important contributions of tails, tails-of-tails (Weakly dependent on past history) and the non-linear memory (Strongly dependent on past history)



ARE WE THERE???

Contributions to the accumulated number $\mathcal{N} = \frac{1}{\pi}(\phi_{\rm ISCO} - \phi_{\rm seismic})$ of gravitational-wave cycles. Frequency entering the bandwidth is $f_{\rm seismic} = 10$ Hz; terminal frequency is assumed to be at the Schwarzschild innermost stable circular orbit $f_{\rm ISCO} = \frac{c^3}{6^{3/2}\pi G m}$. A$\equiv 2 \times 1.4 M_\odot$ B$\equiv 10 M_\odot + 1.4 M_\odot$ C$\equiv 2 \times 10 M_\odot$

RR Order	A	B	C
Newtonian	16031	3576	602
1PN	441	213	59
1.5PN	−211	−181	−51
2PN	9.9	9.8	4.1
2.5PN	−12.2	−20.4	−7.5
3PN	2.6	2.3	2.2
3.5PN	−1.0	−1.9	−0.9

Blanchet, Faye, BRI and Joguet

Blanchet, Damour, Esposito-Farese and BRI

Extensions to..

- Spinning BH Binaries (BH have spin)
- Quasi-eccentric binaries (there exist Ap mechanisms leading to binaries that could have eccentricity - Kozai mechanism)
- Tidal Effects (NSNS binaries would tidally distort in their final stages..Allow one to determine EOS of NS material)
- BH Horizon Flux (Though small could be important as one goes to very high PN order)

MPM-PN formalism

- The Multipolar Post Minkowskian (MPM) formalism matching to a PN source with its genesis to compute 1PN corrections in the Binary pulsar case is a good example of the advantage that a complete and mathematically rigorous treatment of a problem can eventually bring in the future for more demanding applications that could be around the corner. Consistent algorithmic approach..

MPM-PN FORMALISM

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- MPM: Currently the most successful since it can deal with *all* aspects: the Conservative EOM, Radiation field at infinity, Non-linear efffects related to Tails. Starting point, in data analysis, to construct templates for double neutron-star binaries and crucial input to validate the early inspiral phase of the numerical relativity waveforms for black-hole binaries.

Other approaches

- Direct Integration of Relaxed Einstein Eqns -
  DIRE (Epstein, Thorne, Will and Wiseman, Pati); Perfect Fluid, Split volume integrals - 2PN EOM, FLUX
  2.5PN EOM in Scalar Tensor gravity (Mirshekari, Will); 2PN Waveform (Lang), 1PN energy flux (Lang) - Phasing in other theories
- Strong field point particle limit (Schutz, Futamase, Asada, Itoh); EIH like - 3PN EOM
- Effective Field Theory (EFT)
- Gravitational Self Force (GSF) approaches for EMRI's...
- NR-AR (Cornell-Caltech, Goddard, Jena, RIT, IHES, Maryland) and Self force-PN (Blanchet, Tiec, Whiting, Detweiler, Johnson-McDaniel..) comparisons....
- RR from balance equations (BRI, Will, Gopakumar, Sai Iyer, Zeng ....
  3.5PN, 4.5PN EOM

Recent results: partial high order amplitudes

Binaries of non-spinning objects

What we know

	PN orders	
quantities	circular	eccentric
EOM	4PN	3.5PN
E, $\mathbf{J}$	4PN	3.5PN
E, $\mathbf{J}$ flux	3.5PN	3PN
$\phi(t)$	3.5PN	3PN
$h_{+,\times}$	3PN	2PN

[Jaranowski, Schäfer, Damour; Blanchet, Faye, Damour, Esposito-Farése,; Itoh, Futamase; Foffa, Sturani]

[Blanchet, Iyer, Joguet, Damour, Esposito-Farèse, Faye]

[Gopakumar, Iyer, Arun, Qusailah, Sinha, Mishra, Faye, Blanchet]

Courtesy: G. Faye

Recent partial results for $h_{+,\times}$

- Eccentric case (no spin)
  inst. part of the waveform at 3PN [Chandra Mishra, Arun, Iyer (2015)]
- Circular case:
  1. mode (2,2)
     [F., Marsat, Blanchet, Iyer (2012)]
  2. mode (3,3), (3,1)
     [F., Blanchet, Iyer (2014)]

Completion of those works in progress...

Spinning binaries: state of the art

Near-zone dynamics

	Leading	Known
NS	N	4PN (ADM)1
SO	1.5PN	3.5PN (ADM, PNISH)2
SS	2PN	3PN (SS) – 4PN (S_1S_2) (ADM, EFT, PNISH)3
SSS	3.5PN	3.5PN (ADM/EFT, PNISH) 4
SSSS	4PN	4PN (ADM/EFT)5

ADM: reduced Hamiltonian in ADM gauge

EFT: effective field theory

PNISH: PN Iteration Scheme in Harmonic coord.

Energy flux

	Leading	Known
NS	N	3.5PN (PNISH)
SO	1.5PN	3.5PN+4PN (PNISH)
SS	2PN	3PN (SS, S_1S_2) (partial EFT, PNISH)
SSS	3.5PN	3.5PN (PNISH)

waveform

	Leading	Known
NS	N	3PN (3.5PN) (PNISH)
S	1PN	2PN (PNISH) 6

1 [Jaranowski, Schäfer (2013); Damour, Jaranowski,Schäfer (2014)]

2 [Hartung, Steinhoff (2011); Marsat, Bohé, F., Blanchet (2013)]
3 EFT: [Porto, Rothstein (2008a, 2008b); Levi (2010, 2012)]
ADM: [Hergt, Steinhoff, Schäfer (2010); Hartung, Steinhoff (2011)

PNISH: Bohé, F., Marsat, Porter (2015)]

4 [Levi, Steinhoff (2015); Marsat (2015)]

5 [Levi, Steinhoff (2015)]

6 [Buonanno, F., Hinderer (2013)]

[A. Gupta, A. Gopakumar (2014)]

Courtesy: G. Faye

Test particle limit

- Test particle limit - Analytical: 22PN energy flux for Schwarzschild (Fujita), 11PN for Kerr case (Fujita) using Mano, Suzuki, Takasugi functional series formalism (Confluent Heun; Expn in infinite series of confluent hypergeometric fns using parameter ν determined by continued fraction eqn). 20PN numerically for the Kerr case (Shah). Important for PE in eLISA.. Where reqd achieved by calculating the fluxes numerically with an accuracy greater than 1 part in 10^{600}!!

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- Productive Interplay of analytical PN, self force programs for EMRI. GSF facilitates computation of linear in mass ratio pieces of high order PN coefficients and binding energy for circular orbits..

  Experimental mathematics (high accuracy numerics+ PSLQ integer relation algorithm) can obtain analytic forms of high PN coeffs (involving Euler-Mascheroni gamma, logs of primes, Riemann zeta fn at integers) of linear in mass ratio of binding energy. (Detweiler, Whiting, Blanchet, Friedman, Shah, Johnson-McDaniel, Bini, Damour..)

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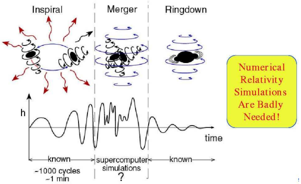
- Frequency domain waveforms and orbital dynamics (Yunes et al; Tessmer, Schäfer; Huerta et al)

Complete Waveform from EOB - Inspiral, Plunge, Merger, Ringdown

(Buonanno and Damour, 2000)



How Complicated will it be??

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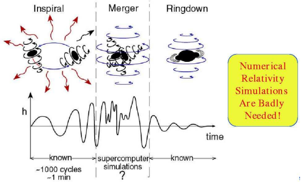

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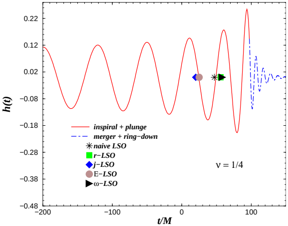

It is rather simple!!
A smooth continuation of inspiral & sharp transition around merger to ringdown