S. G. Dani's work on homogeneous dynamics

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- François Ledrappier and Riddhi Shah, Dani's work on probability measures on groups

Homogeneous dynamics: Background

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- Examples: the geodesic and horocycle flows on the modular surface
- These form a very rich class of dynamical systems
- With extensive connections to number theory and geometry

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- Dani (82): Dense orbits of unipotent flows equidistribute on $\mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$

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- Dani's measure conjecture: this continues to hold in much greater generality for unipotent flows on G/Γ
- Raghunathan's topological conjecture (appears in print in a paper of Dani): The closure of every u_t-orbit on G/Γ is a finite-volume homogeneous space

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- Dani's work on unipotent flows is very influential

- For example, the *linearization* technique introduced by Dani-Margulis is a fundamental and widely used tool

Divergent trajectories of flows on homogeneous spaces and Diophantine approximation

By *S. G. Dani**) at Bombay

Let G be a connected Lie group and Γ be a lattice in G; that is, Γ is a discrete subgroup of G such that G/Γ admits a finite G-invariant measure. Let $\{g_t\}_{t\in\mathbb{R}}$ be a one-parameter subgroup of G. The action of $\{g_t\}$ on G/Γ (on the left) induces a flow on G/Γ. The ergodic theory of these flows is extensively studied and, at least from a certain point of view, satisfactorily understood (cf. [6] and its references). Thus, for instance, it is possible to determine, in terms of the position of $\{g_t\}$ in G relative to Γ, whether the flow admits dense trajectories $\{g_t x\Gamma \mid t \geqq 0\}$, where $x \in G$, and whether a generic trajectory (either with respect to the measure or topologically) is dense in G/Γ. In general, however, there exist exceptional trajectories which are not dense, but to describe their set is a very difficult task; for an arbitrary one-parameter subgroup this is known only when G is a nilpotent Lie group (cf. [18] for that case and [16] for results on horocycle flows).

In this paper we assume G/Γ to be non-compact and investigate a special class of such exceptional trajectories: 'divergent' trajectories. A trajectory is said to be divergent if eventually it leaves every compact subset of G/Γ (cf. § 1 for precise definition). In §§ 2 and 3 we also get some results on bounded trajectories of certain flows.

It was proved by G. A. Margulis in [21] that if $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$ and $\{g_t\}$ is a one-parameter subgroup consisting of unipotent elements then there are no divergent trajectories (cf. [11] and [14] for stronger results). A similar situation can be seen to hold if all the eigenvalues of g_t, $t \in \mathbb{R}$ are of absolute value 1 (cf. Proposition 2. 6). However if g_1 (or any g_t, $t \neq 0$) has some eigenvalue λ with $|\lambda| \neq 1$ then there exist at least certain 'obvious' divergent trajectories. For instance, if $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and $g_t = \operatorname{diag}(e^{-t}, e^t)$, then the trajectory starting from any point of $P\Gamma/\Gamma$, where P is the subgroup consisting of all upper triangular matrices in G, is divergent for simple geometric reasons. We call these degenerate divergent trajectories (cf. § 2 for details). In § 2 we also consider the one-parameter subgroups of G of the form

$$\operatorname{diag}(e^{-t}, \ldots, e^{-t}, e^{\lambda t}, \ldots, e^{\lambda t}),$$

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On orbits of unipotent flows on homogeneous spaces, II

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Abstract. We show that if (u_t) is a one-parameter subgroup of $\mathrm{SL}(n, \mathbb{R})$ consisting of unipotent matrices, then for any $\varepsilon > 0$ there exists a compact subset K of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ such that the following holds: for any $g \in \mathrm{SL}(n, \mathbb{R})$ either $m(\{t \in [0, T] \mid u_t g\, \mathrm{SL}(n, \mathbb{Z}) \in K\}) > (1-\varepsilon)T$ for all large T (m being the Lebesgue measure) or there exists a non-trivial $(g^{-1}u_t g)$-invariant subspace defined by rational equations.

Similar results are deduced for orbits of unipotent flows on other homogeneous spaces. We also conclude that if G is a connected semisimple Lie group and Γ is a lattice in G then there exists a compact subset D of G such that for any closed connected unipotent subgroup U, which is not contained in any proper closed subgroup of G, we have $G = D\Gamma U$. The decomposition is applied to get results on Diophantine approximation.

0. *Introduction*

Let (u_t) be a one-parameter subgroup in $\mathrm{SL}(n, \mathbb{R})$ consisting of unipotent matrices. In **[3]**, strengthening a result of G. A. Margulis **[10]**, it was proved that for any $x \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ there exists a compact set K such that the set $\{t \geq 0 \mid u_t x \in K\}$ has positive lower density. Recently while studying density of orbits of horospherical flows (cf. **[7]**) the author found that in certain contexts it is necessary to know whether the compact set K can be chosen so that the above assertion holds (simultaneously) for all x such that the (u_t)-orbit of x is not contained in a proper *closed* subset of the form $Hx\Gamma/\Gamma$, where $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and H is a closed subgroup.

On the other hand in **[6]** it was proved that if G is a simple Lie group of $\mathbb{R}$-rank 1 and Γ is a lattice in G then given $\varepsilon > 0$ and a unipotent one-parameter subgroup (u_t) there exists a compact subset K of G/Γ such that for any $x \in G/\Gamma$ whose (u_t)-orbit is unbounded the lower density of $\{t \geq 0 \mid u_t x \in K\}$ exceeds $(1-\varepsilon)$. This raises the question of whether a similar stronger assertion about the lower density is possible in the above-mentioned case (and more generally for all arithmetic lattices).

The answers to both the questions turn out to be in the affirmative; namely, we

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- "Dani correspondence" translates Diophantine properties into dynamics on G/Γ. Kleinbock-Margulis and many many others

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- Dani: the set of points on a torus whose forward orbit under a semisimple surjective endomorphism does not contain an element of finite order is winning

Nondivergence of unipotent flows

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- Shah, Eskin-Mozes-Shah, Minsky-Weiss, Lindenstrauss-Mirzakhani

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- "Elementary" proof