Emergent geometry from RG

Gautam Mandal

Tata Institute of Fundamental Research

ICTS Discussion Meeting, ICTS, Bangalore
September 26, 2012

(with Daniel Elander and Hiroshi Isono)

References

I. Heemskerk and J. Polchinski (1010.1264)
T. Faulkner, H. Liu and M. Rangamani (1010.4036)
Daniel Elander, GM, Hiroshi Isono (1109.3366,12xx.yyyy)
Sung-Sik Lee (0912.5223,1011.1474, 1204.1780)

Large N=Classical

- It has long been known that large N field theories are classical; e.g. correlators of (gauge)-invariant observables is described by a functional integral which has a classical limit, and at $N=\infty$, are described by a classical saddle point.
- E.g., in the Gross-Neveu model

$$S_\psi = \int d^2x[\bar{\psi}\,\partial\!\!\!/\psi(x) - g/(2N)(\bar{\psi}(x)\psi(x))^2]$$

$$= \int d^2x[\bar{\psi}\,\partial\!\!\!/\psi(x) + \sigma\bar{\psi}\psi(x) + N/(2g)\sigma^2]$$

$$= S_\sigma = N[Trln(\partial\!\!\!/ + \sigma) + \int d^2x\sigma^2/(2g)]$$

Correlators of the fermion bilinear are

$$\langle\bar{\psi}\psi(x).....\rangle = \sigma_{cl}(x)... + O(1/\sqrt{N})$$

where $\sigma_{cl}(x)$ is a solution of S_σ. At $N=\infty$, the theory S_σ is classical, but it is non-local.

large N= classical geometry

- Imagine computing correlators of $T_{\mu\nu}$:

$$Z[h_{\mu\nu}] = \int D[...] \exp[-S + \int h_{\mu\nu} T_{\mu\nu}]$$

- In a GN-like example, in a background metric $\bar{g}_{\mu\nu}$, schematically

$$Z[h] = \int D\sigma \exp[N(Tr \ln(\bar{\not{D}} + \sigma + h_{\mu\nu}\gamma^\mu \bar{D}^\nu) + \int \sigma^2/(2g))]$$

At large N, the σ-functional integral is evaluated at a saddle point: $\sigma = \sigma_{cl}[h] + O(1/\sqrt{N})$.

- The conservation law $\bar{D}^{\mu} T_{\mu\nu} = 0$ implies a gauge symmetry of $h_{\mu\nu}$, that of linearized general covariance:

$$\delta h_{\mu\nu} = \bar{D}_{\mu}\xi_{\nu} + \bar{D}_{\nu}\xi_{\mu}$$

- The theory $Z[h]$ possibly defines a generally covariant theory, with metric $g = \bar{g} + h$, except that the theory is non-local. An example of a typical non-local, covariant term in $\ln Z[h]$, for a 2D CFT, is

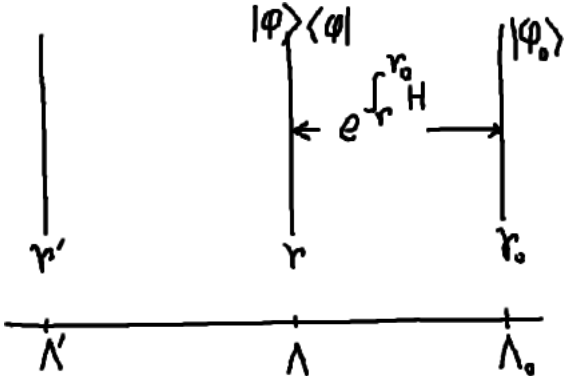
$$c \int R \frac{1}{D^2} R$$

Lesson from the example of AdS/CFT

- For certain large N theories, the classical geometry (=metric+ other fields) IS local, albeit in higher dimensions.
- radial coordinate $r \sim$ cut-off scale Λ of FT
- radial derivative $d\phi/dr \leftrightarrow \beta(g) = dg/d\Lambda$
- A very precise correspondence beteen the radial hamiltonian evolution of the path-integral, and Wilsonian RG evolution has been obtained. (see picture).
- E.g. for double trace flows $\int gO^2$, this technology gives

$$\dot{\delta g} = \pm 2\nu\delta g + \delta g^2$$

which gives the correct beta-function expected near the IR (upper sign) and UV (lower sign) fixed points of a Wilson-Fisher system.


$|\varphi\rangle\langle\varphi|$
$|\varphi_0\rangle$
$e^{\int_r^{r_0} H}$
r'
r
r_0
Λ'
Λ
Λ_0

Wilson locality = locality in r

- As Wilson showed us, the physics at any scale can be understood in terms of an effective hamiltonian at that scale. the new vertices that are generated in going from Λ to $\Lambda - d\Lambda$, have couplings $g + \delta g$, where $dg/d\Lambda$ is given by a function $\beta(g)$ where g is the coupling at THAT scale (Λ).
- There is a direct AdS/CFT map of the above statements: the bulk theory has a radial Hamiltonian evolution (in the ADM sense), and the theory must be local in r.

Holography from ANY large N field theory?

- **The question**: Can we reformulate a garden variety large N field theory, in terms of an equivalent higher dimensional theory, with at least locality in r?
- **Answer**: Yes.

- The spacetime theory is in one higher dimension.
- It is a 2-derivative theory in terms of radial derivatives.
- The higher dimensional theory has non-trivial metric; the radial component of the metric is related to field theory beta-functions.
- We do not require criticality or near-criticality for the proof of the above statements.
- The $d+1$-dimensional theory is generically non-local in the original d dimensions and the metric is not that of AdS.
- At a fixed point, the radial component of the metric coincides with that of AdS (with the conventional identification between radius and scale). The m^2-Δ relation also works out.
- (Work in Progress): Near a fixed point, the theory appears to be local in the d dimensions as well. The overall metric g_{MN} coincides with AdS.

Plan

- An example
- General large N field theory: emergence of higher dimensional spacetime
- Emergence of AdS near CFT

Example

Consider a matrix field theory in $d = 3$ (cut-off Λ)

$$Z(\Lambda) = \int D\Phi \exp[-S],$$

$$S = N \int d^d x [Tr \partial_\mu \Phi^2 + g_2 Tr \Phi^2 + g_{2,2}/(2N)(Tr \Phi^2)^2 + g_6 Tr \Phi^6]$$

where we have tuned Φ^4 to zero for simplicity.
We will get rid of the double trace coupling by introducing an additional, Gaussian auxiliary variable:

$$S = S_{ST} + S_{source} + S_{UV},$$

$$S_{ST} = N \int d^d x [Tr \partial_\mu \Phi^2 + g_2 Tr \Phi^2 + g_6 Tr \Phi^6]$$

$$S_{source} = N \int \sigma_0 (Tr \Phi^2), \; S_{UV} = -N \int \sigma_0^2/(2g_{2,2})$$

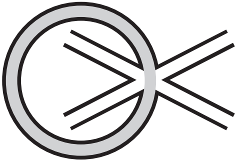
We now apply Wilsonian RG procedure to derive an effective theory at a lower cutoff $\Lambda' = \Lambda - d\Lambda$.

One-step RG computation

- We split $\Phi \to \{\Phi, \hat{\Phi}\}$ where the 'fast' variables $\hat{\Phi}$ have momenta in the shell $\{\Lambda', \Lambda\}$.
- We integrete out the $\hat{\Phi}$. Additional vertices are induced. After we do the usual rescaling of momenta, the original couplings become

$$g_2', g_6', g_{2,2}' = \delta g_{2,2} = -b_{2,2} g_6 \Lambda^{d-3} d\Lambda$$

- The last term arises because of the following diagram



New auxiliary variables

Like for the original bare couplings, we again get rid of the double trace couplings, by introducing a new auxiliary variable $\tilde{\sigma}_0$:

$$S = N \int d^d x [Tr \partial_\mu \Phi^2 + g_2' Tr \Phi^2 + g_6' Tr \Phi^6 - N \sigma_0^2 / (2 g_{2,2})]$$
$$+ \int d^d x [N^2 A_0 \frac{\tilde{\sigma}_0^2}{d\Lambda} + N(\sigma_0 + \tilde{\sigma}_0)(Tr \Phi^2)], \; A_0 \equiv \Lambda^{3-d} / (b_{2,2} g_6)$$

The blue term can be made to look more natural, by making a field redefinition

$$\{\sigma_0, \tilde{\sigma}_0\} \to \{\sigma_0, \sigma_1 \equiv \sigma_0 + \tilde{\sigma}_0\},$$

to get

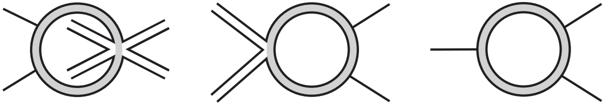
$$A_0 (\sigma_1 - \sigma_0)^2 / d\Lambda$$

As we continue with the RG steps, introducing the variables $\sigma_2, \sigma_3, ..$ for every new Wilson slice, we get

$$A_0 (\sigma_1 - \sigma_0)^2 / d\Lambda + A_1 (\sigma_2 - \sigma_1)^2 / d\Lambda + ... = \int d\Lambda \, A(\Lambda) \, (\partial_\Lambda \sigma)^2$$

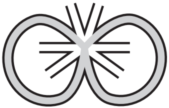
Other vertices

The structure of A is slightly more complicated, actually, because of new vertices which are generated,e.g.



$\sigma^n (Tr\,\Phi^2)^2$ $\sigma^n\, Tr\,\Phi^2$ σ^n

Triple (and higher) traces involve $(d\Lambda)^2$ (and higher powers); since we are working here with infinitesimal $d\Lambda$ in the one-step RG, we ignore these vertices:



Collecting all...

Collecting all the terms, and ignoring derivatives in the x^μ directions, we get

$$Z(\Lambda_0) = \int D\sigma(\Lambda) Z[\Lambda', \sigma(\Lambda')] \exp[-S_{bulk}(\sigma) - S_{UV}(\sigma(\Lambda_0))]$$

$$S_{bulk} = N^2 \int_{\Lambda'}^{\Lambda_0} d\Lambda d^d x [A(\Lambda, \sigma) (\partial_\Lambda \sigma)^2 + V(\sigma)]$$

$$Z[\Lambda', \sigma(\Lambda')] = D_{\Lambda'} \Phi \exp[-S_{ST}(\Phi, \Lambda') + \int d^d x \sigma(\Lambda') Tr \phi^2],$$

$$A \propto \Lambda^{3-d}$$

Collecting all...

Rewriting $\Lambda = r$, and making the couplings and fields dimensionless $\sigma(\Lambda) \to \phi(r)$,

$$Z(\Lambda_0) = \int D\phi(r) Z[r', \phi(r')] \exp[-S_{bulk}(\phi) - S_{UV}(\phi(r_0))]$$

$$S_{bulk} = N^2 \int_{r'}^{r_0} dr d^d x [A(r,\phi)\, (\partial_r \phi)^2 + V(\phi)]$$

$$Z[\Lambda' = r', \phi(r')] = \int D_{\Lambda'=r'}\Phi \exp[-S_{ST}(\Phi, r') + \int \phi(r') Tr \Phi^2],$$

$$A(r) = r^{d+1}\, \bar{f}(g_2(r), g_6(r), \phi(r)),$$

$$S_{ST}(\Phi, r') = \int d^d x [Tr(\partial_\mu \Phi)^2 + g_2(r')\Phi^2 + g_6(r')\Phi^6]$$

General construction

$$S_{ST}[\Phi_0,\lambda_0;\Lambda_0] = \int d^dx\; N\;\mathrm{Tr}\left[-\frac{1}{2}\partial_\mu\Phi_0\partial^\mu\Phi_0 + \sum_a \Lambda_0^{d-a[\Phi]}\lambda_{0a}\Phi_0^a\right]$$

$$S[\Phi_0,g_0;\Lambda_0] = S_{ST}[\Phi_0,g_0;\Lambda_0] + \int d^dx\left[\sum_a \Lambda_0^{d-a[\Phi]} g_a \mathrm{Tr}\Phi_0^a + \sum_{a,b}\Lambda_0^{d-a[\Phi]-b[\Phi]}\frac{g_{a,b}}{2}\mathrm{Tr}\Phi_0^a\mathrm{Tr}\Phi_0^b\right]$$

$$Z = \int D\sigma_0 D\Phi_0 \exp\left[-S_{ST}[\Phi_0,g_0;\Lambda_0] - \int d^dx\,\sigma_{0a}\mathrm{Tr}\Phi_0^a\right]\Psi_{\mathrm{UV}}(\sigma_0),$$

$$\Psi_{\mathrm{UV}}[\sigma_0] = \exp\left[\int d^dx\left(\Lambda_0^{d-[\sigma_a]-[\sigma_b]}\frac{g^{a,b}\sigma_{0a}\sigma_{0b}}{2} - \Lambda_0^{d-[\sigma_b]}g_a g^{a,b}\sigma_{0b} + \Lambda_0^d\frac{g_a g^{a,b} g_b}{2}\right)\right]$$

$$Z(\Lambda_0) = \int D\phi_a(r) Z[r', \phi(r')] \exp[-S_{bulk}(\phi_a) - S_{UV}(\phi_a(r_0))]$$

$$S_{bulk} = N^2 \int_{r'}^{r_0} dr d^d x \; [A^{ab}(r, g_a(r), \phi_a(r)) \, \partial_r \phi_a \partial_r \phi_b + V(\phi_a)]$$

$$Z[r', \phi(r')] = \int D_{\Lambda'=r'} \Phi \exp[-S_{ST}(\Phi, r') + \int \phi_a(r') Tr \Phi^a],$$

$$A^{ab}(r) = r^{d+1} \bar{f}^{ab}(g_a(r), \phi_a(r)), \; S_{ST} = \int_x [Tr(\partial_\mu \Phi)^2 + g_a(r') Tr \Phi^a]$$

where $\bar{f}^{ab}$ is dimensionless. All x-derivatives are ignored at the moment; we will discuss them later.

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d + 1$ dimensions.

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d + 1$ dimensions. $\checkmark$

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d+1$ dimensions. $\checkmark$
- It is a 2-derivative theory in terms of radial derivatives.

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d + 1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d + 1$ dimensions. $\checkmark$
- It is a 2-derivative theory in terms of radial derivatives. $\checkmark$
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d + 1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d + 1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d + 1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d+1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d+1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓
- We do not require criticality or near-criticality for the proof of the above statements.

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d+1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d+1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓
- We do not require criticality or near-criticality for the proof of the above statements.✓

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d+1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d+1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓
- We do not require criticality or near-criticality for the proof of the above statements.✓
- The $d+1$-dimensional theory is generically non-local in the original d dimensions and the metric is not that of AdS.

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d+1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d+1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓
- We do not require criticality or near-criticality for the proof of the above statements.✓
- The $d+1$-dimensional theory is generically non-local in the original d dimensions and the metric is not that of AdS. ✓

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d + 1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d + 1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓
- We do not require criticality or near-criticality for the proof of the above statements.✓
- The $d + 1$-dimensional theory is generically non-local in the original d dimensions and the metric is not that of AdS. ✓
- In the neighbourhood of a fixed point, the g_{rr} component of the metric coincides with that of AdS (with the conventional identification between radius and scale).

Back to Claims

- A spacetime theory has emerged, whose radial slice $r \in \{r', r_0\}$ captures all the quantum fluctuations in the field theory between the momentum scales $\Lambda \in \{\Lambda', \Lambda_0\}$. It is in $d+1$ dimensions. ✓
- It is a 2-derivative theory in terms of radial derivatives. ✓
- The spacetime theory has a non-trivial metric; it has the general form of a nonlinear sigma-model in a $d+1$ dimensional spacetime, with target space metric $\bar{f}^{ab}$. The radial component of the metric is related to field theory beta-functions.✓
- We do not require criticality or near-criticality for the proof of the above statements.✓
- The $d+1$-dimensional theory is generically non-local in the original d dimensions and the metric is not that of AdS. ✓
- In the neighbourhood of a fixed point, the g_{rr} component of the metric coincides with that of AdS (with the conventional identification between radius and scale). $\rightarrow$ ...

Conformality and AdS

- Recall the emergent bulk action

$$S_{bulk} = N^2 \int_{r'}^{r_0} dr d^d x \, [r^{d+1} \, \bar{f}^{ab}(r, g_a(r), \phi_a(r)) \, \partial_r \phi_a \partial_r \phi_b + V(\phi_a)]$$

- As it stands, the term with the radial derivatives defines a non-linear sigma model, i.e. a map from a $d+1$ dimensional spacetime to a target space with metric $\bar{f}^{ab}$.

Conformality and AdS

- Recall the emergent bulk action

$$S_{bulk} = N^2 \int_{r'}^{r_0} drd^d x \, [r^{d+1} \, \bar{f}^{ab}(r, g_a(r), \phi_a(r)) \, \partial_r \phi_a \partial_r \phi_b + V(\phi_a)]$$

- As it stands, the term with the radial derivatives defines a non-linear sigma model, i.e. a map from a $d+1$ dimensional spacetime to a target space with metric $\bar{f}^{ab}$.
- However, note the N^2 in front; this means that at large N, the NLSM has a classical saddle point: $\phi_a(r,x) = \overline{\phi}_a(r,x) + \frac{1}{N}\varphi_a(r,x)$
- E.g. if the ϕ_a represents gravitons, the choice of the saddle point represents the choice of a background spacetime: $h_{\mu\nu}(r,x) = \overline{h}_{\mu\nu}(r,x) + \sqrt{\kappa}\delta h_{\mu\nu}(r,x)$

Conformality and AdS

- At large N, we get

$$S_{bulk} = N^2 S_{cl}[\overline{\phi}_a] + \int_{r'}^{r_0} dr d^d x \, [r^{d+1} \, \bar{f}^{ab}(r, g_a(r), \overline{\phi}_a(r,x)) \, \partial_r \varphi_a \partial_r \varphi_b$$
$$+ r^{d-1} m^{ab}(r, g_a(r), \overline{\phi}_a(r,x)) \varphi_a \varphi_b + O(1/N)]$$

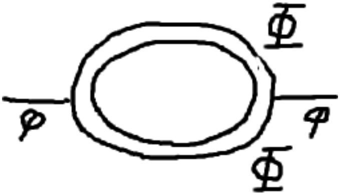
- As we approach conformal point, we have the fixed point values $g_a(r) \to g_a^*$. Also we must have $\overline{\phi}_a(r) \to \overline{\phi}_a$ (spacetime-independent); so that $S_{source} = N \int d^d x \overline{\phi}_a(x) Tr \Phi^a$ is scale-invariant.
- With this, we have

$$S_{bulk} = \int_{r'}^{r_0} dr d^d x, r^{d+1} \, \partial_r \varphi_a \partial_r \varphi_a + r^{d-1} m^{ab} \varphi_a \varphi_b O(1/N)]$$

  where we have set the constant $\bar{f}^{ab} = \delta^{ab}$ by appropriate diagonalization and rescaling.
- Note that for AdS, $\sqrt{g} = r^{d-1}$, $\sqrt{g} g^{rr} = r^{d+1}$. Hence, at the conformal point, the AdS structure emerges. [Note that we have ignored x-derivatives so far.]

x-derivatives

- In our little example considered before, let us revisit the diagram contributing to the φ^2 term



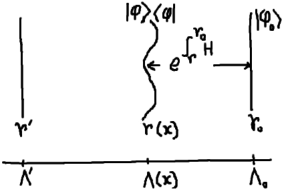
- The loop integral $\int_{k=0}^{\Lambda}$ schematically evaluates to $\Lambda^2 f(k^2/L^2)$, $f(k^2/L^2) = 1 + k^2/L^2 + k^4/\Lambda^4 + \ldots$
- In the interval $\int_{k=\Lambda-d\Lambda}^{\Lambda}$, we should get $\Lambda d\Lambda \tilde{f}(k^2/L^2)$, $\tilde{f}(k^2/L^2) = 1 + k^2/L^2 + k^4/\Lambda^4 + \ldots$
- Thus, we get

$$r^{d-1}[r^2(\partial_r\varphi_a)^2 + m^{ab}\varphi_a\varphi_b + k^2/r^2\varphi(k)\varphi(-k)]$$

- There is a conceptual issue regarding Wilson-Polchinski RG (smooth vs hard cut-off).

Conclusions and open issues

- Need to complete the x-derivatives.
- Vasiliev
- New geometries?
- Does the higher dimensional viewpoint help technically?
- General covariance.



- Compare with other approaches: Entanglement-DMRG, Matrix models, ...