On a Hecke Algebra isomorphism of Kazhdan

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Outline of the talk

- Kazhdan isomorphism over close local fields
- Local class field theory and the local Langlands correspondence
- Applications of Kazhdan's theory to the local Langlands correspondence
- Generalizing Kazhdan's theory to non-split groups

Notation

F: a non-archimedean local field

$\mathfrak{O}_F$: ring of integers

$\mathfrak{p}_F$: its maximal ideal

$\mathfrak{f} = \mathfrak{O}_F/\mathfrak{p}_F$ denote the residue field of F.

A non-archimedean local field is:

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A non-archimedean local field is:

is a finite extension of $\mathbb{Q}_p$ (these are the non-archimedean local fields of characteristic 0) or

is isomorphic to $\mathbb{F}_q((t))$, (where $q = p^n$), the field of formal Laurent series in the indeterminate t. (these are the non-archimedean local fields of characteristic p).

Smooth representations of $G(F)$

Let G be a connected, reductive group over F.

A *smooth* representation of $G(F)$ is a pair (σ, V) where

- V is a $\mathbb{C}$-vector space.
- $\sigma : G(F) \to \mathrm{GL}(V)$ such that for each $v \in V$, there is a compact open subgroup K of $G(F)$ such that $\sigma(k) \cdot v = v$ for all $k \in K$.

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It is called *admissible* if V^K is finite dimensional for each compact open subgroup K of $G(F)$.

The Hecke algebra $\mathcal{H}(G(F), K_m)$

Let G be split, connected reductive group defined over $\mathbb{Z}$ (Some examples are $G = \mathrm{GL}_n, \mathrm{GSp}_4, \mathrm{SO}_{2n+1}$).
Let $K_m = \mathrm{Ker}(G(\mathfrak{O}_F) \to G(\mathfrak{O}_F/\mathfrak{p}_F^m))$ be the m-th usual congruence subgroup of G.

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The Hecke algebra $\mathcal{H}(G(F), K_m)$ is the $\mathbb{C}$-span of $\{\mathrm{vol}(K_m; dg)^{-1}\,\mathrm{char}(K_m x K_m) \mid x \in G\}$. This is an algebra with product given by convolution.

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Given an irreducible representation (σ, V) of $G(F)$, the space V^{K_m} is a simple $\mathcal{H}(G, K_m)$-module. More generally, the functor $V \to V^{K_m}$ is an eqvivalence of catagories between the category of representations of $G(F)$ that are generated by their K_m-fixed vectors and the category of modules over $\mathcal{H}(G, K_m)$.

Close local fields

Definition

Let $m \geq 1$. Two non-archimedean local fields F and F' are m-*close* if the quotient rings $\mathfrak{O}_F/\mathfrak{p}_F^m$ and $\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ are isomorphic.

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Example

The fields $\mathbb{F}_p((t))$ and $\mathbb{Q}_p(p^{1/m})$ are m-close.

In fact,

$$\mathbb{Z}_p[p^{1/m}]/(p) \cong \mathbb{Z}_p[X]/(X^m - p, p) \cong \mathbb{F}_p[X]/(X^m) \cong \mathbb{F}_p[[X]]/(X^m).$$

Note

Given a local field F' of characteristic p and an integer $m \geq 1$, there is a non-archimedean local field F of characteristic 0 such that F' is m-close to F.

Kazhdan isomorphism

G - split, connected reductive group defined over $\mathbb{Z}$.

Let $K_m = \mathrm{Ker}(G(\mathfrak{O}_F) \to G(\mathfrak{O}_F/\mathfrak{p}_F^m))$ be the m-th usual congruence subgroup of G. Consider the Hecke algebra $\mathcal{H}(G(F), K_m)$. Note that

$$G(\mathfrak{O}_F)/K_m \cong G(\mathfrak{O}_{F'})/K'_m$$

when the fields F and F' are m-close.

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Theorem (Kazhdan)

Let F be a non-archimedean local field and let $m \geq 1$. There exists $l \geq m$ such that for any local field F' that is l-close to F, the Hecke algebras $\mathcal{H}(G(F), K_m)$ and $\mathcal{H}(G(F'), K'_m)$ are isomorphic.

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Kazhdan isomorphism

An irreducible representation (σ, V) of $G(F)$ such that $V^{K_m} \neq 0$ is a simple $\mathcal{H}(G(F), K_m)$-module. If the fields F and F' are sufficiently close, the Kazhdan isomorphism gives a bijection between

$$\{\text{Irreducible representations } (\sigma, V) \text{ of } G(F) \text{ such that } V^{K_m} \neq 0\}$$

$$\xleftrightarrow{\mathrm{Kaz}_m}$$

$$\{\text{Irreducible representations } (\sigma', V') \text{ of } G(F') \text{ such that } V'^{K'_m} \neq 0\}.$$

So the Kazhdan isomorphism enables us to compare representations of p-adic groups over close local fields.

This isomorphism has come applications in the study of the local Langlands conjectures, which we now recall.

The Galois group and the Weil group

Note that we have an exact sequence

$$1 \longrightarrow \mathrm{Gal}(\bar{F}/F^{\mathsf{un}}) \longrightarrow \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{Gal}(F^{\mathsf{un}}/F) \longrightarrow 1$$

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W_F is called the Weil group of F.

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W_F is called the Weil group of F. The group $\mathrm{Gal}(\bar{F}/F^{\mathsf{un}})$ is denoted by I_F and is called the inertia group of F.

Local class field theory

Local class field theory is a study of finite abelian extensions of a local field.

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The main theorem gives a topological isomorphism

$$\phi_F : F^\times \to W_F^{\mathsf{ab}}$$

that induces an isomorphism

$$\hat{F}^\times \xrightarrow{\cong} \mathrm{Gal}(\bar{F}/F)^{\mathsf{ab}}.$$

Here $\hat{F}^\times \cong \mathfrak{O}_F^\times \times \hat{\mathbb{Z}}$ is the profinite completion of $F^\times$ (Note that $F^\times \cong \mathfrak{O}_F^\times \times \mathbb{Z}$).

Local class field theory

The inertia group I_F admits a nice descending filtration of ramification subgroups with *upper numbering* $\{I_F^m\}$ and the isomorphism

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in fact maps

$$\mathfrak{O}_F^\times \to I_F^{\mathsf{ab}} \text{ and } 1 + \mathfrak{p}_F^m \xrightarrow{\phi_F} (I_F^m)^{\mathsf{ab}}.$$

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Hence local class field theory gives an isomorphism

$$\mathrm{Hom}(F^\times, \mathbb{C}^\times) \cong \mathrm{Hom}(W_F^{\mathsf{ab}}, \mathbb{C}^\times) \cong \mathrm{Hom}(W_F, \mathbb{C}^\times).$$

The local Langlands correspondence

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Let $\mathbf{G} = \mathrm{GL}_n$. To describe the correspondence, we replace

$$\mathrm{Hom}(F^\times, \mathbb{C}^\times) \rightsquigarrow \{ \text{ Irreducible smooth representations of } \mathrm{GL}_n(F)\}$$

and

$$\mathrm{Hom}(W_F, \mathbb{C}^\times) \rightsquigarrow \{ \text{ semi-simple } n\text{-dim. representations of } \mathrm{WD}_F\}.$$

Here $\mathrm{WD}_F := W_F \times \mathrm{SL}_2(\mathbb{C})$ is the Weil-Deligne group of F.

Semi-simple representations of W_F

On the other side of the Langlands correspondence, we have to take n-dimensional semisimple representations of WD_F.

First, an n-dimensional irreducible representation of W_F is a pair (ϕ, V) where

- $\dim_{\mathbb{C}}(V) = n$,
- $\phi : W_F \to \mathrm{GL}(V)$ is such that every vector $v \in V$ has open stabilizer in W_F.
- It has no non-zero proper W_F-invariant subspaces.

It is *semisimple* if it is a sum of its irreducible subspaces.

Representations of WD_F?

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Next, why should be consider representations of $\mathrm{WD}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$? It turns out that the n-dimensional semisimple representations of W_F are not enough to account for all irreducible smooth representations of $\mathrm{GL}_n(F)$ under the Langlands correspondence.
One can use parabolic induction to obtain a representation for $\mathrm{GL}_n(F)$ using irreducible representations of $\mathrm{GL}_{n_1}(F) \times \cdots \mathrm{GL}_{n_k}(F)$, $n_1 + n_2 + \cdots n_k = n$, and such a representation is, in general, not irreducible. One needs semisimple n-dimensional representations of WD_F to account for the irreducible summands of such representations under the LLC.

The local Langlands correspondence for GL_n

The local Langlands correspondence for GL_n can be described a follows: There is a bijection between

$$\{\text{ Irreducible smooth representations of } \mathrm{GL}_n(F)\}$$
$$\overset{\text{LLC}}{\longleftrightarrow} \{\text{ semi-simple } n\text{-dim. representations of } \mathrm{WD}_F\}.$$

$$\sigma \to \phi_\sigma \tag{0.1}$$

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Given representations (σ, V) of $\mathrm{GL}_n(F)$ and (τ, W) of $\mathrm{GL}_t(F)$, and a non-trivial additive character $\psi : F \to \mathbb{C}^\times$, Jacquet, Piatetski-Shapiro annd Shalika gave a theory of L- and ϵ-factors

$$L(s, \sigma \times \tau)$$
$$\epsilon(s, \sigma \times \tau, \psi)$$

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On the Artin side, for parameters ϕ_σ and ϕ_τ of WD_F, Artin, Deligne, and Langlands defined

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The LLC has the property that for each (τ, W) of $\mathrm{GL}_t(F),\ 1 \leq t \leq n-1$

$$L(s, \sigma \times \tau) = L(s, \phi_\sigma \otimes \phi_\tau)$$
$$\epsilon(s, \sigma \times \tau, \psi) = \epsilon(s, \phi_\sigma \otimes \phi_\tau, \psi)$$

and furthermore, there is a unique map (0.1) that satisfies this property.

Proofs

There is a unique bijection between $\sigma \to \phi_\sigma$,

$$\{ \text{ Irreducible smooth representations of } \mathrm{GL}_n(F)\}$$

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such that

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- Proof over local function fields was done in 1993 (Laumon-Rapoport-Stuhler).
- Proof in characteristic 0 was completed in 2000 (Harris-Taylor, Henniart), and recently Scholze (2013).

Beyond GL_n

For other split reductive groups $G(F)$ (like $\mathrm{GSp}_4(F), \mathrm{SO}_{2n+1}(F)$), the local Langlands correspondence will no longer be a bijection and will only be a surjective finite-to-one map:

$\{$Irr. smooth reps. of $G(F)\} \twoheadrightarrow \{$Homomorphisms $\phi : \mathrm{WD}_F \to^L G\}$

where ${}^L G$ is the "Langlands dual group" of G (the complex group associated to the dual root datum).

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The LLC has been established for

- For GSp_4 (Gan-Takeda in char 0, (-) in characteristic $p > 2$)
- For classical groups (Arthur in char 0, Ganapathy - Varma in sufficiently large characteristic))

Deligne's theory

Kazhdan's theory enables us to compare representations of p-adic groups over close local fields. A similar story on the Galois side is due to Deligne, which we now review.

Deligne's theory

For an object X associated to the field F, we use the notation X' to denote the corresponding object associated to F'. Let

- $\bar{F}$ - a separable closure of F.
- I_F - the inertia group.
- I_F^m - the m-th higher ramification subgroup with upper numbering.

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- $\bar{F}$ - a separable closure of F.
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Theorem (Deligne)

If F and F' are m-close, then

$$\mathrm{Gal}(\bar{F}/F)/I_F^m \xrightarrow{\mathrm{Del}_m} \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m,$$

is an isomorphism and is unique upto inner automorphisms.

Properties of Del_m: Local class field theory

The Deligne isomorphism is compatible with local class field theory.

Deligne proved that if the fields F and F' are m-close, then the following diagram is commutative:

$$\begin{array}{ccc} (\mathrm{Gal}\,\bar{F}/F)/I_F^m)^{ab} & \xrightarrow{\mathrm{Del}_m} & (\mathrm{Gal}(\bar{F}'/F')/I_{F'}^m)^{ab} \\ \text{LCFT}\downarrow & & \downarrow\text{LCFT} \\ (F^\times/(1+\mathfrak{p}_F^m))\widehat{} & \xrightarrow[\mathrm{cl}_m]{} & (F'^\times/(1+\mathfrak{p}_{F'}^m))\widehat{} \end{array}$$

In the above, we have used that if F and F' are m-close, then

$$F^\times/1+\mathfrak{p}_F^m \cong F'^\times/1+\mathfrak{p}_{F'}^m.$$

Properties of Del_m: Representations of the Galois group

Now let $\phi : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(V)$ be an irreducible n-dimensional representation such that $\phi|_{I_F^m} = 1$. Then ϕ factors through $\mathrm{Gal}(\bar{F}/F)/I_F^m$. If the fields F and F' are m-close, then

$$\mathrm{Gal}(\bar{F}/F)/I_F^m \overset{\mathrm{Del}_m}{\cong} \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m.$$

Hence

$$\phi' = \phi \circ \mathrm{Del}_m^{-1} : \mathrm{Gal}(\bar{F}'/F')/I_{F'}^m \to \mathrm{GL}(V).$$

The isomorphism Del_m induces a bijection

$\{$Isomorphism classes of representations of $\mathrm{Gal}(\bar{F}/F)$ trivial on $I_F^m\}$

$\leftrightarrow$

$\{$Isomorphism classes of representations of $\mathrm{Gal}(\bar{F}'/F')$ trivial on $I_{F'}^m\}$.

Summary

- Deligne's result enables us to compare representations of Galois groups over close local fields.
- Kazdhan's result and its variant enables us to compare representations of p-adic groups over close local fields.
- Given F' in characteristic p and $m \geq 1$, there exists a local field of characteristic 0 that is m-close to F.

Summary

- Deligne's result enables us to compare representations of Galois groups over close local fields.
- Kazdhan's result and its variant enables us to compare representations of p-adic groups over close local fields.
- Given F' in characteristic p and $m \geq 1$, there exists a local field of characteristic 0 that is m-close to F.

Question: Is the Deligne-Kazhdan philosophy compatible with the local Langlands correspondence?

The Deligne-Kazhdan theory and Local Langlands Correspondence

Question: Assume that F and F' are two sufficiently close local fields, and consider the following diagram:

$$\begin{array}{ccc} \{(\sigma, V) \text{ of } G \mid \text{depth}(\sigma) < m\} & \xrightarrow{\text{LLC}} & \{\phi : \mathrm{WD}_F \to {}^L G \mid \text{depth}(\phi) < m\} \\ \text{Kazhdan} \downarrow & & \downarrow \text{Deligne} \\ \{(\sigma', V') \text{ of } G' \mid \text{depth}(\sigma') < m\} & \xrightarrow{\text{LLC}} & \{\phi' : \mathrm{WD}_{F'} \to {}^L G \mid \text{depth}(\phi') < m\} \end{array}$$

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\{(\sigma', V') \text{ of } G' \mid \text{depth}(\sigma') < m\} & \xrightarrow{\text{LLC}} & \{\phi' : \mathrm{WD}_{F'} \to {}^L G \mid \text{depth}(\phi') < m\}
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$$\mathrm{depth}(\sigma) < m \implies \sigma^{K_{m+1}} \neq 0.$$

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$\mathrm{depth}(\sigma) < m \implies \sigma^{K_{m+1}} \neq 0.$

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$\mathrm{depth}(\sigma) < m \implies \sigma^{K_{m+1}} \neq 0.$

$\mathrm{depth}(\phi) < m \implies \phi|_{I_F^m} = 1.$

Is this diagram commutative? For GL_n? For GSp_4? For classical groups like $\mathrm{SO}_{2n+1}(F), \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n}(F)$?

The Deligne-Kazhdan theory and Local Langlands Correspondence

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\end{array}$$

For GL_n: (G) 2012, ABPS (2013)

For GSp_4: (G) 2013

For split classical groups: Joint with Sandeep Varma (2015).

The Kazhdan isomorphism

Recall that G is a split, connected, reductive group over $\mathbb{Z}$ and K_m is the m-th filtration subgroup of the $G(\mathfrak{O}_F)$. If F and F' are sufficiently close, then $\mathcal{H}(G, K_m) \cong \mathcal{H}(G', K'_m)$. Some key ingredients in the proof of this isomorphism:

(1) The Hecke algebra $\mathcal{H}(G(F), K_m)$ is finitely presented.

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(1) The Hecke algebra $\mathcal{H}(G(F), K_m)$ is finitely presented.

(2) The group $G(F)$ admits a *Cartan decomposition*, that is

$$G(\mathfrak{O}_F)\backslash G(F)/G(\mathfrak{O}_F) = W(G,T)\backslash X_*(T)$$

where T is a maximal $\mathbb{Z}$-split torus in G, $X_*(T)$ its cocharacter lattice and $W(G,T)$ the Weyl group of T in G.

The Kazhdan isomorphism

Recall that G is a split, connected, reductive group over $\mathbb{Z}$ and K_m is the m-th filtration subgroup of the $G(\mathfrak{O}_F)$. If F and F' are sufficiently close, then $\mathcal{H}(G, K_m) \cong \mathcal{H}(G', K'_m)$. Some key ingredients in the proof of this isomorphism:

(1) The Hecke algebra $\mathcal{H}(G(F), K_m)$ is finitely presented.

(2) The group $G(F)$ admits a *Cartan decomposition*, that is

$$G(\mathfrak{O}_F)\backslash G(F)/G(\mathfrak{O}_F) = W(G,T)\backslash X_*(T)$$

where T is a maximal $\mathbb{Z}$-split torus in G, $X_*(T)$ its cocharacter lattice and $W(G,T)$ the Weyl group of T in G.

(3) We have obvious isomorphisms

$$G(\mathfrak{O}_F)/K_m \cong G(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong G(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m) \cong G(\mathfrak{O}_{F'})/K'_m.$$

if the fields F and F' are m-close.

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2. Assuming this can be done, we need to understand what the analogues of $K = G_0(\mathfrak{O}_F)$ and K_m are.
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4. Generalize Kazhdan's proof of the Hecke algebra isomorphism.

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Answer: Let (R, Δ) be a based root datum and let $(G_0, T_0, B_0, \{u_\alpha\}_{\alpha \in \Delta})$ be a pinned, split, connected, reductive $\mathbb{Z}$-group with based root datum (R, Δ). Let $E_{qs}(F, G_0)_m$ be the set of F-isomorphism classes of quasi-split groups G that split (and become isomorphic to G_0) over an atmost m-ramified extension K of F (i.e. $I(K/F)^m = 1$).

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- $E_{qs}(F, G_0)_m$ is parametrized by $H^1(\Gamma_F / I_F^m, Aut(R, \Delta))$.
- There is a bijection $E_{qs}(F, G_0)_m \to E_{qs}(F', G_0)_m, \; G \to G'$, provided F and F' are m-close.

Generalizing Kazhdan isomorphism to quasi-split groups

On the three crucial ingredients that go into the proof of the Kazhdan isomorphism for split reductive groups.

(1) The Hecke algebra $\mathcal{H}(G(F), K_m)$ is finitely presented.

- is true for any pair (G, K) where G is a connected reductive group over F and K is a compact open subgroup of $G(F)$.

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$$G(\mathfrak{O}_F)\backslash G(F)/G(\mathfrak{O}_F) = W(G,T)\backslash X_*(T) \tag{0.2}$$

where T is a maximal $\mathbb{Z}$-split torus in G, $X_*(T)$ its cocharacter lattice and $W(G,T)$ the Weyl group of T in G.

- For a pair (G, K) where G is a connected reductive group over F and K a special maximal parahoric subgroup of $G(F)$, the Cartan decomposition has been established in the work of Haines-Rostami.

Generalizing Kazhdan isomorphism to quasi-split groups

(3) If the fields F and F' are m-close,

$$G(\mathfrak{O}_F)/K_m \cong G(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong G(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m) \cong G(\mathfrak{O}_{F'})/K'_m$$

- We note that (3) is not obvious when G is not necessarily split. It has been established for a pair (G, P_m) where G is a connected reductive group over F and P_m is the m-th Moy-Prasad filtration subgroup of a parahoric subgroup P of $G(F)$ (- , 2019).

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With these ingredients in place for general G, we follow the strategy of Kazhdan to prove that if the fields F and F' are sufficiently close, then

$$\mathcal{H}(G(F), K_m) \cong \mathcal{H}(G'(F'), K'_m)$$

where G is a connected reductive group over F, and $K_m = \mathrm{Ker}(\mathcal{K}(\mathfrak{O}_F) \to \mathcal{K}(\mathfrak{O}_F/\mathfrak{p}_F^m))$ where K is as in (2) and $\mathcal{K}$ is the underlying smooth affine $\mathfrak{O}_F$-group scheme constructed by Bruhat-Tits.

Analogues of $K = \mathbf{G}_0(\mathfrak{O}_F)$, K_m, I and I_m

Let $\mathcal{B}(G, F)$ denote the Bruhat-Tits building of G over F. This is a simplicial complex with an action of $G(F)$ on it.

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Let $\mathcal{B}(G, F)$ denote the Bruhat-Tits building of G over F. This is a simplicial complex with an action of $G(F)$ on it.

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- For example, when G is split, semisimple and simply connected, $G(\mathfrak{O}_F)$ is the stabilizer of a certain nicely chosen vertex in the building, and the Iwahori subgroup I is the stabilizer of an alcove (facet of maximal possible dimension) in the building.

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- In general, the subgroups of interest are finite index subgroups of stabilizers of facets in the building of $G(F)$. These are called *parahoric subgroups*.

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- In general, the subgroups of interest are finite index subgroups of stabilizers of facets in the building of $G(F)$. These are called *parahoric subgroups*. With $\mathcal{F}$ denoting a facet in the building, $P_{\mathcal{F}}$ denotes the corresponding parahoric subgroup.

$K/K_m \cong K'/K'_m$

Given a facet $\mathcal{F}$ in the building, Bruhat-Tits have constructed a smooth, affine, $\mathfrak{O}_F$-group scheme $\mathcal{P}_{\mathcal{F}}$ with generic fiber G and whose $\mathfrak{O}_F$-points $\mathcal{P}_{\mathcal{F}}(\mathfrak{O}_F) = P_{\mathcal{F}}$.

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To prove these isomorphisms, we

- Understand how to compare certain facets of the building $\mathcal{B}(G, F)$ and certain facets of the building $\mathcal{B}(G', F')$.

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- Understand how to compare certain facets of the building $\mathcal{B}(G, F)$ and certain facets of the building $\mathcal{B}(G', F')$. Let us denote this as $\mathcal{F} \to \mathcal{F}'$.
- Study the reduction $\mathcal{P}_\mathcal{F} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ and prove that

$$\mathcal{P}_\mathcal{F} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \cong \mathcal{P}_{\mathcal{F}'} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$$

  provided the fields F and F' are sufficiently close.

Thank you for your attention!