Representations of finite groups over arbitrary fields

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Group Algebras, Representations and Computation

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Outline of the Talk

Notations

- $G =$ a finite group.
- $F =$ a field.
- $F[G] =$ group algebra of G over F.
- $\overline{F} =$ the algebraic closure of F.
- $V =$ a finite dimensional vector space over F.
- $\mathrm{End}\, V =$ the ring of endomorphisms of V
- $\mathrm{Aut}\, V = \mathrm{GL}(V)$- the group of linear automorphisms of V.
- For an irreducible F-representation $\rho : G \mapsto GL(V)$, its corresponding primitive central idempotent **(pci)** in $F[G]$, we denote it by e_ρ.

Ferdinand Georg Frobenius (1849 – 1917)

The **Frobenius theory** of representations of finite groups over algebraically closed fields, is standard. Frobenius, when he started the whole theory in 1896, started with characters, and **not** representations.

Issai Schur (1875 – 1941)

Schur's 1902-work handles the case $F = \mathbb{C}$, or an algebraically closed field of characteristic 0. Schur clearly understood that to extend this work in the case of general F, which is much deeper and involved **arithmetic** aspects.

- In 1906, he wrote a paper “Aritlhmetische Untersuchungen über endliche Gruppen linearer Substitutionen" and obeserved most of the salient features.
- However he did not have the advantage of the **Wedderburn Theory** which came around 1914. To realize its relevance for the representation theory took some more years.

Richard Brauer (1901 – 1977)

In 1952, at the International Congress of Mathematics, Brauer gave an account of this development, in which he also removed the **"semisimplicity"** restriction and allowed **"char(F) | |G|"**. This is the **"modular representation theory"**, which is one of the Brauer's major contributions to mathematics.

(1) Let F be a field of characteristic 0 or prime to $|G|$. When F is not algebraically closed, there arise arithmetic aspects depending on how the cyclotomic polynomials split over F. We first review, and reformulate, the works of **Schur, Witt**, and **Berman**, in this case.

(2) Let G contain a normal subgroup H of index p, a prime. Let F be a field of characteristic 0 or prime to $|G|$. In 1955, Berman computed that, in case F is an algebraically closed, a pci of $F[G]$ corresponding to an irreducible F-representation of G, in terms of pci's of $F[H]$. We extend this result when **F is not necessarily algebraically closed.**

Schur Index

- Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible F-representation of G.
- The representation $\rho : G \to \mathrm{GL}(V)$ canonically extends to an F-algebra homomorphism $\rho : F[G] \to \mathrm{End}\, V$.
- Let $D = \{A \in \mathrm{End}(V) \mid A$ commutes with $\rho(g)$ for all $g \in G\}$. By Schur's lemma, D is a finite-dimensional F-algebra which is a **division ring**. We call D, the centralizer of G in $\mathrm{End(V)}$.

- $V \otimes_F \overline{F}$ is a $\overline{F}$-representation of G.
- Schur asserted that $V \otimes_F \overline{F}$ decomposes into distinct irreducible $\overline{F}$-represnetations $\rho_1, \rho_2, \ldots, \rho_k$ which occur with the same multiplicity, say, m.
- If $\rho_i : G \to \mathrm{GL}_{\overline{F}} W_i$, then $V \otimes_F \overline{F} \approx m(W_1 \oplus W_2 \oplus \cdots \oplus W_k)$. Its isotypic components mW_i are canonically defined as submodules of $V \otimes_F \overline{F}$. The number m was later called the **Schur index** of ρ.

Main new ingredients from the theory of semisimple F-algebras are the following.

- Let D be a finite dimensional F-algebra which is a division ring. Let Z be its centre. Then Z is a finite dimensional field extension of F, and D has a structure of a Z-algebra.
- Then $[D : Z] = m^2$, for some natural number m. D contains maximal subfields E such that $[E : Z] = m$.
- $D \otimes_Z E$ is isomorphic to $M_m(E)$, which is a **split** simple E-algebra. The **Schur index of D (or more generally, $M_n(D)$, for any n)** over F may be defined to be m.

- The minimal 2-sided ideal of $F[G]$ corresponding to ρ be abstractly isomorphic to $M_n(D)$, for suitable n.
- Let Z be the centre of $D, [Z : F] = k$, and E a maximal subfield of D. Then E contains Z.
- $Z[G] = F[G] \otimes_F Z$ is a sum of certain number of minimal 2-sided ideals, which correspond to irreducible Z-representations of G.
- $M_n(D) \otimes_F Z$ is isomorphic to a summand of $Z[G]$. Since Z is the center of D, we have

$$M_n(D) \otimes_F Z \approx M_n(D \otimes_F Z) \approx M_n(D + \cdots + D) \approx M_n(D) + \cdots + M_n(D),$$

where in the last two terms there are k summands. Each of these summands is isomorphic to a minimal 2-sided ideal of the group algebra $Z[G]$.

- In terms of representations, $V \otimes_F Z$ splits into k many simple $Z[G]$-summands, $V_1 \oplus V_2 \oplus \cdots \oplus V_k$. Each V_i is a Z-vector space, which by restriction of scalars may be regarded as an F-vector space.
- $\dim_F V = (\dim_F Z)(\dim_Z V) = k(\dim_Z D)(\dim_D V) = km^2n = k\dim_Z V_i$ for each $i = 1, 2, \ldots, k$.
- So, $\rho \otimes_F Z$ splits into k distinct irreducible Z-representations of G. For more precision, we shall write $M_n(D)_i$ for the i-th summand in $\approx M_n(D) + M_n(D) + \cdots + M_n(D)$, which itself occurs as a summand in $Z[G]$. We have $M_n(D)_i \approx M_n(D)$, for each $i = 1, 2, \ldots, k$, and the corresponding representation-space is V_i.

- Consider $V_i \otimes_Z E$. It is an E-representation space of G.
- From the above description, its irreducible components are E-representation spaces which is isomorphic to W_i.
- Since $\dim_Z V_i \otimes_Z E = nm^2$, it follows that $V_i \otimes_Z E$ is a sum of m copies of W_i.
- **To summarise:** Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible F-representation of G. With D, Z, E, as defined above, we have

$$V \otimes_F E = V \otimes_F Z \otimes_Z E = (V_1 + \cdots + V_k) \otimes_Z E = m(W_1 + \cdots + W_k),$$

a decomposition into distinct representation-spaces $W_1, W_2, \ldots, W_k$ of E-irreducible representations, each occuring with the **same multiplicity** m, where m **is the Schur index of** ρ, and $V_i = mW_i$, for each $i = 1, 2, \ldots, k$.

- I. Reiner, actually gives a different definition of the Schur index. If we consider F as the "bottom" and $\overline{F}$ as the "top", Reiner gives a definition of the Schur index from a viewpoint of the "top". The fields Z, E mentioned above arise as subfields of $\overline{F}$, which depend on a specific irreducible F-representation.
- Let $\tilde{\rho}$ be an irreducible representation of G over $\overline{F}$. Let $\chi = \chi_{\tilde{\rho}}$ be the character of $\tilde{\rho}$. Then for each g in G, $\chi(g)$ is the sum of eigenvalues of $\tilde{\rho}(g)$. Let u be the exponent of G, that is the l.c.m. of the orders of elements of G. Then $\chi(g)$ is a sum of u^{th} roots of unity. So $\chi(g)$ is an element of the field $F(\zeta_u)$, where ζ_u is a primitive u^{th} root of unity.

- Let $F(\chi) = F(\chi_{\tilde{\rho}})$ be the extension field of F obtained by adjoining all $\chi(g)$'s for g in G. It is a subfield of $F(\zeta_u)$. It is called The **character field** of $\tilde{\rho}$ over F. Since $F(\zeta_u)$ is an abelian Galois extension of F, it follows that $F(\chi)$ is also an abelian Galois extension field of F.

- Let A be the Galois group of $F(\chi)$ over F. It is easy to see that for each α, an element of A, the values $\alpha(\chi(g))$ are also values of a character of a representation of G over $\overline{F}$. In effect, starting with $\tilde{\rho}$, we have obtained $|A|$ distinct representations of G, over $\overline{F}$. Any two of these representations are called **algebraically conjugate**. In this way we have obtained a class of mutually inequivalent algebraically conjugate representations of G. These representations have different characters, but they all have the same character field.

- Let Z_1 denote the field $F(\chi)$, and $e = e_{\tilde{\rho}}$ be the pci corresponding to $\tilde{\rho}$. Then e is in $Z_1[G]$, and $Z_1[G]e$ is a minimal $2-$sided ideal of $Z_1[G]$. By the Artin-Wedderburn theorem, $Z_1[G]e \approx M_n(D_1)$, for some n and some division ring D_1. Then $V_1 \approx D_1^n$ is a representation space of G over Z_1. Then Z_1 is the center of D_1 and the dimension of D_1 over Z_1 is m^2 for some m. If E_1 is a maximal subfield of D_1, it can serve as a splitting field for D_1, and so the representation $\tilde{\rho}$, is *realisable over* E_1. That is, we can choose a basis of the representation space $V_1 \otimes E_1$, w.r.t. which all the entries of the matrices $\tilde{\rho}(g)$ for all g in G lie in E_1.

- Since the dimension of E_1 over Z_1 is m, and m is the least such dimension, we can take the second definition, due to I. Reiner, of **Schur index**, as the minimum of the dimensions of fields $\tilde{E}$ over which the representation $\tilde{\rho}$, is **realisable over** $\tilde{E}$. Let V be the vector space over F obtained from V_1 by restriction of scalars from Z_1 to F. Consider it as a representation space of a representation ρ of G. It is easy to see that D_1 is the centraliser of G in $EndV$. Then the Schur index of ρ according to the first definition, equals the Schur index of the representation of $\tilde{\rho}$ according to the second definition.

F-conjugacy

Definition. Two elements x, y in G are said to be ***F*-conjugate**, if for all finite dimensional F-representations (ρ, V) with the characters χ_ρ, we have $\chi_\rho(x) = \chi_\rho(y)$, and is denoted by $x \sim_F y$.

- $\sim_F$ is an equivalance relation on G.
- The **F-conjugacy class** of an element $x \in G$ consists of all those elements in G, which are F-conjugate to x. We denote the F-conjugacy class of x by $C_F(x)$ and the conjugacy class of x by $C(x)$.
- F-conjugacy class of an element of G is union of certain conjugacy classes.

Berman-conjugacy

Let u be the least common multiple of the orders of the elements of G. Let ω be a primitive u-th root of unity in $\overline{F}$. Let $K = \mathrm{Gal}(F(\omega)/F)$, which is an abelian group. We have a homomorphism θ from the Galois group K, into the multiplicative group $\mathbb{Z}_u^*$, defined as follows. If $\sigma \in K$, then $\sigma(\omega) = \omega^a$, where $a \in \mathbb{Z}_u^*$, and we define $\theta(\sigma) = a$. Let $A = \theta(K) \subseteq \mathbb{Z}_u^*$. We say that two elements $x, y \in G$ are conjugate in the sense of Berman, or **"Berman-conjugate"**, if there exists $g \in G$ and $j \in A$ such that $g^{-1}xg = y^j$, that is, x is conjugate to y^j.

Remark. For $x, y \in G$, $x \sim_F y$ iff x, y are Berman-conjugate.

Decomposition of Cyclotomic Polynomials

Proposition. Let n be a positive integer. Let F be a field of characteristic 0 or prime to n. Let $\Phi_n(X) = f_1(X)f_2(X)\cdots f_k(X)$ be the decomposition of $\Phi_n(X)$ into irreducible monic polynomials over F. Then

1. The degrees of all $f_i(X)$'s are the same.

2. Let ζ be a root of one $f_i(X)$. Then all the roots of $f_i(X)$ are $\{\zeta^{r_1}, \zeta^{r_2}, \ldots, \zeta^{r_s}\}$, where all r_i's are natural numbers with $r_1 = 1$, and the sequence $\{r_1, r_2, \ldots, r_s\}$ is independent of irreducible factors of $\Phi_n(X)$ and any root of $\Phi_n(X)$.

Witt-Berman Theorem

Theorem 1. Let G be a finite group and F be a field of characteristic 0 or prime to the order of G. Then the number of inequivalent irreducible F-representations of G is equal to the number of F-conjugacy classes of elements of G.

Remark. The set of inequivalent irreducible F-characters of G forms a basis of the space of all functions $f : G \longrightarrow F$, which are constant on each F-conjugacy class of G.

Theorem 2. Let F be a field of characteristic 0 or prime to the order of G. Let x be an element of order n in G. Then

$$C_F(x) = C(x^{r_1}) \cup C(x^{r_2}) \cup \cdots \cup C(x^{r_s}),$$

where $r_1 = 1, r_2, \ldots, r_s$ is the sequence associated with $\Phi_n(X)$ as in the previous Proposition.

Corollary. Let x be an element in G, and of order n. Then the F-conjugacy class of x is uniquely determined by the roots of just one irreducible factor of $\Phi_n(X)$ over F.

F-character Table

- By Witt-Berman theorem, the number of F-conjugacy classes of elements of G is equal to the number of F-irreducible representations of G. We can list the F-character values on F-conjugacy classes in the form of a square matrix over F, which is called the ***F*-character table**.
- The columns of F-character table are parametrized by F-conjugacy classes, and the rows are parametrized by irreducible F-characters.

Remark: Since the number of F-conjugacy classes in general is less than or equal to the number of conjugacy classes, the size of the matrix representing F-character table is smaller than the usual character table.

Remark. Consider the important case $F = \mathbb{Q}$, and G abelian. Then the character table over $\overline{F}$ is a $|G| \times |G|$ square matrix. One the other hand, let $|G| = \Pi_i p^i$. Then the character table of G over F, has size only $\Pi_i p(i)$, where $p(i)$ is the number of partitions of i, which depends only on the exponents of primes occurring in the prime factorization of $|G|$, and not on the actual primes themselves.

F-idempotents

- Let $R = F[G]$.
- Let V denote one of these simple R-modules and e be the corresponding pci.
- By Schur's lemma, $\mathrm{End}_{F[G]}(V)$ is a division ring D, whose center Z contains F. Then Re is abstractly isomorphic to $M_n(D^o)$, V is isomorphic to $(D^o)^n$, and Re is isomorphic to the direct sum of n copies of V.
- Let $\dim_F Z = \delta$ and $\dim_Z D = m^2$. Then $\dim_F V = nm^2\delta$, and so $\dim_F Re = n^2 m^2 \delta$.

- Let $L_1, L_2, \ldots, L_r$ be the F-conjugacy classes of G. Let χ be any F-character of G. Let $\chi(L_i)$ denote the common value of χ over L_i. Let L_i be the F-conjugacy class of x. We denote the F-conjugacy class of x^{-1} by L_i^{-1}. For any subset S of G, S^* denotes the formal sum of elements of S.

Theorem 3. Let F be a field of characteristic 0 or prime to the order of G. Let (ρ, V) be an irreducible F-representation of G, χ be its character and e be the corresponding pci in $F[G]$. Let n be the reduced dimension of V. Then

$$e = \frac{n}{|G|} \sum_{i=1}^{r} \chi(L_i^{-1}) L_i^*.$$

Remark. One can read the complete set of pci's of $F[G]$ from the F-character table.

Berman's Thorem

Before stating Berman's theorem, we define the following setup.

- Let G be a finite group and H a normal subgroup of index p, a prime. Let $G/H = \langle \overline{x} \rangle$, and x be a lift of $\bar{x}$ in G.
- Let F be an algebraically closed field of characteristic either 0 or coprime to $|G|$. Let $\overline{C(x)}$ be the conjugacy class sum of x in $F[G]$.

  Since $\overline{C(x)}$ is a central element in $F[G]$, then $\overline{C(x)}^p$ is a central element of $F[H]$.
- Let (η, W) be an irreducible F-representation of H and e_η be the pci of η in $F[H]$.

- Suppose that $\eta \cong \eta^x$. Then $\overline{C(x)}^p e_\eta$ belongs to the center of $F[H]e_\eta$. By Schur's lemma, $\overline{C(x)}^p e_\eta = \lambda e_\eta$, where $\lambda \in F$, and x in $G - H$ may be chosen so that $\lambda \neq 0$.
- Let μ be any p-th root of λ, in F. Let $c = \overline{C(x)} e_\eta / \mu$. Then $c^p = e_\eta$. Let ζ be a primitive p-th root of unity in F. $e_{\zeta^i c} e_\eta$, where $e_X = (1 + X + \cdots + X^{p-1})/p$, are p mutually orthogonal central idempotents in $F[G]$, and e_η **is a sum of** $e_{\zeta^i c} e_\eta$ **in** $F[G]$.
- Since e_η can split into at most p central idempotents, then $\eta \uparrow_H^G$ splits into p distinct irreducible representations of G. In fact each of these representations are **extensions** of η, that is, the H-action on the representation space W extends to p distinct G-actions on the same vector space W.

Theorem 4. (Berman-1955) Let G be a finite group and H be a normal subgroup of index p, a prime. Let $G/H = \langle xH \rangle$, for some x in G. Let F be an algebraically closed field of characteristic either 0 or prime to the order of G. Let η be an irreducible representation of H over F. We distinguish two cases:

(1) If $\eta \not\cong \eta^x$, then $\rho \cong \eta \uparrow_H^G$ is irreducible, $\rho \downarrow_H^G \cong \eta \oplus \eta^x \oplus \cdots \oplus \eta^{x^{p-1}}$, and $e_\rho = e_\eta + e_{\eta^x} + \cdots + e_{\eta^{x^{p-1}}}$.

(2) If $\eta \cong \eta^x$, then η extends to p distinct irreducible representations $\rho_0, \rho_1, \ldots, \rho_{p-1}$ of G over F, and $\eta \uparrow_H^G \cong \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_{p-1}$. Correspondingly, $e_\eta = e_{\rho_0} + e_{\rho_1} + \cdots + e_{\rho_{p-1}}$, where $e_{\rho_i} = e_{\zeta^i c} e_\eta, i = 0, 1, \ldots, p-1$.

Remark. Let G be a **solvable group**. Then every irreducible representation ρ of G is obtained by a 1-dimensional representation of an abelian subgroup by a sequence of extensions and inductions.

Extension of Berman's Theorem

- H is a normal subgroup of prime index p in G.
- F is a field of characteristic 0 or prime to the order of G.
- $\overline{F}$ is the algebraic closure of F.
- $G/H = \langle \overline{x} \rangle$.
- x is a lift of $\overline{x}$ in G, which may be taken to be order a power of p.
- η is an irreducible F-representation of H, ψ be its character and e_η be its corresponding pci in $F[H]$.
- $\overline{C(x)}$ is the conjugacy class sum of x in $F[G]$.
- Z is the center of $F[H]e_\eta$. So, $\overline{C(x)}^p e_\eta = \lambda e_\eta,$ where $\lambda \in Z.$

- By the classic theorem of Schur
$$\eta \otimes_F \overline{F} \cong m(\tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \cdots \oplus \tilde{\eta}_\delta),$$
where $\tilde{\eta}_i$'s are algebraically conjugates over F and m is the **Schur index** of $\tilde{\eta}_i$ w.r.t. F.

- Let $\tilde{\psi}_i$ be the character of $\tilde{\eta}_i$, and $L = F(\tilde{\psi}_i)$ be the common character field of $\tilde{\eta}_i$ over F.

- Let $\mathcal{G}$ be the Galois group of L over F. Then we have $|\mathcal{G}| = [L : F] = \delta$. In terms of the characters, we write
$$\psi = m(\tilde{\psi}_1 \oplus \tilde{\psi}_2 \oplus \cdots \oplus \tilde{\psi}_\delta) = m\sum_{\sigma \in \mathcal{G}} \sigma o \tilde{\psi}_1,$$
where $\tilde{\psi}_i$'s are algebraically conjugates over F.

- Let D be the H-centraliser of η. Then D is a division ring. So, the center of D is a field. In fact, the center of D, is isomorphic to Z, which is a field. Then
$$\eta \otimes_F Z \cong \eta_1 \oplus \eta_2 \cdots \oplus \eta_\delta$$
where η_i's are mutually ineqiuvalent irreducible Z-representations of H.
- If $\eta^x \ncong \eta$, then $\eta \uparrow_H^G$ is irreducible, $\eta \uparrow_H^G \cong \eta^x \uparrow_H^G \cong \ldots \cong \eta^{x^{p-1}} \uparrow_H^G \cong \rho$, say, and $e_\rho = e_\eta \oplus e_{\eta^x} \oplus \cdots \oplus e_{\eta^{x^{p-1}}}$. Now onwards we restrict our attention to the case $\eta^x \cong \eta$.

- Suppose that $\eta^x \cong \eta$ Then e_η is central in $F[G]$. If e_η remains a pci in $F[G]$, we say e_η **does not split** in $F[G]$, otherwise we say e_η **splits** in $F[G]$.
- Let $\Phi_p(X)$ denote the p-th cyclotomic polynomial. Let $\Phi_p(X) = f_1(X) \dots f_k(X)$ be the factorization into monic irreducible polynomials over Z. Then $X^p - 1 = f_0(X) f_1(X) \dots\ f_k(X)$, where $f_0(X) = X - 1$, be the factorization into monic irreducible polynomials over Z.
- Recall that the degree of $f_i(X)$'s are the same, say, d. If ζ is a root of $f_i(X)$, then all the roots of $f_i(X)$ are $\zeta, \zeta^{r_2}, \dots, \zeta^{r_d}$, and also the sequence $\{r_1 = 1, r_2, \dots, r_d\}$ is independent of $f_i(X)$ and the roots of $f_i(X)$.

Theorem 5. Let G be a finite group, and H be a normal subgroup of prime index p in G. Let $G/H = \langle \overline{x} \rangle$. Let x be a lift of $\overline{x}$ in G. Let F be a field of characteristic 0 or prime to the order of G. Let η be an irreducible F-representation of H, and e_η be its corresponding pci in $F[H]$. Suppose that $\eta^x \cong \eta$. Let Z denote the center of $F[H]e_\eta$. Then

(1) If $\eta_1^x \ncong \eta_1$, then e_η does not split in $F[G]$.

(2) If $\eta_1^x \cong \eta_1$, then it follows that $\overline{C(x)}^p e_\eta = \lambda e_\eta$, where $\lambda \in Z - \{0\}$. We consider three subcases:

(A) If λ has no p-th root in Z, then e_η does not split in $F[G]$.

(B) If λ has two distinct p-th roots in Z, then e_η splits into p pci's in $F[G]$. They are given by $e_{c_i} e_\eta, i = 1, 2, \ldots, p$.

Theorem Cont.

(C) If λ has only one p-th root in Z, then e_η splits into $1+k$ pci's in $F[G]$. Let $\zeta_{ij}, j = 1,2,\ldots,d$ be the roots of $f_i(X)$ in $\bar{F}$. Let

$$c_{ij} = \frac{\overline{C(x)}e_\eta}{\mu\zeta_{ij}}, e_{c_{ij}} = (1 + c_{ij} + \cdots + c_{ij}^{p-1})/p, i = 1,\ldots,k; j = 1,2,\ldots,d$$

and

$$e_{f_i(X)} = \sum_{j=1}^{d} e_{c_{ij}} e_\eta.$$

Then e_η splits into $e_{f_i(X)}, i = 1,2,\ldots,k$, together with $e_c e_\eta$.

Theorem 6. Let G be a finite group, and H a normal subgroup of prime index p in G. Let $G/H = \langle \overline{x} \rangle$. Let x be a lift of $\overline{x}$ in G. Let F be a field of characteristic 0 or prime to the order of G. Let η be an irreducible F-representation of H, and e_η be its corresponding pci in $F[H]$. Suppose that $\eta^x \cong \eta$. Let Z denote the center of $F[H]e_\eta$. Then

(1) If $\eta_1^x \not\cong \eta_1$, then $\eta \uparrow_H^G$ is either irreducible or equivalent to $p\rho$, where ρ is the unique extension of η to G.

(2) If $\eta_1^x \cong \eta_1$, then it follows that $\overline{C(x)}^p e_\eta = \lambda e_\eta$, where $\lambda \in Z - \{0\}$. Then we consider three subcases:

(A) If λ has no p-th root in Z, then $\eta \uparrow_H^G$ is either irreducible or equivalent to $p\rho$, ρ is the unique extension of η to G.

(B) If λ has two distinct p-th roots in Z, then η extends to p distinct irreducible F-representations of G, say, $\rho_0, \rho_1, \dots, \rho_{p-1}$. Then $\eta \uparrow_H^G \cong \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_{p-1}$.

Theorem Cont.

(C) If λ has only one p-th root in Z, then $\eta \uparrow_H^G$ decomposes into $1+k$ distinct irreducible F-representations of G. Let $\rho_0, \rho_1, \ldots, \rho_k$ be the $1+k$ distinct irreducible F-representations of G appear in $\eta \uparrow_H^G$. Then $\eta \uparrow_H^G \cong \rho_0 \oplus s(\rho_1 \oplus \cdots \oplus \rho_k)$, where ρ_0 is unique extension of η and s divides d, the common degree of irreducible factors of $\Phi_p(X)$ over Z.

Illustrative Examples

Example 1. Consider G to be Q_8 with prsentation:

$$G = \langle x, y, z \mid x^2 = 1, y^2 = x, z^2 = y^2, z^{-1}yz = xy \rangle.$$

Take $H = C_4 = \langle x, y \mid x^2 = 1, y^2 = x \rangle$ and $F = \mathbb{Q}$.

- η = the unique faithful irreducible $\mathbb{Q}$-representation of H of degree 2.
- We check that $e_\eta = 1 - e_x$, where $e_x = (1 + x)/2$ and $Z \approx \mathbb{Q}(i)$, $i = \sqrt{-1}$.
- In this case, e_η **does not split**, and $\eta \uparrow_H^G$ **is irreducible**, say, ρ.
- Note that the simple components of η, ρ in their Wedderburn decompositions are $\mathbb{Q}(i)$ and $\mathbb{H}_\mathbb{Q}$ respectively.
- The Schur index of ρ, according to its today's definition, is 2, which is the square root of 4.

- The irreducible $\mathbb{Q}$-representations of G are of degrees $1, 1, 1, 1$, and 4, whereas the degrees of irreducible $\bar{\mathbb{Q}}$-representations of G are $1, 1, 1, 1$ and 2.
- Let V be an irreducible representation space of G of dimension 4 over $\mathbb{Q}$.
- Here $V \approx \mathbb{H}_{\mathbb{Q}}$ (as a $\mathbb{Q}$-vector space) has reduced dimension 1, that is, it has dimension 1 regarded as a vector space over $\mathbb{H}_{\mathbb{Q}}$ (as a division ring).
- As rings, $\mathbb{H}_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$ is $M_2(\overline{\mathbb{Q}})$, the ring of 2×2 matrices over $\overline{\mathbb{Q}}$.
- Each of the two columns of $M_2(\overline{\mathbb{Q}})$ can serve as an irreducible representation space of G over $\overline{\mathbb{Q}}$.
- Notice that $\dim V \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}} = 4$ also, and it is a direct sum of 2 copies of the same irreducible 2-dimensional representation space of G over $\overline{\mathbb{Q}}$.
- So the Schur index is 2. It was the genius of Schur to realise that this indeed was a general phenomenon!

Example 2. Consider G to be $\mathrm{SL}_2(3)$ with presentation:

$$\langle x, y, z, t | x^2 = 1, y^2 = x, y^2 = z^2, z^{-1}yz = xy, t^3 = 1, t^{-1}yt = z, t^{-1}zt = yz \rangle.$$

Take $H = Q_8$ and $F = \mathbb{Q}$.

- η = the degree 4 unique faithful irreducible $\mathbb{Q}$-representation of H.
- In this case, **e_η splits into two pci's in $\mathbb{Q}[G]$.** Correspondingly, $\eta \uparrow_H^G$ decomposes into two distinct irreducible $\mathbb{Q}$-representations of G, of degrees 4, and 4, say, ρ_0, ρ_1, respectively.
- In fact, $\eta \uparrow_H^G \cong \rho_0 \oplus 2\rho_1$.
- Note that the simple component corresponding to η is $\mathbb{H}_{\mathbb{Q}}$, and the simple components corresponding to ρ_0, ρ_1 are $\mathbb{H}_{\mathbb{Q}}$, $\mathbb{H}_{\mathbb{Q}(\omega)} \cong M_2(\mathbb{Q}(\omega))$, where ω is a primitive 3-rd root of unity, respectively.

Example 3. Let p be a prime. Consider G to be

$C_{p^2} = \langle x, y \mid x^p = 1, y^p = x \rangle$. Take $H = C_p = \langle x \mid x^p = 1 \rangle$, and $F = \mathbb{Q}$.

- η = the unique faithful irreducible $\mathbb{Q}$-representation of H of degree $p-1$.
- In this case, e_η **does not split in** $\mathbb{Q}[G]$ and $\eta \uparrow_H^G$ **is irreducible**.

Example 4. Let p be a prime. Consider G to be $C_p \times C_p$ with presentation: $\langle x, y \mid x^p = y^p = 1, xy = yx \rangle$.

Take $H = C_p = \langle x | x^p = 1 \rangle$ and $F = \mathbb{Q}$.

- η = the degree $p-1$ unique faithful irreducible $\mathbb{Q}$-representation of H.
- In this case, e_η **splits into** p **pci** in $\mathbb{Q}[G]$.
- Correspondingly, $\eta \uparrow_H^G$ **decomposes into** p **distinct extensions of** η to G.

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THANK YOU FOR YOUR ATTENTION!