Free Group Rings

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I would like to dedicate this talk to the memory

of

Chander Kanta Gupta and Narain Gupta

Free Group Rings

Narain Gupta: *Free Group Rings*,

Contemporary Mathematics, Vol 66, Amer. Math. Soc., 1987.

Joint work with Roman Mikhailov (St. Petersburg).

Roman Mikhailov and Inder Bir S. Passi:

Narain Gupta's three normal subgroup problem and group homology,

Journal of Algebra, **526** (2019), 243 - 265.

Roman Mikhailov and Inder Bir S. Passi:

Dimension quotients, Fox subgroups and limits of functors,

Forum Mathematicum, **31** (2019), 385 - 401.

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Let F be a free group and $\mathbb{Z}[F]$ its integral group ring. For every two-sided ideal $\mathfrak{a}$ in $\mathbb{Z}[F]$, we have a normal subgroup

$$D(F, \mathfrak{a}) := F \cap (1 + \mathfrak{a})$$

of F. The identification of such normal subgroups of free groups is a recurring problem in the theory of group rings.

- Passi [1979]: Group Rings and Their Augmantation Ideals.
- Gupta [1987]: Free Group Rings.
- Mikhailov-Passi [2009]: Lower Central and Dimension Series of Groups.

Notation

For a subgroup $H \subseteq G$, let H' and $\sqrt{H}$ denote respectively the derived subgroup $[H, H]$ of H and the isolator

$$\{g \in G \mid g^n \in H \text{ for some } n > 0\}$$

in G of the subgroup H, and let $\{\gamma_i(G)\}_{i \geq 1}$ denote the lower central series of G:

$$\gamma_1(G) = G,$$

$$\gamma_{n+1}(G) = [G, \gamma_n(G)], \ \ n \geq 1.$$

Magnus [1937]

Theorem (Fundamental Theorem of Free Group Rings)

For a free group F,

$$D(F, \mathbf{f}^n) = \gamma_n(F), \text{ for all } n \geq 1,$$

where $\gamma_n(F)$ denotes the n^{th} term of the lower central series of F.

Given a normal subgroup R of F, let $\mathbf{r}$ denote the two-sided ideal of $\mathbb{Z}[F]$ generated by the augmentation ideal $\Delta(R)$ of the integral group ring $\mathbb{Z}[R]$, i.e.,

$$\mathbf{r} = \Delta(R)\mathbb{Z}[F].$$

Two famous problems

Let F be a free group and R a normal subgroup of F.

Dimension subgroup problem: Identify $D(F, \mathbf{f}^n + \mathbf{r}), \; n \geq 1$.

Identify $D_n(G) = G \cap (1 + \mathfrak{g}^n), \; n \geq 1$, where $\mathfrak{g}$ denotes the augmentation ideal of the group G.

Fox subgroup problem: Identify $D(F, \mathbf{f}^n\mathbf{r}), \; n \geq 1$.

$$D(F, \mathbf{r}) := F \cap (1 + \mathfrak{r}) = R.$$

$$D(F, \mathbf{f}^2) = [F, F], \text{ the derived subgroup of F.}$$

$$D(F, \mathbf{f}^2 + \mathbf{r}) = F'R.$$

$$D_2(G) := G \cap (1 + \mathfrak{g}^2) = G'.$$

$$D(F, \mathbf{f}^3 + \mathbf{r}) = \gamma_3(F)R \quad \text{(Higman; Rees; see Passi [1968]).}$$

$$D_3(G) = \gamma_3(G).$$

$$D(F, \mathbf{fr}) = R' \quad \text{(Schumann [1935])}.$$

For two normal subgroups R, S of F,

$$D(F, \mathbf{rs}) = [R \cap S, R \cap S],$$

the derived subgroup of $R \cap S$ (Enright [1968]; see Gupta [1987], Theorem 1.6, p. 3).

Narain Gupta's three normal subgroup problem

If R, S, T are three normal subgroups of F, a currently open problem formulated by Narain Gupta [1987] asks for the identification of the normal subgroup

$$D(F, \mathbf{rst}) := F \cap (1 + \mathbf{rst}).$$

Free group rings and derived functors

Homology of groups and derived functors of non-additive functors can provide a useful tool for investigating normal subgroups determined by ideals in free group rings.

Free group rings and derived functors

Roman Mikhailov and Inder Bir S. Passi: Generalized dimension subgroups and derived functors, *J. Pure Appl. Algebra*, **220** (2016), 2143–2163

Roman Mikhailov and Inder Bir S. Passi: The subgroup determined by a certain ideal in a free group ring, *J. Algebra*, **449**, (2016), 400–407

Roman Mikhailov and Inder Bir S. Passi: *Free group rings and derived functors*, Mehrmann, Volker (ed.) et al., European congress of mathematics. Proceedings of the 7th ECM (7ECM) congress, Berlin, Germany, July 18 - 22, 2016. Zürich: European Mathematical Society (EMS), 407-425 (2018).

Narain Gupta's three normal subgroup problem

$$D(F, \mathbf{rfs}) = \sqrt{[R' \cap S', R \cap S][R \cap S', R \cap S'][R' \cap S, R' \cap S]}$$

Kanta Gupta[1978], see Gupta [1987], p. 114 - 116; MP [2016]

$$D(F, \mathbf{rfr}) = \gamma_3(R)$$

$$D(F, \mathbf{frf}) = \sqrt{[R', F]}$$

Stöhr [1984],

Narain Gupta's three normal subgroup problem

$$D(F, \mathbf{f}^2\mathbf{r}) = [R \cap F', R \cap F']\gamma_3(R).$$

(Enright [1968]; Hurley [1973[; Gupta [1987], p. 48)

$$D(F, \mathbf{rrs}) = [R' \cap S, R' \cap S]\gamma_3(R \cap S)$$

(Ram Karan, Deepak Kumar and L. R. Vermani [2002])

Narain Gupta's three normal subgroup problem

Given a triple of subgroups R, S, T of a free group F, set

$$I(R, S, T) :=$$
$$\sqrt{[(R \cap S)' \cap (S \cap T)', R \cap T](R \cap (S \cap T)')'((R \cap S)' \cap T)'}.$$

$$D(F, \mathbf{rst}) = I(R, S, T)?$$

Narain Gupta's three normal subgroup problem

Consider the case when $R \subseteq T$ (or equivalently, in view of the canonical anti-automorphism of $\mathbb{Z}[F]$, when $T \subseteq R$). It is easy to see that a complete answer for this case, together with the above known results, provides identification of $D(F, \mathbf{rst})$ whenever one of the three normal subgroups R, S, T is contained in either of the other two.

Narain Gupta's three normal subgroup problem

Our main contribution to the Narain Gupta's three normal subgroup problem is the following

Theorem (Mikhailov - Passi)

If $R,\ S,\ T$ are normal subgroups of a free group F, such that $R \subseteq T$, and the integral homology groups $H_i(R/R\cap S)$, are torsion groups for $i = 3, 4, 5$, then

$$D(F, \mathbf{rst}) = I(R,\ S,\ T) = \sqrt{(R \cap (S\cap T)')'[(R\cap S)',\ R]}.$$

Our proof of the above theorem involves a mix of homological and combinatorial arguments. A striking feature to note here is the role played by homology in the identification of normal subgroups determined by ideals in free group rings.

Theorem

If F is a free group, and R, S its normal subgroups with $S \subseteq R$, then there is a natural isomorphism

$$H_2(F/S, \mathbf{f}/\mathbf{r}) \cong \frac{\mathbf{fs} \cap \mathbf{rf}}{\mathbf{fsf} + \mathbf{rs}}.$$

Proof. Consider the Gruenberg free resolution (Gruenberg [1970], p. 34)

$$\cdots \to \mathbf{sf}/\mathbf{s}^2\mathbf{f} \to \mathbf{s}/\mathbf{s}^2 \to \mathbf{f}/\mathbf{sf} \to \mathbb{Z}[F/S] \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ viewed as a trivial left F/S-module. On tensoring this resolution with the right F/S-module $\mathbf{f}/\mathbf{r}$, we have the complex

$$\cdots \to \mathbf{f}/\mathbf{r} \otimes_{F/S} \mathbf{sf}/\mathbf{s}^2\mathbf{f} \to \mathbf{f}/\mathbf{r} \otimes_{F/S} \mathbf{s}/\mathbf{s}^2 \to \mathbf{f}/\mathbf{r} \otimes_{F/S} \mathbf{f}/\mathbf{sf} \to \mathbf{f}/\mathbf{r}.$$

For a free group F, and ideals $\mathfrak{b} \subset \mathfrak{a}$, $\mathfrak{d} \subset \mathfrak{c}$, we have (Ivanov - Mikhailov], Lemma 4.9)

$$(\mathfrak{a}/\mathfrak{b}) \otimes_F (\mathfrak{c}/\mathfrak{d}) \cong \frac{\mathfrak{ac}}{\mathfrak{bc} + \mathfrak{ad}}.$$

Thus the above complex reduces to the following complex

$$\cdots \to \frac{\mathbf{fsf}}{\mathbf{fs}^2\mathbf{f} + \mathbf{rsf}} \to \frac{\mathbf{fs}}{\mathbf{fs}^2 + \mathbf{rs}} \to \frac{\mathbf{f}^2}{\mathbf{fsf} + \mathbf{rf}} \to \mathbf{f}/\mathbf{r}.$$

Hence

$$H_2(F/S, \mathbf{f}/\mathbf{r}) \cong \frac{\mathbf{fs} \cap \mathbf{rf}}{\mathbf{fsf} + \mathbf{rs}}.$$

□

Theorem

(Vermani - Ram Karan [1995]) If R, S, T are normal subgroups of a free group F and $R \subseteq T$, then

$$\Delta(R)\Delta(S)\Delta(T) \cap \Delta(R)\Delta(R \cap S) = \Delta(R)\Delta(R \cap S)\Delta(R) + \Delta(R)\Delta(R \cap (S \cap T)').$$

Proof. We can assume that $F = ST = TS$. Let

$$u \in \Delta(R)\Delta(S)\Delta(T) \cap \Delta(R)\Delta(R \cap S).$$

Since the ideal $\Delta(R)\mathbb{Z}[F]$ is a free right $\mathbb{Z}[F]$-module,

$$u = \sum_{y \in \mathcal{Y}} (y-1)u_y,$$

where $\mathcal{Y}$ is a free basis of R and

$$u_y \in \Delta(S)\Delta(T)\mathbb{Z}[F] \cap \Delta(R \cap S)\mathbb{Z}[F] = \Delta(S)\Delta(T) \cap \mathbb{Z}[F]\Delta(R \cap S).$$

Since $\Delta(R)\Delta(R \cap S) \subseteq \mathbb{Z}[T]$, projecting the above equation under the map $\theta : \mathbb{Z}[F] \to \mathbb{Z}[T]$ induced by

$$f = ts \mapsto t$$

for $f \in F$. $t \in T$, s in a right transversal for $S \cap T$ in S, it follows that

$$u_y \in \mathbb{Z}[T].$$

Similarly, using the projection $\mathbb{Z}[F] \to \mathbb{Z}[T]$ induced on using left transversal for $S \cap T$ in S, it follows further that

$$u_y \in \Delta(S \cap T)\Delta(T) \cap \mathbb{Z}[T]\Delta(R \cap S).$$

Therefore, $u_y \in \Delta(R \cap S)\Delta(T) + \Delta(R \cap (S \cap T)')$ (Vermani - Ram Karan [1995], Lemma 2.1).

Consequently

$$u \in \Delta(R)\Delta(R \cap S)\Delta(T) + \Delta(R)\Delta(R \cap (S \cap T)').$$

Because $u \in \Delta(R)$, projecting under $\mathbb{Z}[T] \to \mathbb{Z}[R]$, it follows that

$$u \in \Delta(R)\Delta(R \cap S)\Delta(R) + \Delta(R)\Delta(R \cap (S \cap T)').$$

We have thus proved that the left hand side of the asserted equality is contained in the right hand side; the reverse inclusion being obvious, the proof of the Theorem is complete. $\square$

Relation module: Let $1 \to R \to F \to G \to 1$ be a free presentation of the group G. Then R/R' can be viewed as right G module via conjugation in F:

$$rR'.g = f^{-1}rfR',\ r \in R,\ f \in F,\ g \in G,\ f \mapsto g.$$

Analogously, R/R' can be viewed as a left G-module.

For a free group F and its normal subgroup R, and any left $\mathbb{Z}[F/R]$-module M, there are isomorphisms

$$H_i(F/R,\, R/R' \otimes M) \cong H_{i+2}(F/R,\, M),\ \ i \geq 1.$$

This is a well-known fact which follows easily from the Magnus embedding $R/R' \hookrightarrow \mathbf{f}/\mathbf{fr}$ of the relation module R/R'.

Relation sequence: If F is a free group and $R \lhd F$, then we have an exact sequence of left F/R-modules

$$1 \to R/R' \to \mathbf{f}/\mathbf{rf} \to \mathbf{f}/\mathbf{r} \to 1,$$

with the middle term $\mathbf{f}/\mathbf{rf}$ a free F/R-module.

Homological and Combinatorial Preliminaries

Let R, S be normal subgroups of the free group F. For the group

$$G := R/R \cap S \cong RS/S,$$

one can consider two different relation modules and a natural map between them:

$$\frac{R \cap S}{(R \cap S)'} \to S/S'.$$

This map can be naturally extended to a map between the corresponding relation sequences (Gruenberg:1976, p. 7): The $\mathbb{Z}[G]$-modules $\frac{\Delta(R)}{\Delta(R\cap S)\Delta(R)}$ and $\frac{\Delta(RS)}{\Delta(S)\Delta(RS)}$ are free, hence for any $\mathbb{Z}[G]$-module M, there are natural isomorphisms

$$H_i\left(R/R \cap S, \frac{R \cap S}{(R \cap S)'} \otimes M\right) \cong H_i(R/R \cap S, S/S' \otimes M)$$

$$\cong H_{i+2}(R/R \cap S, M), \; i \geq 1.$$

Quadratic Endofunctors on the category of abelian groups

$\otimes^2$ *tensor square*,

SP^2 *symmetric square*,

Λ^2 *exterior square*,

$\tilde{\otimes}^2$ *antisymmetric square*,

Γ^2 *divided square*.

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Quadratic Endofunctors on the category of abelian groups

Recall that, for an abelian group A, by definition,

$$\mathrm{SP}^2(A) = \otimes^2(A)/\langle a \otimes b - b \otimes a,\ \ a, b \in A\rangle,$$

$$\Lambda^2(A) = \otimes^2(A)/\langle a \otimes a,\ \ a \in A\rangle,$$

$$\tilde{\otimes}^2(A) = \otimes^2(A)/\langle a \otimes b + b \otimes a,\ \ a, b \in A\rangle.$$

Quadratic Endofunctors on the category of abelian groups

The divided square functor Γ^2 is also known as the **Whitehead quadratic functor**. Given an abelian group A, the abelian group $\Gamma^2(A)$ is generated by symbols $\gamma(x)$, $x \in A$, satisfying the following relations for all x, y, $z \in A$:

$$\gamma(0) = 0;$$

$$\gamma(x) = \gamma(-x);$$

$$\gamma(x+y+z) - \gamma(x+y) - \gamma(x+z) - \gamma(y+z)$$
$$+\gamma(x) + \gamma(y) + \gamma(z) = 0.$$

For a survey of the properties of these functors and their derived functors, see (H.-J. Baues and T. Pirashvili [2000], L. Breen and R. Mikhailov [2011])

The exterior and the antisymmetric squares are connected as follows:

For every abelian group A, we have a short exact sequence

$$0 \to A \otimes \mathbb{Z}/2 \to \tilde{\otimes}^2(A) \to \Lambda^2(A) \to 0.$$

Similarly, connecting the symmetric and divided square functor, we have the following short exact sequence:

$$0 \to \mathrm{SP}^2(A) \to \Gamma^2(A) \to A \otimes \mathbb{Z}/2 \to 0.$$

Derived functors of non-additive functors

Let E be a free abelian group, I its subgroup and $E/I = A$. Then there is a natural exact sequence

$$0 \to L_1\mathrm{SP}^2(A) \to \Lambda^2(E)/\Lambda^2(I) \to E \otimes E/I \to \mathrm{SP}^2(A) \to 0$$

where $L_1\mathrm{SP}^2$ is the first derived functor of SP^2 in the sense of Dold-Puppe, and $L_1\mathrm{SP}^2(A)$ is equal to the quotient of $\mathrm{Tor}(A, A)$ by the subgroup generated by the diagonal elements, i.e. the elements $(a, {}_n, a)$, $na = 0$, $a \in A$.

For the proof and applications of above kind of sequences in the theory of group rings, see Roman Mikhailov and Inder Bir S. Passi [2016].

Another ingredient that we need is the following analog of the results from Köck [2001] (Proposition 2.4 and Remark 2.7) on Koszul sequences.

Lemma

For a free abelian group E, $I \subseteq E$, $E/I = A$, the homology of the naturally defined complex

$$\mathrm{SP}^2(I) \rightarrowtail I \otimes E \to \tilde{\otimes}^2(E)$$

where the left hand map is given as $ij \mapsto i \otimes f(j) + j \otimes f(i),\ i, j \in I,\ f : I \hookrightarrow E$*, the right hand map is the composite of the natural inclusion* $I \otimes E \hookrightarrow E \otimes E$ *and the projection* $E \otimes E \twoheadrightarrow \tilde{\otimes}^2(E)$*, satisfies the following:*

$$H_0 = \tilde{\otimes}^2(A),$$

$$0 \to \mathrm{Tor}(A, \mathbb{Z}/2) \to H_1 \to L_1\Lambda^2(A) \to 0.$$

Here we place the term $\tilde{\otimes}^2(E)$ on the zeroth degree, $I \otimes E$ on the first degree in the considered complex.

Since

$$F \cap (1 + \Delta(R)\Delta(S)) = [R \cap S, R \cap S] \subset (1 + \Delta(R \cap S)\Delta(R)),$$

Vermani - Ram Karan [1995] Theorem implies that

$$D := D(F, \Delta(R)\Delta(S)\Delta(F)) = F \cap (1 + \Delta(R)\Delta(R \cap S)\Delta(R) + \Delta(R)\Delta(R \cap (S \cap T)'))$$

Observe that, for $w \in D$, $w - 1 \in \Delta(R \cap S)\Delta(R)$, and

$$\Delta(R)\Delta(R \cap S)\Delta(R) \subset \Delta(R \cap S)\Delta(R).$$

Therefore

$$D = F \cap (1 + \Delta(R)\Delta(R \cap S)\Delta(R) + \Delta(R)\Delta(R \cap (S \cap T)') \cap \Delta(R \cap S)\Delta(R))$$

We have to prove that the quotient

$$\frac{F \cap (1 + \Delta(R)\Delta(R \cap S)\Delta(R) + \Delta(R)\Delta(R \cap (S \cap T)') \cap \Delta(R \cap S)\Delta(R}{(R \cap (S \cap T)')'[(R \cap S)', R]}$$

is a torsion group.

The subgroup $D(F, \mathbf{rsf})$

Theorem

If the cohomological dimension $cd(R/R\cap S) \leq 3$*, then*

$$\frac{D(F, \mathbf{rsf})}{(R\cap S')'[(R\cap S)', R]} \hookleftarrow H_3(R/R\cap S) \otimes \tilde{\otimes}^2 \left(\frac{S}{(R\cap S)S'}\right).$$

Example.

Let $F = F(x_1, x_2, x_3, x_4, x_5)$,

$$R := \langle x_1, x_2, x_3 \rangle^F,$$
$$S := \langle [x_1, x_2], [x_2, x_3], [x_1, x_3], x_4, x_5 \rangle^F.$$

The quotient group $R/R \cap S$ is then a free abelian group of rank three, with the images of x_1, x_2, x_3 as generators, the group $\frac{S}{(R\cap S)S'}$ is a free abelian group of rank two with the images of x_4, x_5 as generators. Therefore,

$$H_3(R/R \cap S) \otimes \Lambda^2 \left(\frac{S}{(R \cap S)S'} \right) \cong \mathbb{Z}.$$

Hence, the quotient $\frac{D(F, \mathbf{rsf})}{I(R, S, F)}$ is non-zero.

A combinatorial proof of a result of Stöhr

As mentioned earlier, the normal subgroup $D(F, \mathbf{rfr})$, which is a special case of Gupta's three subgroup problem, has been identified by Stöhr [1984]. The following result on free group rings is implicit in Stöhr's work based on using homological mehods.

Theorem

Let F be a free group, and R a normal subgroup of F. If $a \in D(F, \mathbf{frf})$, then $a^2 \in D(F, \mathbf{rrf} + \mathbf{frr})$.

Since the above conclusion is a result purely in group rings, the following questions arise naturally.

- Does there exist a combinatorial proof of the above fact without the use of homology?

- Is it possible to generalize this result to more complicated ideals and generalized dimension subgroups?

We answer the first question affirmatively, and offer some remarks on the second question.

We have the following more general result, from which Stöhr's Theorem follows in case $S = T = F$.

Theorem (Mikhailov - Passi)

Let R, S, T be normal subgroups in F, such that $R \subseteq S, T$. If

$$a \in D(F, \mathbf{srt} + \mathbf{trs}),$$

then

$$a^2 \in D(F, \mathbf{rrs} + \mathbf{srr} + \mathbf{trr} + \mathbf{rrt}).$$

The general problem in free group rings, of which the foregoing are special cases, asks for the identification of normal subgroups $D(F, \mathfrak{a})$, where $\mathfrak{a}$ is a sum of ideals of the form $\mathbf{r_1} \ldots \mathbf{r_n}$ with $R_1, \ldots, R_n$ normal subgroups of the given free group F. As a contribution to this general problem, we present the following two results.

Theorem

Let $R \subseteq S,\ T$ *be normal subgroups of a free group* F*.*

If $a \in D(F, \mathbf{rsst} + \mathbf{tssr})$, *then*

$$a^2 \in D(F, \mathbf{rsss} + \mathbf{sssr} + \mathbf{tsss} + \mathbf{ssst} + ([S, S] - 1)\mathbf{s})$$

and

$$a^6 \in [S, S, S, RT].$$

Our concluding result is a generalization of Kanta Gupta's identification of $D(F, \mathbf{rffr})$ Kanta Gupta [1983].

Theorem

If $R \subseteq S$ are normal subgroups of a free group F, then

$$D(F, \mathbf{rsfr}) = D(F, \mathbf{rfsr}) = [R \cap S', R \cap S', R]\gamma_4(R).$$

Conjecture

For arbitrary normal subgroups $R,\ S$*, we conjecture that*

$$D(F, \mathbf{rssr}) = \sqrt{[\gamma_3(R \cap S),\ R][\gamma_2(R \cap S'), R]}.$$

Dimension Quotients, Fox Subgroups and Limits of Functors

Given a group G, let $\mathbb{Z}[G]$ be its integral group ring and $\mathfrak{g}$ the augmentation ideal of $\mathbb{Z}[G]$. The dimension quotients of G are defined to be its subquotients $D_n(G)/\gamma_n(G)$, $n \geq 1$, where $D_n(G) := G \cap (1+\mathfrak{g}^n)$ is the nth dimension subgroup of G and $\gamma_n(G)$ is the nth term in the lower central series $\{\gamma_i(G)\}_{i\geq 1}$ of G. The evaluation of dimension quotients is a challenging problem in the theory of group rings, and has been a subject of investigation since 1935 (Cohn [1952], Gupta [1987], Losey[1960], Magnus [1935], Magnus [1937], MP [2009], Passi [1968], Passi [1979]). While these quotients are trivial for free groups (Magnus: [1937], Witt [1937]), for $n = 1, 2, 3$ in case of all groups, and for odd prime-power groups Passi [1968], it was first shown in 1972 by E. Rips that $D_4(G)/\gamma_4(G)$ is, in general, non-trivial.

Subsequently, the structure of these fourth dimension quotients has been described by K. I. Tahara (Tahara [1977a], Tahara1[977b]) and Narain Gupta [1987]. Instances of the non-triviality of dimension quotients in all dimensions $n \geq 4$ are now known (Gupta [1987]; MP [2009], p. 111); however, their precise structure still remains an open problem.

Another challenging problem concerning normal subgroups determined by two-sided ideals in group rings is the so-called Fox subgroup problem (Fox, page 557; Birman [1974], Problem 13; Gupta [1987]). It asks for the identification of the normal subgroup

$$D(F, \mathbf{rf}^n) := F(n, R) := F \cap (1 + \mathbf{rf}^n)$$

for a free group F and its normal subgroup R.

A solution to this problem has been given by I. A. Yunus [1984] and Narain Gupta (Gupta [1987], Chapter III).

It turns out that while $F(1,\ R) = \gamma_2(R)$, $F(2,\ R) = [R \cap \gamma_2(F),\ R \cap \gamma_2(F)]\gamma_3(R)$, the identification of $F(n,\ R), n \geq 3$, is given as an isolator of a subgroup. For instance, $F(3,\ R) = \sqrt{G(3,\ R)}$, where

$$G(3,\ R) := \gamma_2(R \cap \gamma_3(F))[[R \cap \gamma_2(F),\ R],\ R \cap \gamma_2(F)]\gamma_4(R).$$

This identification essentially amounts to the one when the coefficients of the group ring are in the field of rational numbers, rather than in the ring of integers, and thus raises the question of the precise determination of the involved torsion.

Our aim in this Math Forum paper is to present an entirely different approach to the above problems via derived functors and limits of functors over the category of free presentations.

Limit of a functor

If $\mathcal{C}$ and $\mathcal{D}$ are two categories and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor, then the limit $\lim \mathcal{F}$ of the functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an object of the category $\mathcal{D}$ together with a universal collection of morphisms $\{\varphi_c : \lim \mathcal{F} \to \mathcal{F}(c)\}_{c \in \mathcal{C}}$ such that $\mathcal{F}(f)\varphi_c = \varphi_{c'}$ for any morphism $f : c \to c'$.

Universality means that for any object $d \in \mathcal{D}$ and any collection of morphisms $\{\psi_c : d \to \mathcal{F}(c)\}_{c \in \mathcal{C}}$ such that $\mathcal{F}(f)\psi_c = \psi_{c'}$ for any morphism $f : c \to c'$ there exists a unique morphism $\alpha : d \to \lim \mathcal{F}$ such that $\psi_c = \varphi_c \alpha$.

The limit does not always exist but if it exists it is unique up to unique isomorphism that commutes with morphisms φ_c.

Our approach is motivated by the connections between the theory of limits of functors with homology of groups, derived functors in the sense of Dold-Puppe [1961], cyclic homology and group rings (Emmanuel - Mikhailov [2008], MP [2016], MP [2017], Quillen [1989]). For instance, the even dimensional integral homology groups turn up as limits (Emmanuel - Mikhailov [2008]):

$$H_{2n}(G) = \varprojlim (R/\gamma_2(R))_F^{\otimes n},$$

where $(R/\gamma_2(R))^{\otimes n}$ is the nth tensor power of the relation module $R/\gamma_2(R))$, and $(R/\gamma_2(R))_F^{\otimes n}$ is the group of F-coinvariants, the action of F on $(R/\gamma_2(R))^{\otimes n}$, via conjugation in R, being diagonal.

Certain derived functors in the sense of Dold-Puppe [1961] turn out as limits (MP [2016]):

$$L_1\mathrm{SP}^2(G_{ab}) = \lim_{\longleftarrow} \frac{\gamma_2(F)}{\gamma_2(R)\gamma_3(F)}, \quad L_1\mathrm{SP}^3(G_{ab}) = \lim_{\longleftarrow} \frac{\gamma_3(F)}{[\gamma_2(R),\, F]\gamma_4(F)},$$

where $L_1\mathrm{SP}^2$ and $L_1\mathrm{SP}^3$ are the first derived functors of the symmetric square and cube functor respectively and $G_{ab} := G/\gamma_2(G)$. The description of derived functors $L_1\mathrm{SP}^n(G_{ab})$ as limits for $n \geq 4$ is given in MP [2017]. An application of the theory of limits to cyclic homology is given in Quillen [1989] where it is shown that the cyclic homology of algebras can be defined as limits over the category of free presentations of certain simply defined functors. We work in the same direction, but consider the category of groups. Our approach brings out a fresh context for the study of dimension subgroups and Fox subgroups.

To describe the main results of the Forum paper, let F be a free group, R a normal subgroup of F, and $G = F/R$. Then there is a natural short exact sequence

$$\frac{R \cap \gamma_2(F)}{\gamma_2(R)(R \cap \gamma_4(F))} \hookrightarrow \frac{\gamma_2(F)}{\gamma_2(R)\gamma_4(F)} \twoheadrightarrow \frac{\gamma_2(G)}{\gamma_4(G)}.$$

Observe that the first two terms can be viewed as functors from the category of free presentations

$$R \hookrightarrow F \twoheadrightarrow G$$

of G to the category of abelian groups.

The limit functor $\varprojlim$ is known to be left exact, i.e., it sends monomorphisms to monomorphisms, however, it is not right exact, and therefore short exact sequences of presentations induce long exact sequences involving higher derived $\varprojlim^i$-terms. For instance, the above short sequence induces the following long exact sequence

$$\varprojlim \frac{R \cap \gamma_2(F)}{\gamma_2(R)(R \cap \gamma_4(F))} \hookrightarrow \varprojlim \frac{\gamma_2(F)}{\gamma_2(R)\gamma_4(F)} \to \frac{\gamma_2(G)}{\gamma_4(G)} \to \varprojlim{}^1 \frac{R \cap \gamma_2(F)}{\gamma_2(R)(R \cap \gamma_4(F))}.$$

Our main result on dimension quotients describes the cokernel of the left monomorphism in the above exact sequence. To be precise, we have

Theorem (Mikhailov - Passi)

There is a natural short exact sequence

$$\varprojlim \frac{R \cap \gamma_2(F)}{\gamma_2(R)(R \cap \gamma_4(F))} \hookrightarrow \varprojlim \frac{\gamma_2(F)}{\gamma_2(R)\gamma_4(F)} \twoheadrightarrow \frac{D_4(G)}{\gamma_4(G)}.$$

Thus we present a description of the fourth dimension quotient purely in functorial terms (not involving the group ring), which to us, is a very surprising result.

We next give a functorial description of the quotient $F(3, R)/G(3, R)$, together with a complete identification (not involving an isolator), of the third Fox subgroup $F(3, R)$.

Theorem (Mikhailov - Passi)

Let F be a free group and R a normal subgroup of F.

(a) There is a natural isomorphism

$$\frac{F(3,\ R)}{G(3,\ R)} \simeq L_1\mathrm{SP}^2\left(\frac{R\cap\gamma_2(F)}{\gamma_2(R)(R\cap\gamma_3(F))}\right).$$

(b)
$$F(3,\, R) = G(3,\, R)W,$$
where W is a subgroup of F, generated by elements

$$[x,\, y]^m[x,\, s_y]^{-1}[y,\, s_x]$$

with

$$x^m = r_x s_x,\;\; r_x \in R \cap \gamma_3(F),\;\; s_x \in \gamma_2(R)$$
$$y^m = r_y s_y,\;\; r_y \in R \cap \gamma_3(F),\;\; s_y \in \gamma_2(R).$$

Finally, we give a description of $\varprojlim \frac{F(3,R)}{G(3,R)}$, where it may be noted that composition of derived functors appears.

Theorem

There is a natural isomorphism

$$\varprojlim \frac{F(3,R)}{G(3,R)} \simeq L_1\mathrm{SP}^2\left(L_1\mathrm{SP}^2(G_{ab})\right).$$

In particular, there a monomorphism

$$L_1\mathrm{SP}^2\left(L_1\mathrm{SP}^2(G_{ab})\right) \hookrightarrow \frac{F(3,R)}{G(3,R)}.$$

Thank you!