Semisimple $\mathbb{Q}$-algebras in algebraic combinatorics

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Definition 1 (D. Higman 1987).

A *coherent algebra* (CA) is a subalgebra $\mathbb{C}\mathbf{B}$ of $M_n(\mathbb{C})$ defined by a special **basis B**, which is a collection of non-overlapping 01-matrices that
(1) is a closed set under the transpose,
(2) sums to J, the $n \times n$ all 1's matrix, and
(3) contains a subset Δ of diagonal matrices summing I, the identity matrix.

The standard basis $\mathbf{B}$ of a CA in $M_n(\mathbb{C})$ is precisely the set of adjacency matrices of a *coherent configuration* (CC) of *order* n and *rank* $r = |B|$.

$\mathbf{B}^\top = \mathbf{B} \implies \mathbb{C}\mathbf{B}$ is semisimple.

When $\Delta = \{I\}$, $\mathbf{B}$ is the set of adjacency matrices of an *association scheme* (AS).

An AS or CC can be illustrated by its *basic matrix*. Here we give the basic matrix of an AS with standard basis $\mathbf{B} = \{b_0, b_1, \ldots, b_7\}$:

$$\sum_{i=0}^{d} i b_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 1 & 0 & 3 & 2 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 \\ 2 & 3 & 0 & 1 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 3 & 2 & 1 & 0 & 6 & 6 & 7 & 7 & 4 & 4 & 5 & 5 \\ 4 & 4 & 7 & 7 & 0 & 1 & 6 & 6 & 5 & 5 & 2 & 3 \\ 4 & 4 & 7 & 7 & 1 & 0 & 6 & 6 & 5 & 5 & 3 & 2 \\ 5 & 5 & 6 & 6 & 7 & 7 & 0 & 1 & 2 & 3 & 4 & 4 \\ 5 & 5 & 6 & 6 & 7 & 7 & 1 & 0 & 3 & 2 & 4 & 4 \\ 7 & 7 & 4 & 4 & 5 & 5 & 2 & 3 & 0 & 1 & 6 & 6 \\ 7 & 7 & 4 & 4 & 5 & 5 & 3 & 2 & 1 & 0 & 6 & 6 \\ 6 & 6 & 5 & 5 & 2 & 3 & 4 & 4 & 7 & 7 & 0 & 1 \\ 6 & 6 & 5 & 5 & 3 & 2 & 4 & 4 & 7 & 7 & 1 & 0 \end{bmatrix}$$

I.3 Example: Basic matrix of a CC of order 10 and $|\Delta| = 2$

$$\sum_{i=0}^{d} ib_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\ 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 \\ 4 & 5 & 6 & 7 & 8 & 7 & 8 & 7 & 8 & 7 \\ 5 & 4 & 7 & 6 & 7 & 8 & 7 & 8 & 7 & 8 \\ 4 & 5 & 8 & 7 & 6 & 7 & 8 & 7 & 8 & 7 \\ 5 & 4 & 7 & 8 & 7 & 6 & 7 & 8 & 7 & 8 \\ 4 & 5 & 8 & 7 & 8 & 7 & 6 & 7 & 8 & 7 \\ 5 & 4 & 7 & 8 & 7 & 8 & 7 & 6 & 7 & 8 \\ 4 & 5 & 8 & 7 & 8 & 7 & 8 & 7 & 6 & 7 \\ 5 & 4 & 7 & 8 & 7 & 8 & 7 & 8 & 7 & 6 \end{bmatrix}$$

$\Delta = \{b_0, b_6\}$, $\delta(b_0) = \delta(b_6) = 1$

Diagonal block valencies: $\delta(b_1) = 1$, $\delta(b_7) = 4$, $\delta(b_8) = 3$;

Off diagonal block valencies:

row valencies: $\delta_r(b_2) = \delta_r(b_3) = 4$, $\delta_r(b_4) = \delta_r(b_5) = 1$

column valencies: $\delta_c(b_2) = \delta_c(b_3) = 1$, $\delta_c(b_4) = \delta_c(b_5) = 4$.

The order $n = \sum$ row valencies $= \sum$ column valencies.

I.4 Example: A finite group is a *thin* AS

If $G = \{g_0 = e, g_1, \dots, g_{n-1}\}$ is a finite group of order n.
Let b_i be the left regular matrix of g_i for each $g_i \in G$.
Then $\{b_0 = I_n, b_1, \dots, b_{n-1}\}$ is an AS with every $\delta(b_i) = 1$.

For example, here is a (possible) basic matrix for Q_8:

$$\sum_{i=0}^{7} ib_i = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 3 & 2 & 0 & 1 & 6 & 7 & 5 & 4 \\ 2 & 3 & 1 & 0 & 7 & 6 & 4 & 5 \\ 5 & 4 & 7 & 6 & 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 & 1 & 0 & 3 & 2 \\ 7 & 6 & 4 & 5 & 3 & 2 & 0 & 1 \\ 6 & 7 & 5 & 4 & 2 & 3 & 1 & 0 \end{bmatrix}$$

I.5 Examples of ASs generated by graphs

(1) The complete graph AS $K_n = \{I_n, J_n - I_n\}$ is the unique rank 2 AS of order n.

(2) A strongly regular graph (SRG) with n vertices produces a rank 3 symmetric AS $\{I_n, A, A^c\}$ of order n.

(3) A distance regular graph (DRG) on n vertices with distance matrices $A_0, A_1, A_2, \dots, A_d$ produces a symmetric AS $\{A_0, A_1, \dots, A_d\}$ of order n and rank $d+1$.

(4) A doubly regular tournament (DRT) of order $n = 4u + 3$ produces an asymmetric rank 3 AS $\{I_n, A, A^\top\}$.

I.6 Adjacency algebras of ASs are subrings of group rings!

Proof: Suppose $S = \{b_0, b_1, \dots, b_{r-1}\}$ is the set of adjacency matrices of an AS of order n.

By the valency condition, each $b_i \neq b_0$ is a doubly stochastic 01-matrix, hence a sum of (non-overlapping) permutation matrices (in fact, derangements) that sum to J. [Schneider, 1959].

Therefore, $\mathbb{Z}S$ is isomorphic to a subring of $\mathbb{Z}G$, where G is a subgroup of S_n generated by a collection of derangements whose $n \times n$ permutation matrices sum to J. $\square$

I.7 The pseudo-inverse condition for ASs

Let $\mathbf{B} = \{b_0 = I_n, b_1, \ldots, b_{r-1}\}$ be the standard basis of an AS. Let $\{\lambda_{ijk}\}$ be the structure constants relative to $\mathbf{B}$, so

$$\forall b_i, b_j \in \mathbf{B}, \quad b_i b_j = \sum_k \lambda_{ijk} b_k.$$

Then $\forall b_i \in \mathbf{B}$, $\exists! b_j \in \mathbf{B}$ such that $\lambda_{ij0} \neq 0$. (For an AS $b_j = b_i^\top$.) If we write this b_j as b_{i^*} for all i, then we have

$$\lambda_{ij0} \neq 0 \iff j = i^*, \text{ and } \lambda_{i^* i0} = \lambda_{ii^*0} > 0. \qquad (1)$$

Definition 2.

Condition (1) is the *pseudo-inverse* condition on the basis $\mathbf{B}$.

A *table algebra* (or *fusion rule algebra*) is an associative algebra with involution $*$ that has a $*$-invariant basis $\mathbf{B}$ containing 1 and satisfying the pseudo-inverse condition.

- [Ziv-Av 2013] CCs are completely classified up to order 13.

- [Hanaki-Miyamoto 2005] ASs are considered classified for orders ≤ 30, 32, 33, and 34. (Order 31 is only obstructed by classification of DRTs of order 31, which should be available soon.)

- [Zieschang 1996] There are no noncommutative rank 5 ASs.
- [H 2019] There are no noncommutative rank 7 ASs with 6 asymmetric elements. They must have at least 2. Are there some with 4?

Question 1: Feasibility Problem

Which TAs are represented by ASs?

- All multiplicities must be positive integers, elements must satisfy graph eigenvalue restrictions, ...

Question 2

(i) [Brouwer] Is there a symmetric AS of rank 3 and order 65 with $b_1^2 = 32b_0 + 15b_1 + 16b_2$?
(This is the smallest SRG that is not known to exist. The missing Moore graph (57-regular on 3250 vertices) is another famous one.)

(ii) [H 2019] Is there a noncommutative AS of rank 7 that has four $*$-asymmetric elements?

Definition 3.

Let $S = \{A_0 = I, A_1, \dots, A_{r-1}\}$ be set of adjacency matrices for a CC (or AS) of rank r and order n.

Fix an $x \in X$. For each A_i, let $E_i^* = E_i^*(x)$ be the diagonal idempotent matrix whose yy-entry is 1 if $(A_i)_{xy} = 1$, and otherwise 0. If $E^*(x) = \{E_i^*(x) : i = 0, 1, \dots, r-1\}$ is this dual idempotent basis then $\mathbb{Q}E^*(x)$ is an r-dimensional semisimple commutative $\mathbb{Q}$-algebra.

The *rational Terwilliger (or subconstituent) algebra at vertex x* is $T_x = \mathbb{Q}\langle S \cup E^*(x) \rangle$.

Remarks: T_x is a semisimple subalgebra of a coherent algebra (its coherent closure), but it is usually not coherent.

T_x depends on x. There is one T_x up to isomorphism for each orbit of $Aut(S)$.

I.11 T-algebras for SRGs and DRGs, importance

- If G is a finite group then $T_x(G) \simeq M_n(\mathbb{Q})$.
- $T(K_n) \simeq M_2(\mathbb{Q}) \oplus \mathbb{Q}$
- [Yamazaki-Tomiyama 1994] For a symmetric rank 3 AS (SRG), $\mathbb{C} \otimes T_x(S) \simeq M_3(\mathbb{C}) \oplus M_2(\mathbb{C})^{m_2(x)} \oplus \mathbb{C}^{m_1(x)}$.
- $T_x(S)$ has been extensively studied for DRGs - this has led to deep connections with

-orthogonal polynomials and their related quantum groups,
-spin models in knot theory and conformal field theory, and
-the recent solution to Bannai's finiteness conjecture for ASs that are both metric and cometric.

I.12 T-algebras of asymmetric ASs? The integral T-alg?

For ASs that are not DRGs, not much is known about T_x!

Question 3: Rank 3A T-alg conjecture [H]

Suppose S is an asymmetric AS of rank 3 and order $4u+3$, $u \geq 0$, (i.e. a DRT). Then $\forall x$,

- dim T_x is odd and $\leq 8u+9$;
- $T_x(S) \simeq M_3(\mathbb{C}) \oplus M_2(\mathbb{C})^{m_2(x)} \oplus \mathbb{C}^{m_1(x)}$

Question 4: [H]

Except for finite groups, are CCs distinguished combinatorially by their rational/integral T_x-algebras (as a list of algebras up to isomorphism)?

Question 5: [Terwilliger 1991] (Conjecture 10a)

Is $\mathbb{Z}\langle S \cup E^* \rangle$ equal to the set of all integral matrices in T_x?

II.1 Let's get integral!

Let $\mathbf{B} = \{b_0, b_1, \ldots, b_{r-1}\}$ be a CC, AS, or the standard basis of an integral TA.

- $\mathbf{B}^2 \subset \mathbb{N}\mathbf{B}$, so $\mathbb{Z}\mathbf{B}$ will be an integral ring.
- $R\mathbf{B} := R \otimes_{\mathbb{Z}} \mathbb{Z}\mathbf{B}$ is an R-algebra for any ring R.

We can now consider representation theory over any ring or field, like $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{F}_{p^n}$, $\hat{\mathbb{Q}}_p$.

- If F is a field, $F\mathbf{B}$ will be semisimple when...(a condition reminiscent of Maschke's theorem)...holds.

- $\mathbb{Z}\mathbf{B}$ is a $\mathbb{Z}$-order in $\mathbb{Q}\mathbf{B}$, so we can consider units, maximal orders, zeta functions, etc.

II.2 Representation-theoretic questions

Question 6: Cyclotomic Eigenvalue Conjecture [Norton 1978]

Suppose $\mathbf{B}$ is the standard basis of a CC. Let $\chi \in Irr(\mathbb{C}\mathbf{B})$.
Then $\forall b_i \in \mathbf{B}$, $\chi(b_i)$ is an element of a cyclotomic extension of $\mathbb{Q}$.

• CEC does not hold for regular graphs of order ≥ 9, nor TAs of order ≥ 13.

• CEC would follow for CCs if it is shown for ASs, it was originally asked for commutative ASs.

• CEC holds for finite groups and ASs of rank 2 or 3 (easy!).

• [Hanaki-Uno 2006,Komatsu 2008] Number fields have been shown to exist that make it possible for CEC to fail for ASs of *prime* order.

$\mathbb{Q}\mathbf{B}$ is semisimple. Wedderburn decomp.? Schur indices $m(\chi)$?

Theorem 4 (H-Rahnamai Barghi 2011).

(i) If $\chi \in Irr(\mathbb{C}\mathbf{B})$ and $\mathbb{Q}(\chi) \subset \mathbb{R}$, then $m(\chi) \leq 2$.
(ii) The collection of Brauer equivalence classes of K-central simple algebras that occur as simple components of $\mathbb{Q}\mathbf{B}$ for some AS (or TA) $\mathbf{B}$ form a subgroup of $Br(K)$. (We call this the AS (or TA) Schur Subgroup of $Br(K)$).

Question 7:

Is there a good bound for $m(\chi)$ in terms of parameters of $\mathbf{B}$?

Theorem 5 (H 2019).

If $\mathbf{B}$ has $r-2$ symmetric elements, then $m(\chi) = 1$ for every $\chi \in Irr(\mathbb{C}\mathbf{B})$.

II.4 Modular representation theory

Suppose F is a finite field of characteristic p for which $F\mathbf{B}$ is not semisimple, and let (K, R, F) be a splitting p-modular system.

- [Hanaki 2002] If $o(\mathbf{B}) = p$ then $F\mathbf{B}$ is a local ring.

Question 8: [Hanaki 2009]

Let S be an AS.
(a) Blocks of FS depend on p *and* the choice of modular system (not just on p). Is there a defect theory for blocks of FS?
(b) FS need not be a symmetric algebra, but its Cartan matrix is symmetric. What associative algebras can be blocks of an FS?
(c) When does FS have finite representation type?

II.5 Units of $\mathbb{Z}\mathbf{B}$?

Let $\mathbf{B}$ be an AS or the standard basis of an integral TA.

Theorem 6 (H-Singh 2017).

Central torsion units of $\mathbb{Z}\mathbf{B}$ are trivial (i.e. $u = \pm b_i$ for some $b_i \in \mathbf{B}$ with $\delta(b_i) = 1$).

Question 9:

If $\mathbf{B}$ is a commutative TA, find the rank of the torsion-free part of $U(\mathbb{Z}\mathbf{B})$ in terms of parameters of $\mathbf{B}$.

Normalized units have valency $\delta(u) = 1$. Is there a partial augmentation condition for normalized torsion units of $\mathbb{Z}\mathbf{B}$?

II.6 The Zeta function of $\mathbb{Z}\mathbf{B}$?

Recall that the *zeta function* of a module M is the function $\zeta_M(s) = \sum_n a_n s^{-n}$, where a_n is the number of $\mathbb{Z}$-submodules of M with index n for all $n \geq 1$.

Theorem 7.

The zeta functions of the order $\mathbb{Z}\mathbf{B}$ have been computed for the following ASs:

- *[Solomon 1977] $\mathbf{B} = C_p$, a cyclic group of prime order.*
- *[Reiner 1980] $\mathbf{B} = C_{p^2}$.*
- *[Hironaka 1981/1985] $\mathbf{B} = C_p \rtimes C_q$, $p \neq q$ distinct primes.*
- *[Takegahara 1987] $\mathbf{B} = C_p \times C_p$.*
- *[Hanaki-Hirasaka 2015] $\mathbf{B} = K_n$.*
- *[Hanaki-Hirasaka 2015] $o(\mathbf{B}) = p$ a prime.*
- *[H-Hirasaka-Oh 2017] $\mathbf{B} = \mathbf{C} \times \mathbf{D}$, where the orders $\mathbb{Z}\mathbf{C}$ and $\mathbb{Z}\mathbf{D}$ are "locally coprime".*

Suppose $\mathbf{S}$ is the standard basis of an AS (or integral TA).

A sub-AS is a *closed subset* $\mathbf{T}$ of $\mathbf{S}$, satisfying $\mathbf{T}^* = \mathbf{T}$, $\mathbf{T}^2 \subset \mathbb{N}\mathbf{T}$, and $\dim \mathbb{C}\mathbf{T} = |\mathbf{T}|$.

The order of the sub-AS is $o(\mathbf{T}) = \sum_{t \in \mathbf{C}} \delta(t)$.

Theorem 8 (Zieschang 1996).

Suppose $\mathbf{T}$ *is a sub-AS of* $\mathbf{T}$.
(i) $o(\mathbf{T})$ *divides* $o(\mathbf{S})$.
(ii) $\mathbf{S}^+$ *is partitioned into left* $\mathbf{T}$*-cosets* $(s\mathbf{T})^+$
(using what is analogous to support in group rings)
(iii) $\mathbf{S}^+$ *is partitioned into* $\mathbf{T}$*-*$\mathbf{T}$*-double cosets* $(\mathbf{T}s\mathbf{T})^+$.

Let **S** be the standard basis of an AS or integral TA.
S is *primitive* if it contains no closed subsets other than $\{b_0 = I\}$ and itself.
The only primitive finite groups are the C_p's where p is prime.
[Hanaki-Uno 2006] Any AS of prime order p will be primitive (and commutative and of rank dividing $p-1$).

[H-Muzychuk-Xu 2017,2018] Every noncommutative TA of rank 5 is primitive. There are primitive noncommutative TAs of rank 6.

Question 10: [Muzychuk]

Is there a primitive noncommutative AS?

Let $\mathbf{S}$ be an AS or the standard basis of an integral TA.

Theorem 9 (Zieschang 1996, Blau 2009).

Let $\mathbf{T}$ *be any closed subset of* $\mathbf{S}$*.*
Then $\mathbf{S}/\!/\mathbf{T} = \{o(\mathbf{T})^{-1}(\mathbf{T}s\mathbf{T})^{+} : s \in \mathbf{S}\}$ *is the set of standard matrices of an AS of order* $o(\mathbf{S})/o(\mathbf{T})$*.*

Definition 10.

An *AS (or TA) homomorphism* $\phi : \mathbf{S} \to \mathbf{U}$ is determined by a composition which (up to positive scalars) takes
$\mathbf{S} \twoheadrightarrow \mathbf{S}/\!/\mathbf{T} \twoheadrightarrow \mathbf{V} \hookrightarrow \mathbf{U}$,
for some closed subset $\mathbf{T}$ of $\mathbf{S}$ ($\mathbf{T} = \ker\phi$)
and some closed subset $\mathbf{V}$ of $\mathbf{U}$.

AS homomorphic images of finite groups G produce *double coset algebras* $\mathbb{C}[G/\!/H]$. These are known as *Schurian ASs*.

III.3 Let's do algebra: ...extensions...

Suppose $\mathbf{B}$ and $\mathbf{C}$ are two ASs (or TA). The following constructions give new ASs (or TAs):

- (direct product) $\mathbf{B} \times \mathbf{C} = \{b_i \otimes c_j\}$
- (wreath product) $\mathbf{B}\, wr\, \mathbf{C} = \{b_i \otimes c_0, \mathbf{B}^+ \otimes c_j : (j \neq 0)\}$
- (semidirect product) $\mathbf{B} \rtimes G$ for (a group Hom) $G \to Aut(\mathbf{B})$
- (wedge/circle product) [Arad-Fisman 1992, Blau 2015] $\mathbf{B} \vee \mathbf{C} = \mathbf{B} \cup (\mathbf{C} - \{c_0\})$
- (wedge-direct product) [Xu 2019] $\mathbf{B} \boxtimes_\phi \mathbf{C} \simeq \ker\phi \oplus \mathbf{C}$

A *fusion* of a TA $\mathbf{B}$ is a TA $\mathbf{D}$ obtained from the characteristic functions of a $*$-compatible partition of $\mathbf{B}$, with one of the subsets of the partition being $\{b_0\}$.

- Fusions of thin ASs (group rings) are precisely *unital Schur rings*.

Remark: The "$*$-compatibility condition" on the partition making the fusion cannot be dropped. For example, in the group algebra $\mathbb{C}S_3$, the set
$\mathbf{D} = \{(1), (1,2,3) + (1,2), (1,3,2) + (1,3), (2,3)\}$
generates a 4-dimensional subalgebra $\mathbb{C}\mathbf{D}$ which is *not* semisimple! This is an example of what we call a *semi-fusion*.

Remark: ASs can be divided into three types:
(i) Schurian ASs = AS homomorphic images of finite groups
(ii) Fusions of Schurian ASs = unital and non-unital Schur rings
(iii) ASs with intransitive automorphism groups. These are homogeneous fusions of CCs with no intermediate AS fusion that are not included in (i) and (ii).

Remark: The CEC holds for (i) but is open for (ii) and (iii).

Question 11:

(i) Does CEC hold for Schur rings that are fusions of non-Abelian finite groups?
(ii) Can an element of a Schur ring basis have noncyclotomic eigenvalues?

The End

Thank You!