Computing Wedderburn decomposition using the concept of Shoda pairs

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1 Introduction

1.1 Shoda pairs

Theorem 1.1 (Shoda's irreducibilty criterion) *Let χ be a linear character defined on a subgroup H of G. Then the induced character χ^G is irreducible, if, and only if, for every $g \in G \setminus H$, there is an element $h \in H \cap H^g$ such that $\chi(ghg^{-1}) \neq \chi(h)$.*

Definition 1.2 (Shoda pair-rephrasing Shoda's criterion in group theoretic terms) *A pair (H, K) of subgroups of G satisfying (i)-(iii) is called a* Shoda pair *of G.*

(i) $K \trianglelefteq H$,

(ii) H/K *is cyclic,*

(iii) *if $g \in G$ and $[H, g] \cap H \subseteq K$, then $g \in H$.*

Theorem 1.3 (PCI associated to a Shoda pair) *Let (H, K) be a Shoda pair and let χ is a linear character of a subgroup H of G with kernel K, then*

$$e_{\mathbb{Q}}(\chi) = \alpha e(G, H, K),$$

for some $\alpha \in \mathbb{Q}$ (known), where $e(G, H, K)$ is the sum of distinct G-conjugates of $\varepsilon(H, K)$ given by

$$\varepsilon(H, K) := \begin{cases} \hat{H}, & H = K; \\ \prod(\hat{K} - \hat{L}), & \text{otherwise}, \end{cases}$$

where L runs over the normal subgroups of H which are minimal over the normal subgroups of H containing K properly.

Remark 1.4 *If χ is the trivial character of G, then $e_{\mathbb{Q}}(\chi) = \hat{G}$.*

Furthermore, $\alpha = 1$, if the distinct G-conjugates of $\varepsilon(H, K)$ are mutually orthogonal.

This led to the definition of strong Shoda pair.

Definition 1.5 (Strong Shoda pair) *A strong Shoda pair of G is a pair (H, K) of subgroups of G with the theoremerty that*

- **(i)** $K \trianglelefteq H \trianglelefteq N_G(K)$;
- **(ii)** *H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$;*
- **(iii)** *the distinct G-conjugates of $\varepsilon(H, K)$ are mutually orthogonal.*

Remark 1.6 *A strong Shoda pair is indeed a Shoda pair.*

Theorem 1.7 *[Simple component associated to strong Shoda pair]Let (H, K) be a strong Shoda pair and $k = [H : K]$, $N = N_G(K)$, $n = [G : N]$, x a generator of H/K and $\phi : N/H \mapsto N/K$ a left inverse of the projection $N/K \mapsto N/H$. Then,*

$$\mathbb{Q}Ge(G, H, K) \cong M_n(\mathbb{Q}(\zeta_k) *^{\sigma}_{\tau} N/H),$$

where the action σ and the twisting τ are given by $\zeta_k^{\sigma(a)} = \zeta_k^i$, if $x^{\phi(a)} = x^i$; $\tau(a, b) = \zeta_k^j$, if $\phi(ab)^{-1}\phi(a)\phi(b) = x^j$, for $a, b \in N/H$ and integers i and j.

Remark 1.8 **(i)** *The action σ in above theorem is always faithful due to the fact that H/K is a maximal abelian subgroup of N/K.*

(ii) *In case the twisting τ is trivial, then $\mathbb{Q}(\zeta_k) *^{\sigma}_{\tau} N/H \cong M_m(F)$, where $m = [N : H]$ and F is the fixed subfield of $\mathbb{Q}(\zeta_k)$ under the action of N/H. Consequently, we have in this case, $\mathbb{Q}Ge(G, H, K) \cong M_{mn}(F)$.*

A strong Shoda pair (H, K) of G with H normal in G has even easier description.

Definition 1.9 *A pair (H, K) of subgroups of G is called an extremely strong Shoda pair, if it satisfies*

(i) $K \trianglelefteq H \trianglelefteq G$,

(ii) *H/K is cyclic and a maximal abelian subgroup of $N_G(K)/K$.*

An extremely strong Shoda pair is indeed a strong Shoda pair and hence $e(G, H, K)$ is a primitive central idempotent of $\mathbb{Q}G$.

2 Shoda pairs (SP) do exist!

Example 2.1 $G = C_2 \times C_2 = \langle a, b\rangle$.

all possible pairs of subgroups (possible candidates for SP):

$\{(G, G), (G, \langle a\rangle), (G, \langle b\rangle), (G, \langle ab\rangle), (G, \langle 1\rangle)\}$

SP

$\{(G, G), (G, \langle a\rangle), (G, \langle b\rangle), (G, \langle ab\rangle), (G, \langle 1\rangle)\}$.

Example 2.2 $G = C_4 = \langle a\rangle$.

pairs of subgroups (possible candidates for SP):

$\{(G, G), (G, \langle a^2\rangle), (G, \langle a^4\rangle), (G, \langle 1\rangle), (\langle a^2\rangle, \langle 1\rangle)\}$

SP

$\{(G, G), (G, \langle a^2\rangle), (G, \langle a^4\rangle), (G, \langle 1\rangle), (\langle a^2\rangle, \langle 1\rangle)\}$

Observations

1. (G, G) is always an ESSP and hence a SP. What is the simple component associated to (G, G)?
2. (H, H) is an ESSP if and only if $H = G$.
3. If $H = G$, then (G, K) is an ESSP of G if and only if $K \trianglelefteq G$ and G/K is cyclic.
4. If G is abelian then all ESSPs are of the form (G, K) such that G/K is cyclic. In fact calculating simple component for each of these pairs yield Perlis Walker theorem.

Example 2.3 $G = S_3 = D_6 := \langle a, b | a^3 = b^2 = 1, a^b = a^-1 \rangle$.

Direct check as above yields the following set of ESSPs

$$\{(G, G), (G, \langle a \rangle), (\langle a \rangle, \langle 1 \rangle)\}$$

Example 2.4 $G = D_8 := \langle a, b | a^4 = b^2 = 1, a^b = a^-1 \rangle$.

Repeating above process yields the following set of ESSPs

$$\{(G, G), (G, \langle a^2, ab \rangle), (G, \langle a^2, b \rangle), (G, \langle a \rangle), (\langle a^2, ab \rangle, \langle ab \rangle), (\langle a^2, ab \rangle, \langle a^3 b \rangle), (\langle a^2, b \rangle, \langle b \rangle), (\langle a^2, b \rangle, \langle a^2 b \rangle), (\langle a \rangle, \langle 1 \rangle)\}$$

Count the number of ESSPs obtained for each group. Do you see a problem?

3 Some obvious questions

Question 3.1 *Is their a refined method to find the set of ESSPs of G.*

Question 3.2 *Each ESSP yields a simple component. How to choose a subset of ESSPs, so as to obtain exact Wedderburn decomposition?*

Question 3.3 *Are these all the Shoda pairs/strong Shoda pairs?If yes, for what groups. If not, how do we find the remaing?*

4 Answers to the questions above

Algorithm to find a set $\mathcal{S}_G$ of ESSPs of G, such that

$$\mathbb{Q}G \cong \oplus A_{(H,K)},$$

where $A_{(H,K)}$ is simple component given by Theorem 1.7

Step 1 Find $\mathcal{N}$: the set of all the distinct normal subgroups of G.

Step 2 For $N \in \mathcal{N}$, let A_N be a normal subgroup of G containing N such that A_N/N is an abelian normal subgroup of maximal order in G/N [The choice of A_N

is not unique but choice does not affect the output, fix one].

Step 3 For a fixed A_N, set

$\mathcal{D}_N$: the set of all subgroups D of A_N containing N such that $\text{core}_G(D) = N$, A_N/D is cyclic and is a maximal abelian subgroup of $N_G(D)/D$.

$\mathcal{T}_N$: a set of representatives of $\mathcal{D}_N$ under the equivalence relation defined by conjugacy of subgroups in G.

$\mathcal{S}_N$: $\{(A_N, D) \mid D \in \mathcal{T}_N\}$.

$$\mathcal{S}_G := \cup_{N \in \mathcal{N}} \mathcal{S}_N.$$

Apparently, it is a big algorithm, as it is for an arbitrary finite group. There are quick reductions for applications. For instance,

- If $N \in \mathcal{N}$ is such that G/N is abelian, then

$$\mathcal{S}_N = \begin{cases} \{(G, N)\}, & \text{if } G/N \text{ is cyclic;} \\ \emptyset, & \text{otherwise.} \end{cases} \tag{1}$$

- If G/N is non abelian and A_N/N is cyclic, then

$$\mathcal{S}_N = \begin{cases} \{(A_N, N)\}, & \text{if } A_N/N \text{ is a maximal abelian subgroup of } G/N; \\ \emptyset, & \text{otherwise.} \end{cases}$$

- If $N \in \mathcal{N}$ is such that the centre of G/N is not cyclic, then $\mathcal{S}_N = \emptyset$.
- If $\mathcal{M} \subseteq \mathcal{N}$ is such that

$$\sum_{N \in \mathcal{M}} \sum_{(A_N, D) \in \mathcal{S}_N} dim(A_N, D) = |G|,$$

then $\mathcal{S}_N = \emptyset$ for all $N \in \mathcal{N} \setminus \mathcal{M}$.

Groups for which the set of ESSPs suffices

Theorem 4.1 **(i)** *If G is a normally monomial group, then $\mathcal{S}_G := \cup_{N \in \mathcal{N}} \mathcal{S}_N$ is a complete irredundant set of Shoda pairs/strong Shoda pairs of G.*

(ii) *$\{e(G, A_N, D) \mid (A_N, D) \in \mathcal{S}_N,\ N \in \mathcal{N}\}$ is the complete set of primitive central idempotents of $\mathbb{Q}G$, if, and only if, G is normally monomial.*

(iii) *A finite group G is normally monomial if, and only if,*

$$|G| = \sum_{N \in \mathcal{N}} \sum_{D \in \mathcal{D}_N} [G : A_N]\varphi([A_N : D]). \tag{2}$$

But, what if a group is not normally monomial?

Monomial groups, strongly and normally monomial groups.

Recall that a group G is *monomial*, if all its complex irreducible characters are monomial.

Theorem 4.2 (Shoda pairs and monomial groups) *A finite group G is monomial if, and only if, every primitive central idempotent of $\mathbb{Q}G$ is of the form $\alpha e(G, H, K)$, for $\alpha \in \mathbb{Q}$ and a Shoda pair (H, K) of G.*

A group G is *normally monomial*, if all its complex irreducible characters are normally monomial, i.e., induced from a linear character of a normal subgroup of G.

Theorem 4.3 (Extremely strong Shoda pairs and normally monomial groups) *A finite group G is normally monomial if, and only if, every primitive central idempotent of $\mathbb{Q}G$ is of the form $e(G, H, K)$, for an extremely strong Shoda pair (H, K) of G.*

Therefore one can construct all primitive central idempotents of $\mathbb{Q}G$ from extremely strong Shoda pairs of G.

And, a group G is *strongly monomial*, if each primitive central idempotent of $\mathbb{Q}G$ is of the form $e(G, H, K)$, for a strong Shoda pair (H, K) of G.

5 Some associated `GAP` (`wedderga`) commands

```
# G-a group, (H,K) a pair of subgroups of $G$, QG-rational group algebra of $G$.
IsShodaPair(G,H,K);
IsStrongShodaPair(G,H,K);
IsExtremelyStrongShodaPair(G,H,K);
IsMonomial(G);
IsStronglyMonomial(G);
```

```
IsNormallyMonomial(G);
StrongShodaPairs(G);
ExtremelyStrongShodaPairs(G);
# ShodaPairs(G); No
ShodaPairsAndIdempotents(QG);
IsCompleteSetOfOrthogonalIdempotents(R,list); #R a unital ring.
PrimitiveCentralIdempotentsByStrongSP(QG);
# There is also PrimitiveCentralIdempotentsByCharacterTable(QG);
PrimitiveCentralIdempotentsByESSP(QG);
Idempotent_eGsum(QG,G,H)[2];
CrossedProduct( <R>, <G>, act, twist );
WedderburnDecomposition(FG);
WedderburnDecompositionInfo(FG);
```

6 Exercises

1. (c.f. Exercises of GA tutorial) Compute for some groups of small order a complete set of primitive central idempotents and a complete set of primitive idempotents of $\mathbb{Q}G$. Do it, using above process and for groups considered above, i.e., $C_4, C_2 \times C_2, S_3, D_4$ and also write the Wedderburn decomposition of their rational group algebras.

2. Abelian-by-supersolvable groups are strongly monomial. Using `GAP` or otherwise, find a strongly monomial group which is not abelian by supersolvable.

3. Find a group of least order, which is not strongly monomial.

4. Metabelian groups are normally monomial. Using `GAP` or otherwise, find a normally monomial group which is not metabelian.

5. Find a group of least order, which is not normally monomial.

6. Find a group of least order, which is strongly monomial but not normally monomial.

7. Find a group of least order, which is monomial but not strongly monomial.