(1) Prove that if A and B are $n \times n$ matrices with entries in a commutative ring R then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Deduce that if A and B are conjugate in $M_n(R)$ then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

(2) Let A be an $m \times m$ matrix with entries in the field of complex numbers. Prove that

  (a) If $A^n = I$ then A is diagonalizable and the eigenvalues of A are n-th roots of unity. Deduce that $|\mathrm{tr}(A)| \leq m$ and if the equality holds then A is a scalar matrix.

  (b) If $A^2 = A$ then A is diagonalizable and every eigenvalue of A is either 1 or 0. Deduce that $\mathrm{tr}(A)$ is the rank of A.

(3) Let R be a ring, G be a group and let N, N_1 and N_2 be normal subgroups of G with $N_1 \subseteq N_2$. Let $\Phi : RG/N_2 \to R\left(\frac{G/N_1}{N_2/N_1}\right)$ be the R-linear extension of the natural isomorphism $G/N_2 \cong \frac{G/N_1}{N_2/N_1}$. Prove the following statements:

  (a) $\mathrm{Aug}_N(RG) = RG\,\mathrm{Aug}(RN) = \mathrm{Aug}(RN)RG$.

  (b) $\mathrm{Aug}(RG) = \bigoplus_{g \in G \setminus \{1\}} R(g-1)$.

  (c) $\Phi \circ \mathrm{aug}_{G,N_2} = \mathrm{aug}_{G/N_1, N_2/N_1} \circ \mathrm{aug}_{G,N_1}$.

  (d) $\mathrm{Aug}_{N_1}(RG) \subseteq \mathrm{Aug}_{N_2}(RG)$.

  (e) $V(RG, G) = V(RG)$, $V(RG, 1) = 1$ and $V(RG, N_1) \subseteq V(RG, N_2)$.

(4) Let F be a field of characteristic $p > 0$ and let G be a group. Prove the following statements:

  (a) If G is a p-group then $\mathrm{Aug}(FG)$ is the Jacobson radical of FG.

  (b) If P is a normal p-subgroup of G then $\mathrm{Aug}_P(FG)$ is nilpotent.

(5) Let G be a finite group, let p be a prime integer and let P be a normal p-subgroup of G. Prove that every torsion element of $V(\mathbb{Z}G, P)$ is a p-element.

(6) Prove that A and B are two abelian finite groups then $\mathbb{C}A \cong \mathbb{C}B$ if and only if A and B have the same cardinality.

(7) Let R be a commutative ring and let G and H be groups. Let $f : RG \to RH$ be a ring homomorphism. Prove that there is a unique ring homomorphism $f' : RG \to RH$ such that $f'(g) = \mathrm{aug}(f(g))^{-1} f(g)$ for every $g \in G$ and $\mathrm{aug}(f(x)) = \mathrm{aug}(x)$ for every $x \in RG$. Show that if f is an isomorphism then so is f'.

(8) Prove that the following statements are equivalent for a finite group G and a finite subgroup H of $V(\mathbb{Z}G)$:

  (a) $|H| = |G|$.

  (b) $\mathbb{Z}G = \mathbb{Z}[H]$. (Recall that $\mathbb{Z}[H]$ denotes the additive sugroup of $\mathbb{Z}G$ generated by H.)

  (c) H is an basis of $\mathbb{Z}G$ over $\mathbb{Z}$.

(9) Prove that the Isomorphism Problem holds for a finite group G if and only if every subgroup of $V(\mathbb{Z}G)$ with the same cardinality as G is isomorphic to G.

(10) Consider $S_3 = \langle a \rangle \rtimes \langle b \rangle$, the symmetric group on three symbols, where $a = (1,2,3)$ and $b = (1,2)$. Prove the following:

  (a) $\rho(a) = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$ and $\rho(b) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ defines an irreducible representation of S_3. (Hint: Let χ denote the character afforded by ρ. By Character Theory, ρ is irreducible if and only if $\sum_{g \in S_3} \chi(g)\chi(g^{-1}) = |S_3|$.)

  (b) Let $\phi : \mathbb{C}S_3 \to \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$ be the unique linear map with $\phi(g) = (1, \mathrm{sgn}(g), \rho(g))$ for $g \in S_3$. Then ϕ is a isomorphism of complex algebras and

$$\phi(\mathbb{Z}S_3) = \left\{ \left(x, y, \begin{pmatrix} a & 3b \\ c & d \end{pmatrix} \right) : x, y, a, b, c, d \in \mathbb{Z}, \begin{array}{c} x \equiv y \bmod 2, \\ x \equiv a \bmod 3, \\ y \equiv d \bmod 3 \end{array} \right\}.$$

(c) $V(\mathbb{Z}S_3)$ contains an element u of order 2 with $\phi(u) = (1, -1, \text{diag}(1, -1))$.
(d) u is not conjugate in the units of $\mathbb{Z}S_3$ to any element of S_3.
(e) Every torsion element of $V(\mathbb{Z}S_3)$ is conjugate in the units of $\mathbb{Q}S_3$ to an element of S_3.

(11) Let p and n be positive integers with p prime integer. Prove the following statements:
(a) If x and y are elements of a ring R then

$$(x+y)^{p^n} \equiv x^{p^n} + y^{p^n} \mod (pR + [R, R]).$$

Moreover, if $x \in [R, R]$ then $x^p \in pR + [R, R]$.
(b) If G is a finite group and u is a torsion element of $V(\mathbb{Z}G)$ then $|u| = p^n$ if and only if $\varepsilon_{G[p^n]}(u) \not\equiv 0 \mod p$.

(12) [Cohn-Livingstone] If G is a finite group then $V(\mathbb{Z}G)$ and G have the same primary spectrum, i.e. for every prime and every positive integer G contains an element of order p^n if and only if so does $V(\mathbb{Z}G)$. In particular, the least common multiple of the orders of the torsion elements of $V(\mathbb{Z}G)$ is the exponent of G, i.e. the smallest positive integer e such that $g^e = 1$ for every $g \in G$.

(13) Let G a finite group and let u be an element of prime order p in $V(\mathbb{Z}G)$. Prove that if all the elements of order p in G are conjugate then u is rationally conjugate to an element of G. Conclude that, in general, G is a subgroup of a group H such that u is conjugate in $\mathbb{Q}H$ to an element of G.

(14) Use the Luthar-Passi Method to prove that (ZP1) has a positive solution for S_3, A_4, S_4 and A_5.

(15) Prove that the Luthar-Passi Method does not provide a positive answer to (ZP1) for A_6. (Hint: First prove that every torsion element of $V(\mathbb{Z}A_6)$ of order 2 or 3 is rationally conjugate to an element of A_6 and then apply the Luthar-Passi Method to units of order 6.)