A New Stochastic Schrödinger-Newton Equation

Sayantani Bera

Tata Institute of Fundamental Research, Mumbai

@FPQP 2016, ICTS Bangalore

Outline

- Stochastic gravity - Why we need it in the context of *the measurement problem*
- Gravitational backreaction in semiclassical gravity : R. Penrose
  - Semiclassical Einstein's equation
  - The Schrödinger-Newton Equation
  - Drawbacks
- A stochastic gravity model: L. Diósi
  - A brief description
  - Diósi-Penrose criterion
- Unification of the two ideas
  - A unified stochastic description
  - Evolution of Double-Gaussian wave packets
  - Results
- Scopes of Improvement

Why Stochastic gravity?

- Gravity a universal force.
- must possess an intrinsic fluctuation in its structure due to the uncertainty principle.
- The wavefunctions of micro/macro objects propagate through the spacetime fabric.

Ideas due to F. Karolyhazy, Frenkel, L. Diósi, R. Penrose

A possible solution to the "**Measurement Problem**" ?

Gravitational backreaction : Penrose

The Schrödinger equation for a particle should also take into account the effect of "self-gravity" on its wavefunction evolution

- an idea put forward by Roger Penrose in the context of wavefunction collapse.

Penrose's hypotheses

- The Schrödinger equation should contain a term which represents the gravitational backreaction - the Schrödinger-Newton Equation.
- The final collapsed state will be one of the stationary state solutions of this Schrödinger-Newton equation.
- The inclusion of the backreaction term makes the Schrödinger equation non-linear and deterministic, thus breaking the linear superposition.
- The collapse time will be inversely proportional to the gravitational energy difference of the two states - Diósi-Penrose criterion

Framework

→ The background spacetime is treated as classical, and matter sources are quantum.

→ The energy-momentum tensor, in this formalism, becomes a quantum operator $\hat{T}_{\mu\nu}$

→ The R.H.S of the Einstein's field equation is replaced by the quantum expectation value of the energy-momentum operator i.e.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \langle \hat{T}_{\mu\nu} \rangle$$

The Schrödinger-Newton Equation

- Perturbed metric in the linearized gravity regime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}; \qquad |h_{\mu\nu}| \ll 1 \tag{1}$$

- The perturbed field equation upto first order in h :

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(\langle\Psi|\hat{T}_{\mu\nu}|\Psi\rangle - \frac{1}{2}\eta_{\mu\nu}\langle\Psi|\hat{T}_{\alpha\beta}\eta^{\alpha\beta}|\Psi\rangle\right) \tag{2}$$

- In the non-relativistic limit, we recover Poisson's equation

$$\nabla^2 V = \frac{4\pi G}{c^2}\langle\Psi|\hat{T}_{00}|\Psi\rangle \tag{3}$$

- "Schrödinger-newton Equation"(SNE):

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) - Gm^2\int\frac{|\Psi(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\Psi(\mathbf{r},t) \tag{4}$$

Drawbacks of Penrose's hypothesis

- SNE is not exactly solvable analytically, only numerical solutions with certain assumptions has been possible.
- The final stationary states to which the wavefunction collapses is not derivable.
- Not compatible with collapse scenario.
- SNE results in a non-linear evolution of the density matrix which allows faster-than-light signal propagation.
- SNE is a deterministic equation, which can cause the localization of wavefunction, but the random collapse to one of the stationary states can only be explained if one introduces stochasticity.
- Born probability rule cannot be recovered.

Model of Diósi

- The gravitational potential/field of an object cannot be measured better than $\rightarrow (\delta\tilde{\mathbf{g}})^2 \geq G\hbar/VT$.
- Due to the stochastic gravitational background, the Schrödinger equation of the object becomes stochastic.

• Schrödinger equation for a mass distribution in Diósi model:

$$i\hbar\dot{\psi}(t) = \left(\hat{H}_0 + \int \phi(\mathbf{r},t)\hat{f}(\mathbf{r})d^3r\right)\psi(t) \tag{5}$$

• Noise two point correlation shown to be white noise:

$$\langle\phi(\mathbf{r},t)\phi(\mathbf{r}',t')\rangle = \frac{\hbar G}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t') \tag{6}$$

[*L. Diósi and B. Lukács, Annalen der Physik* **44**, *488 (1987)*]

Diósi-Penrose criterion

$$[\tau_d(X, X')]^{-1} = \frac{G}{2\hbar} \int \int d^3r d^3r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot [f(\mathbf{r}|X) - f(\mathbf{r}|X')][f(\mathbf{r}'|X) - f(\mathbf{r}'|X')] \tag{7}$$

An alternative form:

$$[\tau_d(X, X')]^{-1} = \frac{\Delta E_g}{2\hbar}$$

Unification of the two ideas

- A new stochastic SN equation for considering the joint effect of self-gravity and the extrinsic spacetime fluctuations.
- New because unlike the previous proposals, the stochasticity does not arise due to quantum two-point fluctuations of the energy-momentum tensor operator.
- A combination of the SN equation and the Diósi model, with the matter distribution $f(\mathbf{r}'|X)$ now replaced by the quantum mass density $m|\Psi|^2$.

We propose the following form of stochastic SN equation:

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) - \left[\underbrace{Gm^2\int\frac{|\Psi(\mathbf{r}',t)|^2}{|\mathbf{r}'-\mathbf{r}|}d^3r'}_{\text{usual SN term}} + \underbrace{m\int|\psi_{\mathbf{r}}(\mathbf{r}',t)|^2\;\phi(\mathbf{r}',t)\;d^3r'}_{\text{stochastic contribution}}\right]\Psi(\mathbf{r},t)$$

Phase variance: A mathematical tool to calculate the decoherence time

Karolyhazy's prescription

- Approximate solution : $\Psi(\mathbf{r},t) = \psi(\mathbf{r},t)\exp[i\Phi_{st}(\mathbf{r},t)]$
- The phase is given by: $\Phi_{st}(\mathbf{r},t) = -\frac{1}{\hbar}\int_0^t V(\mathbf{r},t')dt'$
- Stochastic variance of the phase difference between the two configurations:
$$\Delta\Phi^2 = \langle[\Phi_{st}(\mathbf{r_1},t) - \Phi_{st}(\mathbf{r_2},t)]^2\rangle_s$$
- $\Delta\Phi^2$ too large ($\gg \pi^2$) $\rightarrow$ coherence is lost.
- Length scale corresponding to $\Delta\Phi^2 \sim \pi^2$ is called the "coherence length"(a_c).
- The "decoherence time"(τ) is of the order of $ma_c^2/\hbar$.

Decoherence time in stochastic SN

- The phase depends on the perturbing potential and is given as
  Note that $V(\mathbf{r}, t)$ has a form

$$V(\mathbf{r}, t) = -Gm^2 \int \frac{|\psi_{\mathbf{r}}(\mathbf{r}', t)|^2}{|\mathbf{r} - \mathbf{r}'|} d^3r' + m \int |\psi_{\mathbf{r}}(\mathbf{r}', t)|^2 \phi(\mathbf{r}', t)\ d^3r' \tag{8}$$

- The phase variance between two wavepackets at $\mathbf{r_1}$ and $\mathbf{r_2}$ is given by,

$$\begin{aligned} \Delta\Phi^2 = \quad & \frac{m^2}{\hbar^2} \Big[\int \langle \phi(\mathbf{r}', t')\phi(\mathbf{r}'', t'') \rangle\ |\psi_{\mathbf{r_1}}(\mathbf{r}', t')|^2\ |\psi_{\mathbf{r_1}}(\mathbf{r}'', t'')|^2 \\ & + \int \langle \phi(\mathbf{r}', t')\phi(\mathbf{r}'', t'') \rangle\ |\psi_{\mathbf{r_2}}(\mathbf{r}', t')|^2\ |\psi_{\mathbf{r_2}}(\mathbf{r}'', t'')|^2 \\ & -2 \int \langle \phi(\mathbf{r}', t')\phi(\mathbf{r}'', t'') \rangle\ |\psi_{\mathbf{r_1}}(\mathbf{r}', t')|^2\ |\psi_{\mathbf{r_2}}(\mathbf{r}'', t'')|^2\ \Big] d^3r'\ d^3r''\ dt'\ dt'' \end{aligned}$$

Decoherence Time

We now calculate the decoherence time in case of two Gaussian wavepackets peaked at $\mathbf{r_1}$ and $\mathbf{r_2}$.

- The Gaussian wavepacket of width a centred at $\mathbf{r_1}$ is given by:

$$\psi_{\mathbf{r_1}}(\mathbf{r}',t) = (\pi a^2)^{-3/4}\left(1+\frac{i\hbar t}{ma^2}\right)^{-3/2}\exp\left(-\frac{|\mathbf{r}'-\mathbf{r_1}|^2}{2a^2(1+\frac{i\hbar t}{ma^2})}\right) \quad (9)$$

- Corresponding probability density is,

$$|\psi_{\mathbf{r_1}}(\mathbf{r}',t)|^2 = \frac{1}{(\pi C_1)^{3/2}}\exp\left(-\frac{|\mathbf{r}'-\mathbf{r_1}|^2}{C_1}\right) \quad (10)$$

with $C_1 = a^2\left(1+\frac{\hbar^2 t^2}{m^2 a^4}\right)$.

- With this form, the phase variance comes out to be

$$\Delta\Phi^2 = \frac{2\sqrt{2}\kappa}{\sqrt{\pi}}\frac{ma}{\hbar}\sinh^{-1}\left(\frac{\hbar T}{ma^2}\right) - \frac{2\kappa}{R}\int_0^T \mathrm{Erf}\left(\frac{R}{\sqrt{2C_1}}\right)dt \equiv I_7 - I_8 \quad (11)$$

where $\kappa = Gm^2/\hbar$ and $R = |\mathbf{r_1}-\mathbf{r_2}|$.

Small T limit:

- Decoherence between the two states takes place when $\Delta\phi^2 \sim \pi^2$ and the separation between the states is the critical length a_c, the decoherence time is T_d.
- For $T_d \ll ma^2/\hbar$, the calculation is trivial, and in this limit, we get T_d as,

$$T_d^{-1} \sim \frac{Gm^2}{\hbar}\left[\sqrt{\frac{2}{\pi}}\frac{1}{a} - \frac{1}{a_c}\mathrm{Erf}\left(\frac{\mathrm{a_c}}{\sqrt{2}\mathrm{a}}\right)\right] \tag{12}$$

- At this transition value, $T_d \approx ma_c^2/\hbar$, which gives a_c in terms of the parameters of the wavepacket.

$$a_c \sim \left(\frac{\hbar^2}{Gm^3}\right)^{1/4} a^{3/4}, \qquad \text{for} \qquad am^3 \gg \hbar^2/G \tag{13}$$

Large T limit

We define the following:

$$\beta(t) = \frac{R}{\sqrt{2}a}\frac{1}{\sqrt{1+\hbar^2 t^2/m^2 a^4}} \tag{14}$$

The phase variance can be written as,

$$\Delta\Phi^2 = \frac{4\kappa}{\sqrt{\pi}R}\int_0^T dt\,\left[\beta(t) - \int_0^{\beta(t)} dx\,\exp(-x^2)\right] \equiv \frac{4\kappa}{\sqrt{\pi}R}\int_0^T dt\, I(t) \tag{15}$$

Note: $\beta(0) = R/\sqrt{2}a \gg 1$
At $t = 0$ the Erf term can be ignored compared to the first term.
For $t = T \gg ma_c^2/\hbar$, we can write $\beta(T) \sim 0$.
Hence in this limit, the time integral is of the order of TR/a and thus we get,

$$T \sim \frac{\hbar a}{Gm^2}, \qquad a_c \sim \left(\frac{\hbar^2}{Gm^3}\right)^{1/2} a^{1/2}, \qquad \text{if} \qquad am^3 \ll \hbar^2/G \tag{16}$$

[S. Bera, P. Giri and T.P. Singh, arXiv:1608.02102]

Results and Scopes of improvement

- Introduced extrinsic stochastic fluctuations (described by Diósi's model) in the usual SN framework.
- Obtained analytical results for decoherence time for double-Gaussian state and recovered Diósi's results.

Issues to be addressed:

1. Solution to the full non-linear equation without a perturbative approach.
2. The issue of superluminal signalling.
3. Explanation for the collapse of the wavefunction in the stochastic SN scenario.
4. A natural derivation of the Born probability rule.

Thank You