Entropy and Geometry of Quantum States

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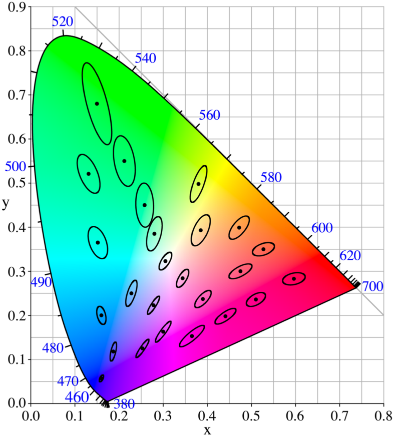
Overview

MacAdam ellipse



Distance Measures

What are distance measures?

Given 2 states, how distinct or similar are they?

What is a state?

Classical Mechanics: Probability distributions.
Quantum Mechanics: Density matrices.

Examples:

Fidelity, Trace distance, Hilbert-Schmidt norm, Relative Entropy, etc.

Relative Entropy (Kullback-Leibler Divergence)

Relative Entropy

$D(p||q) = \sum_i p_i \log\left(\frac{p_i}{q_i}\right), \; S(\rho||\sigma) = Tr\left[\rho \log \rho - \rho \log \sigma\right]$

Properties:

- Always non-negative: $D(p||q) \geq 0, \forall \{p_i, q_i\}$, $D(p||q) = 0 \iff p_i = q_i$
- Not symmetric: $D(p||q) \neq D(q||p)$

Likelihood Theory

The probability that a particular model distribution has generated the observed result.

$$L = \frac{n!}{\prod_i c^i!} \prod_i q_i^{c^i}, \; D(p||q) = -\frac{1}{n} \log_2 L$$

Hessian of Relative Entropy

- Consider 2 mixed states ρ_1 and ρ_2 in the space of density matrices.
- Lets take ρ_2 in the neighbourhood of ρ_1.

Taylor Expansion of Relative Entropy

$$S(\rho_1, \rho_2) = S(\rho_1, \rho_1) + \frac{\partial S}{\partial \rho_2^i}(\delta\rho) + \frac{\partial^2 S}{2\partial \rho_2^i \partial \rho_2^j}(\delta\rho)^2 + \ldots$$

$$g_{ij} = \frac{\partial^2 S}{\partial \rho_2^i \partial \rho_2^j}, \; ds^2 = g_{ij} dx^i dx^j$$

Distance Function for Qubits

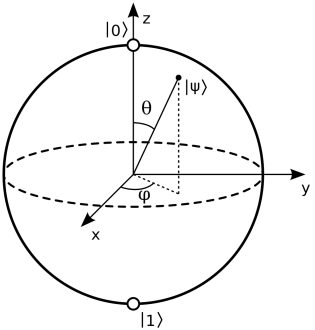
Let us consider the case of qubits, where the density matrices are given by,

$$\rho_1 = \frac{\mathbf{I} + \mathbf{n_1}.\boldsymbol{\sigma}}{2}, \;\; \rho_2 = \frac{\mathbf{I} + \mathbf{n_2}.\boldsymbol{\sigma}}{2}$$

Distance Function

$$ds^2 = d\alpha^2 + F(\alpha)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Where, $F(\alpha) = \frac{\sin\alpha}{2}\log\left[\frac{1+\sin\alpha}{1-\sin\alpha}\right]$



Figures

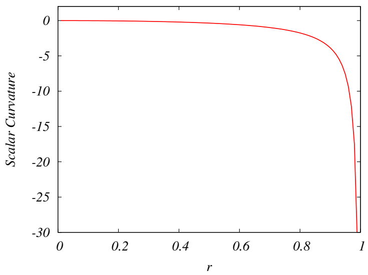


Figure: Scalar Curvature

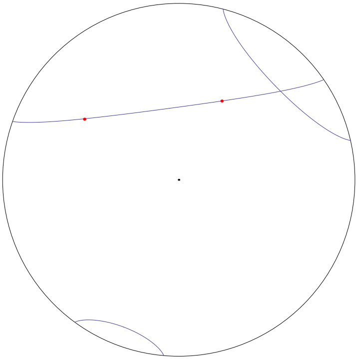


Figure: Geodesic passing through 2 specified points

Thermodynamic Implication Of the Metric

To have a physical understanding of this distinguishability measure we consider two thermal states

$$\rho_1 = \exp\{-\beta(H_1 - F_1)\} \quad \text{and} \quad \rho_2 = \exp\{-\beta(H_2 - F_2)\}$$

It can be shown that:

$$-TS(\rho_1 \| \rho_2) = (F_2 - F_1) - \epsilon\langle V\rangle \tag{1}$$

If ρ_1 and ρ_2 are close to each other we can perturbatively expand F_2 about F_1 and obtain

$$F_2 = F_1 + \epsilon F_1' + \frac{\epsilon^2}{2} F_1'' + \mathcal{O}(\epsilon^3) \quad \text{where} \quad F_1' = \langle V\rangle$$

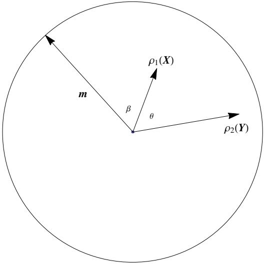
Substituting this in (1) we obtain:

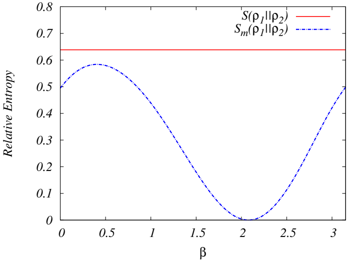
$$\boxed{g_{ij}\Delta\lambda^i\Delta\lambda^j = -\beta\epsilon^2 F_1''}$$

Distinguishability after Projective Measurements

- Let us consider the case in which we are given a string of n qubits, all in same state.
- We make projective measurement of the qubits along $\hat{m}$. This gives us the following probabilities for ρ_1 and ρ_2

$$p_{\pm} = \frac{1 \pm r_1 \cos\beta}{2}, \quad q_{\pm} = \frac{1 \pm r_2 \cos(\theta + \beta)}{2}$$





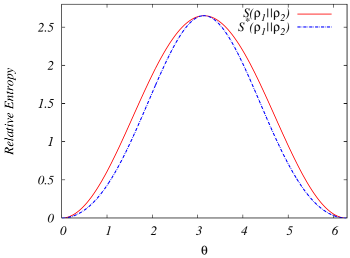
$$S_m(\rho_1 \| \rho_2) = p_+ \log \frac{p_+}{q_+} + p_- \log \frac{p_-}{q_-}$$

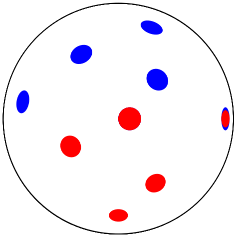
Optimising the Distinguishability w.r.t Measurement basis

Quantum Advantage

$$\Omega(\rho_1, \rho_2) = S(\rho_1, \rho_2) - S^*(\rho_1, \rho_2)$$

Since we are unable to optimise S^* analytically, we use infinitesimal case which leads us to Fisher-Rao metric.





BH metric

$$ds^2 = \frac{dr^2}{1 - r^2} + r^2 d\theta^2$$

Measurement on Multiple Qubits

- Can we beat the Cramer-Rao bound set by BH metric?
- We make measurements on 2 qubits at a time i,e. $\rho \otimes \rho$.
- We now construct a non separable basis $|b_1\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}$, $|b_2\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$, $|b_3\rangle = |++\rangle$ and $|b_4\rangle = |--\rangle$.

S	0.6385
S^*	0.5839
S_b	0.5856
S_{n2}	0.5863
S_{n3}	0.5880

Table: Variation of Relative Entropy

Possibility of Experiments using Cold Atoms

- Our work suggests that experimental realisations of the quantum advantage are within reach.
- The $|b_i\rangle$'s are related to the separable basis $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$ by a unitary transformation U in the four dimensional Hilbert space.
- One can equivalently apply U to the separable state $\tilde{\rho} = \rho \otimes \rho$.
- This creates an entangled state $U^\dagger \tilde{\rho} U$, which can then be measured in the separable basis using a projective measurement.

Thanks!