A Note on Entanglement Entropy, Coherent States and Gravity

Madhavan Varadarajan

Raman Research Institute, Bangalore, India

Introductory Remarks:

- Gravity: Described by Einstein's GR which interprets gravity as the geometry of spacetime. Einstein equations "(Curvature) = G (Matter Stress Energy).
- Freely falling ptcles and light rays move on trajectories determined by the geometry.
- We shall be interested in certain configurations of the gravitational field called Black Holes. Black Hole geometry ($\equiv$ gravitational field) is such that light rays cant escape from the inside the Hole geometry to infinity.
- Turns out that black holes display thermodynamic behavior. The geometric quantity which behaves like thermodynamic entropy is the Surface Area of the BH.
- Thermodyanamic entropy arises from statistical mechanics of underlying microscopic degrees of freedom.

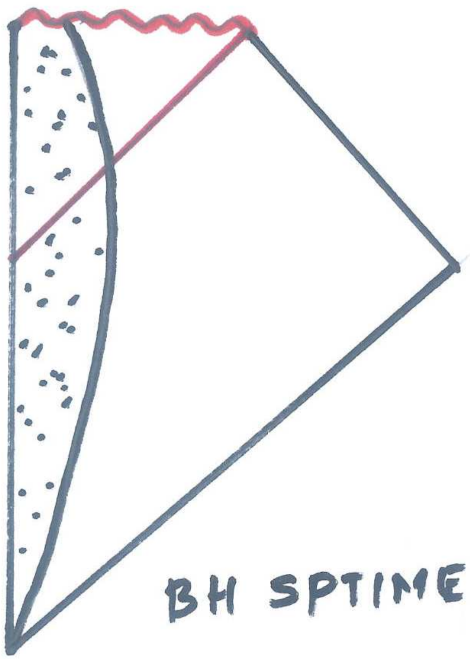
- Question What are the microscopic degrees of freedom which endow black hole configurations of the grav field with entropy equal to area?
- Suggested Answer (**Rafael Sorkin**): the degrees of freedom of quantum matter fields in the vicinity of the surface of the BH. These are entangled, across the surface of the black hole. Their Entanglement Entropy depends on the surface geometry, scales as surface area. However getting physically correct scaling require assumptions on unknown UV physics.
- In thermodynamic processes variations of macroscopic quantities are important. In BH case if variations are due to low energy processes, maybe one can show that variations of ent entropy = variations of BH area indep of any unknown UV physics (**Bianchi**).
- The entanglement entropy of Coherent States: an obstruction to Bianchi's ideas (MV).

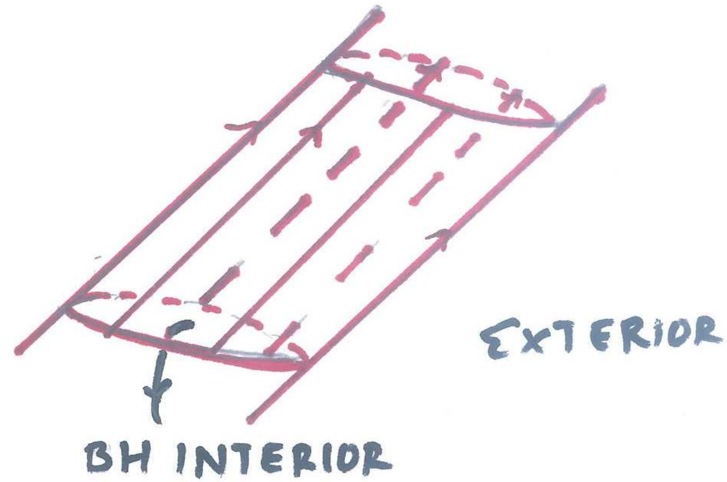
- One may also enquire whether not only black holes, but more general gravitational + matter field configurations exhibit thermodynamic features. One articulation of this question is:
  Can the Einstein equations themselves be interpreted as conditions of thermodynamic equilibrium ?
  Supportive Evidence: **Jacobson's** maximal vacuum entanglement entropy hypothesis.
  Coherent states also provide an obstruction to these ideas. (MV).

Plan of Talk:

- Review definition of black holes and some of their thermodynamic properties.
- Rafael's proposal, its dependence on unknown UV physics and Bianchi's ideas.
- Coherent states as an obstruction to Bianchi's ideas.
- If time, comment on Jacobson's ideas.

Black Hole:

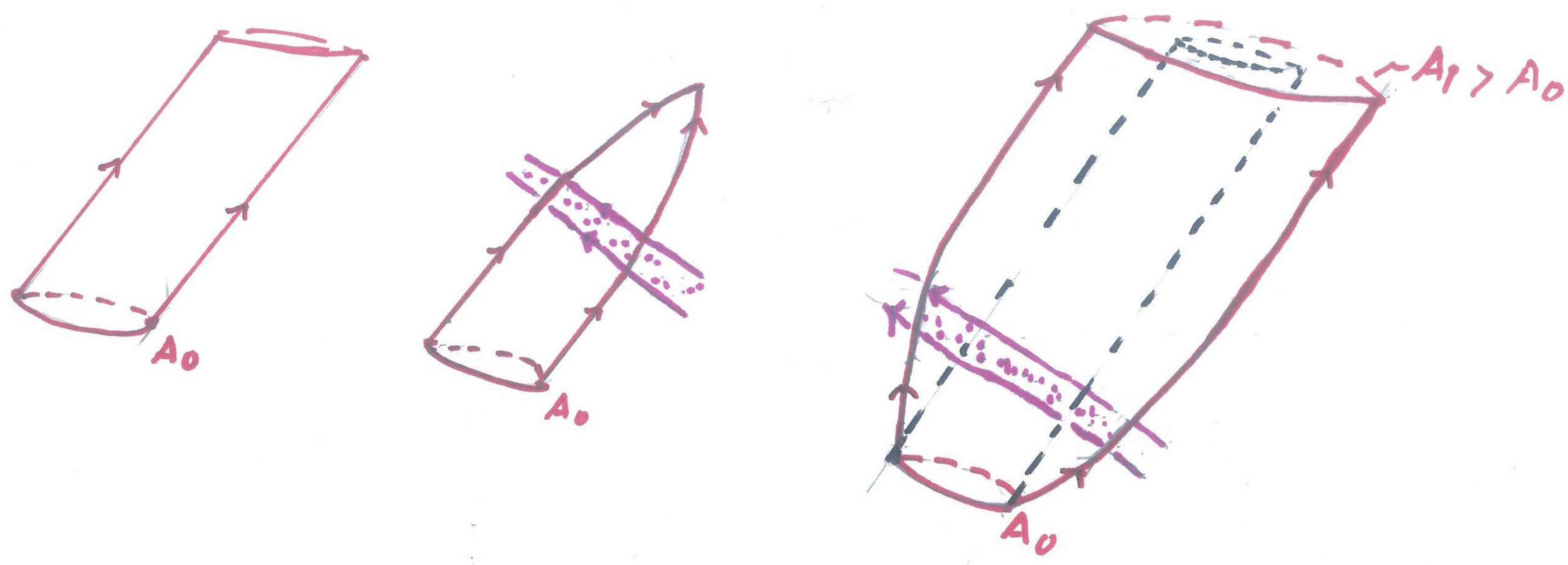




The last set of light rays which make it out to infinity trace out the horizon. The horizon has the topology of a 2d sphere times the real line. At late times when the black hole has settled down, the horizon has a 'time independent' geometry and the light rays along it neither diverge nor converge.

(The 'time' flow vector field along the horizon is called ξ.)

What happens when we perturb the black hole by sending in a small pulse of energy across the horizon? Since gravity is attractive whenever matter crosses the path of light rays it makes them converge. So for light to emerge with zero expansion after encountering the pulse, the rays have to start of diverging. So the final area of the horizon is larger than it would have been without this pulse.



- Greater the amount of stress energy the light encounters, the more is the focussing effect. the larger is the compensatory initial divergence and the larger the final area. Using diffnal geometry + dynamics of gravity and calling the (parallely transported) vector along the light ray trajectory as l^μ, obtain:

  $\kappa\delta A = 8\pi G \int_H \delta T_{\mu\nu}\xi^\nu l^\mu$.

  $\delta(A/8\pi G) = (\int_H \delta T_{\mu\nu}\xi^\nu l^\mu)/\kappa$

  "$dS = \delta Q/T$" with:

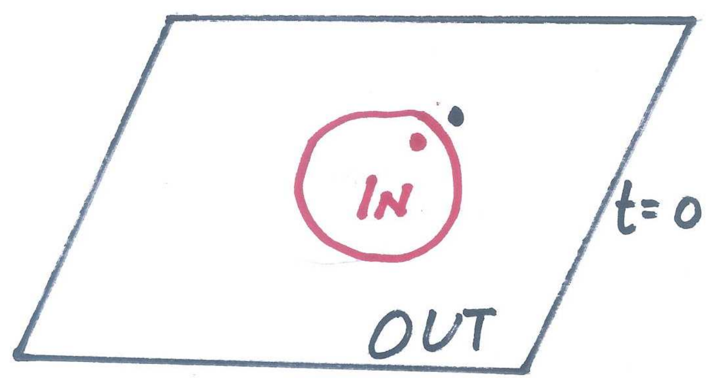
  Area/G ~entropy, surface gravity ~ temperature
- Hawking showed: $T = \frac{\hbar\kappa}{2\pi}$ so that

  Entropy=Area/$4G\hbar$

  $G\hbar \sim 10^{-66} cm^2 = l_P^2$!
- If Area is identified with entropy it also rescues the second law: Throwing entropic objects into a BH reduces entropy; compensated by area increase of BH! (**Bekenstein)**

Sorkin´s Proposal

- The curvature near the horizon goes as $1/M_{BH}^2$. The geometry in the vicinity of (and outside) the horizon of a massive black hole looks almost flat. So consider flat spacetime and quantum fields thereon in their vacuum state.
- Consider, for simplicity, a free scalar field in its vacuum at say, $T = 0$. Consider a spherical region around the origin $(X, Y, Z) = 0$. This models the BH interior.

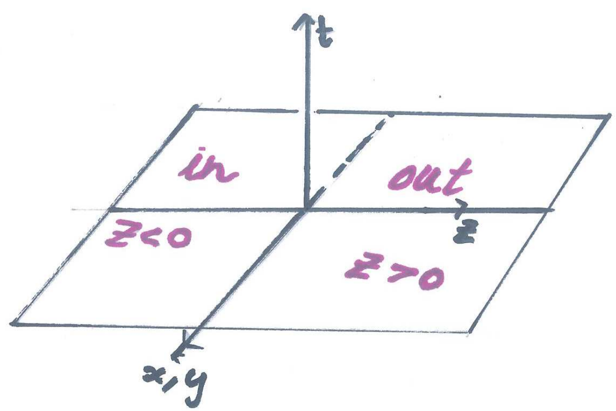


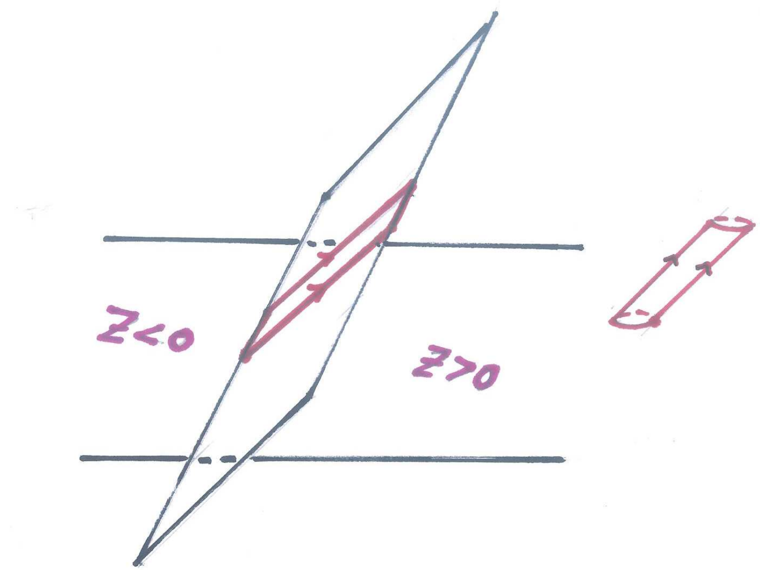
- Define $\rho := \mathrm{Tr}_{in}|0><0|$, $S_{ent} := -\mathrm{Tr}_{out}\rho \ln \rho$. Get S_{ent} =infinity!

- A field operator just outside this sphere has a very strong correlation with the field just inside the sphere : $<0|\hat{\Phi}(\vec{x})\hat{\Phi}(\vec{y})|0> - <0|\hat{\Phi}(\vec{x})|0><0|\hat{\Phi}(\vec{y})|0> \approx 1/|\vec{x}-\vec{y}|^2$
- Put a cutoff by replacing space by a lattice of spacing a. Then $S_{ent} \sim \text{Area of Sphere}/a^2$ If we set $a \sim l_P^2$ we get Bekenstein entropy.
- Why this cut off? Involves unknown UV physics (quantum gravity).
- Bianchi: Thermodynamics deals with changes in entropy. If we vary matter field vacuum to a state $|E>$ which is ‘nice’, maybe variation of S_{ent} is indep of UV physics. Correspondingly need to consider variation of black hole area when some energy E is sent into it. More precisely....

Bianchi' setup

- Once again consider flat spacetime as a model for the near horizon geometry of a large black hole. Let the null plane $t - z =$ constant model the horizon. At the instant $t = 0$, $z > 0$ is the 'exterior' region and $z < 0$ is the 'interior'.
- Consider a bundle of collimated null rays along the 'horizon'. Let cross section of this bundle have area A_0. This is the analog of zero expansion null rays along the BH horizon.





- Let us shoot a pulse of energy at some time $t > 0$ from the exterior so that it crosses the horizon. Due to gravitational field of pulse, the original light beam will converge so that in order to get a collimated beam at late times, we need to send in a diverging beam.
The change in area is exactly given by our previous formula: "change in area = integral of $T_{\mu\nu}$ along horizon"
- Let us now do this with quantum matter. We shall consider a free massless scalar field. The unperturbed case corresponds to scalar field in its vacuum. The perturbation consists of exciting the vacuum at $t = 0$ and thereby sending a pulse of energy into the horizon from the 'out' region. We will compute the change in entanglement entropy at $t = 0$ when we change the state from the vacuum to this excited state and compare it with area change, as computed above, due to the pulse of energy hitting the horizon.

- Free scalar matter in its vacuum state $|0>$ has density matrix $\rho_0 = \mathrm{Tr}_{in}|0><0|$.
  The perturbation consists in varying the vacuum state to some nearby state $|E>$. We have $\rho = \mathrm{Tr}_{in}|E><E|$.

- We want to compute the change in entanglement entropy: $\delta S_{ent} = (-\mathrm{Tr}_{out}(\rho \ln \rho)) - (-\mathrm{Tr}_{out}(\rho_0 \ln \rho_0))$.
- First let us discuss ρ_0.

- ρ_0 is an operator on $\mathcal{H}_{out}$.

  Can show that $\rho_0 = e^{-\hat{K}}$,

  $\hat{K} = \frac{2\pi}{\hbar} \int_{z>0} \hat{T}_{0,\nu} b^\nu d^3x$.

  Here b is the boost Killing vector $zd/dt + td/dz$ on the 'out' part of the $t = 0$ slice.
  (This expression underlies the **Unruh Effect**).

- Assuming no stress energy escapes to infinity we can replace the integral by one over the horizon:

  $\hat{K} = \frac{2\pi}{\hbar} \int_H \hat{T}_{\mu,\nu} b^\nu l^\mu$.

- Now let us compute δS_{ent} when we vary from $|0>$ to $|E>$ and hence form ρ_0 to ρ.

- The variation in entanglement entropy is :

$$\delta S_{ent} = -\delta \mathrm{Tr}_{\mathrm{out}}(\rho \ln \rho)$$

$$= -\mathrm{Tr}_{\mathrm{out}}(\delta\rho \ln \rho_0) - \mathrm{Tr}_{\mathrm{out}}(\delta\rho).$$

$$\mathrm{Tr}_{out}\rho = 1 \Rightarrow \mathrm{Tr}_{out}(\delta\rho) = 0.$$

$$\delta S_{ent} = \mathrm{Tr}_{out}((\rho - \rho_0)\hat{K}) \; = \; \overline{\hat{K}}_E - \overline{\hat{K}}_0$$

- Using the expression for $\hat{K}$ in terms of horizon integral $\hat{T}_{\mu\nu}$ we obtain:

$$\delta S_{ent} = \frac{2\pi}{\hbar} < E| \int_H \hat{T}_{\mu,\nu} b^\nu l^\mu |E >$$

This exactly the same integral along which we got for the change in area except that we have exp value of $\hat{T}_{\mu\nu}$ instead of $T_{\mu\nu}$.

- If we assume Einst eqns `$G_{\mu\nu} = GT_{\mu\nu}$ are valid with $T_{\mu\nu}$ replaced by $< \hat{T}_{\mu\nu} >$ we can replace the horizon integral of $< \hat{T}_{\mu\nu} >$ by $\delta Area/4G$ to get:

  $\delta S_{ent} = \delta Area/4l_P^2$

  No reference to unknown UV physics.

  NOTE: I assumed that the area change was given by the exp value of stress energy. Justification cannot come from classical GR. But assumption can be validated in a computation of the exp value of the area change in the framework of perturbative qg.

Coherent States

- As we shall see the entanglement entropy of any coherent state is the same as the of the vacuum.
- But coherent states can have any stress energy we want! So if the pulse $|E>$ was in a coherent state, its stress energy would gravitate and change the area of the horizon BUT δS_{ent} would vanish! This seems to contradict our result.
- Resolution is that we used a first order variation. $|E>$ was first order perturbation of vacuum. For coherent states which are first order pert of the vacuum, stress energy is nontrivial only at second order. So no contradiction with the math.
- But 'first order' variation means a first derivative with respect to some parameter for a 1 parameter family of states about the vacuum. This has no physical meaning : Physically we want to show, for small but finite variations in state, that $\Delta S_{ent} = \Delta Area/4l_P^2$. So there is an obstruction.

  It remains to show that coherent state ent entropy is same as that of the vacuum...

Coherent State Entanglement Entropy

- Recall oscillator coherent states in QM:
  $|c>= \exp(-\bar{\alpha}\hat{a} + \alpha\hat{a}^\dagger)|0>$.
- Rewrite $\hat{a}, \hat{a}^\dagger$ in terms of $\hat{q}, \hat{p}$:

  $$|c>= \exp i(c_1\hat{q} + ic_2\hat{p})|0>, \qquad c_1, c_2 \text{ real.}$$

- In free field theory also we can write

  $$|f>:= \exp(i \int d^3x f_1(x)\hat{\phi}(x) + f_2(x)\hat{\pi}(x)), \qquad f_1, f_2 \text{ real.}$$

  where $\pi(x)$ is the momentum conjugate to $\phi(x)$ and the integral is on a $t = 0$ slice. $|f>$ is also a coherent state and is a quantum state corresponding to a certain classical solution of the free scalar field wave equation.

- We have that:
  $$|f>= \exp(i \int_{z>0} \int d^3x f_1(x)\hat{\phi}(x) + f_2(x)\hat{\pi}(x)) \exp(i \int_{z<0} d^3x f_1(x)\hat{\phi}(x) + f_2(x)\hat{\pi}(x))|0>. = U_{out}U_{in}|0>$$

- $U_{out} = \exp(i \int_{z>0} \int d^3x f_1(x)\hat{\phi}(x) + f_2(x)\hat{\pi}(x))$ depends only on 'outside fields'.
- $U_{in} = exp(i \int_{z<0} d^3x f_1(x)\hat{\phi}(x) + f_2(x)\hat{\pi}(x))|0 >$. depends only on 'inside fields'.
- Tracing over degrees of freedom "inside" $z = 0$ we have

  $\hat{\rho} = \mathrm{Tr}_{in} U_{out} U_{in} |0 >< 0| U_{in}^{\dagger} U_{out}^{\dagger}$.

  Since only inside degrees are being traced over we can move $U_{out}, U_{out}^{\dagger}$ outside the Trace.

  Using invariance of trace under unitary map U_{in}, we have $\hat{\rho} = U_{out}(\mathrm{Tr_{in}}|0 >< 0|)\mathrm{U_{out}^{\dagger}}$.
- $S_{ent}(f) = -Tr_{out} \rho \ln \rho$. Using unitary inv of Trace under U_{out}, we get $S_{ent}(f)$ = entanglement entropy of the vacuum!

Summary

- BH entropy is proportional to its area. Microscopic explanation could be that BH entropy is entanglement entropy of vaccum fluctuations of quantum fields across horizon. Getting correct (finite) entropy requires cut off at Planck length. Assume QG physics provides this cut off.
- Perhaps variations of area caused by absorption of low energy quanta by BH could be associated with ent entropy variation indep of any UV physics.
- We showed this is only true for first order variations. But physically we should show this for small but finite variations.
- Coherent states provide a counter example to the idea.
- For exactly the same reasons, coherent states also provide an obstruction to Jacobson's ideas for deriving the full non-linear Einstein equations from a maximal entanglement entropy principle.

Jacobson´s maximal entanglement principle

- Consider a neighbourhood of a point p in a 4d manifold. In this neighborhood define pair (g, ψ) g = lorentzian metric, ψ quantum state of matter. The proposed principle is that the entanglement entropy of this state in this geometry in a small spatial geodesic ball within this neighbourhood is maximal (at fixed volume) when the geometry is maximally symmetric and the state is the associated maximally symmetric vacuum.
- The principle applies to all points p and all sufficiently small geodesic balls around p. Aim is to obtain the semiclassical Einst eqns from this principle.
- For simplicity restrict the choice of maximal sym spacetime to be flat and the matter to be a conformal field.
- Variation of S_{ent} under joint variations of geometry (which keep the volume of the geodesic ball fixed) and state gives a contribution from keeping geometry fixed and varying state and another from varying geometry and keeping state fixed.

- Keeping state fixed and varying geometry one gets a contribution $\delta S_{UV} = \delta Area/cutoff$. One can show that change in area relative to flat spacetime is proportional to G_{00} where the '0' direction is orthogonal to the ball.
- Keeping geometry fixed and flat and varying the state allows us to use our variational calculation and get $\delta S_{IR} \sim \delta < T_{00} >$.
- Maximality implies $\delta S_{IR} + \delta S_{UV} = 0$ and after putting factors in one gets G_{00} proportional to $< T_{00} >$ with proportionality constant depending on cutoff. This fixes Newton's constant in terms of cutoff. The relation matches that for obtaining the Bek -Hawking entropy (which we found earlier in this talk). Varying the geodesic ball yields tensor equation $G_{\mu\nu} = 8\pi G\delta < T_{\mu\nu} >$.

- But again, for coherent states δS_{IR} vanishes but we can have gravitating stress energy. Once again, this contradiction is resolved by noticing that Jacobson's derivation is a first order one.
- As before, first order variations are mathematically well defined but not physically appropriate. What we want is a derivation for small but finite variations. This is as yet unavailable.
- There are many other fine points I have glossed over which are not well understood. Modulo these, the coherent state example is an obstruction to deriving the Einstein equations. However it seems likely that the Einstein equations could still be *consistent* with the maximal ent entropy principle.