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Landau-Lifshitz Equations of Radiative Friction

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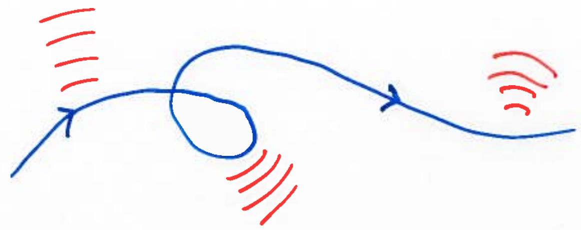
1. The setting

• motion of charge coupled to radiation field



• charged particle charge e, mass m, q, p small size

• Maxwell field – dynamic F

– external F^{ex}

• classical $\hbar = 0$

• temperature 0

⇝ friction ≪ effective hamiltonian motion

H.S. Dynamics of charged particles ... , CUP 2004

Kunze, H.S. Post-Coulombian ... 2001 $v/c \ll 1$

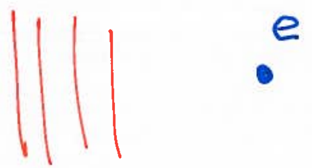
Komech, Imaikin, Kopylova, H.S 1998 – now

$F^{ex} = 0$, asymptotic velocity

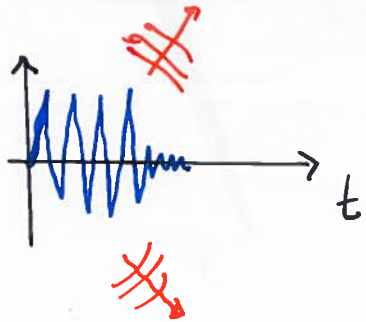
scattering theory

⇒ ultra strong laser sources

⇒ quantitative experimental confirmation



pulse



compare: scattered light

frictionless ⟺ radiative friction

theory

2. Effective equations of motion

Abraham 1903, von Laue 1909 world line $x(\tau)$

$$\underbrace{m_0}_{\text{model-dependent}} \ddot{x} = \frac{e}{c} F^{ex} \cdot \dot{x} + \underbrace{\frac{e^2}{6\pi c^3}}_{\text{small}} \left(\dddot{x} - \frac{1}{c^2} (\ddot{x} \cdot \ddot{x}) \dot{x} \right)$$

universal

Maxwell linear, linear coupling ⇏ explicit memory equation (nonlinear)

Taylor expansion

Dirac 1938 run away solutions

"physical solutions" are singled out by

$$\lim_{t \to \infty} \dot{v}(t) = 0$$

3. Singular perturbation theory

H.S. 1998

highest derivative with small parameter

Example

$$\dot{x} = f(x,y)$$

$$\varepsilon \dot{y} = y - h(x) \qquad \varepsilon \ll 1 \quad , \quad x,y \in \mathbb{R}$$

$\varepsilon = 0$ $\qquad y = h(x) \Leftarrow$ constrained motion

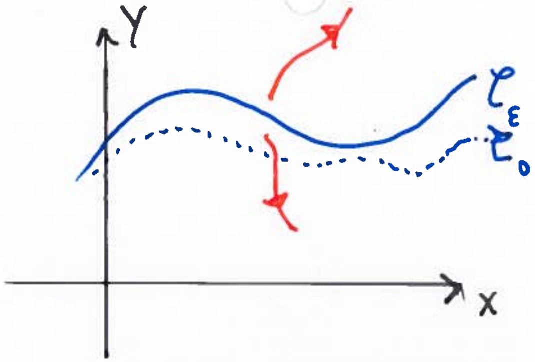
$$\Rightarrow \quad \dot{x} = f(x, h(x))$$

ε small: $\qquad \varepsilon \dot{y} = y - h(x_0)$ run away $\qquad y(t) = (y_0 - h(x_0))e^{t/\varepsilon} + h(x_0)$

|| manifold $C_0 = \{y = h(x)\}$ is deformed to $C_\varepsilon = \{y = h_\varepsilon(x)\}$ ||

locally smooth

$\Rightarrow$ C_ε can be complicated



on $\mathcal{C}_\varepsilon$
$$\dot{x} = \underbrace{f(x, h(x))}_{\text{"hamiltonian"}} + \varepsilon\, \partial_y f(x, h(x))\, \underbrace{h'(x)}_{\text{"friction"}}\, f(x, h(x))$$

apply to ALD

$$m_0 \ddot{x} = \frac{e}{c} F^{ex} \cdot \dot{x} + \frac{e^2}{6\pi c^3} \left(\left(\frac{e}{mc}\right)^2 F^{ex} \cdot F^{ex} \cdot x + \frac{e}{mc} \dot{x} \cdot \nabla F^{ex} \cdot \dot{x} - \left(\frac{e}{mc}\right)^2 \left(F^{ex} \dot{x} \cdot F^{ex} \dot{x}\right) \dot{x} \right)$$

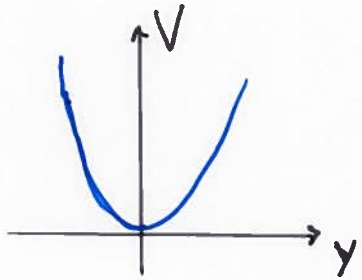
agrees with Landau + Lifshitz, Classical Theory of Fields, all editions $\simeq$ 1950

4. Examples

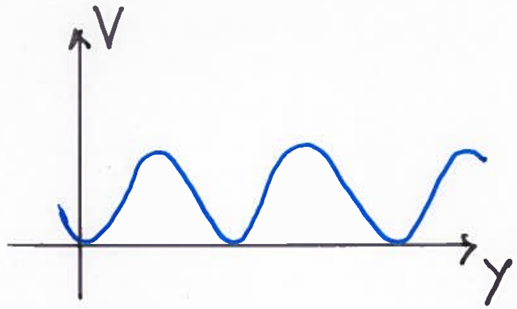
- dimension 1, $B^{ex} = 0$, $E^{ex} = -\nabla V$

$$m\ddot{y} = -V'(y) - \frac{e^2}{6\pi c^3}\frac{1}{m}V''(y)\,\dot{y}$$

friction



+ anti friction



- Penning trap
  - $e\phi(x) = \frac{1}{2}m\omega_z^2\left(-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + x_3^2\right)$
  - uniform magnetic field $\vec{B}$

- ultra relativistic

- numerical simulations

5. Microscopic theory

Abraham model

- rigid charge distribution

$e\varphi(x)$ space scale R

time scale R/c

Maxwell

$$\partial_t B = -\operatorname{rot} E \quad \Big\| \quad \operatorname{div} B = 0 \quad \Big\| \quad \rho = e\varphi(x - q(t))$$

$$\partial_t E = \operatorname{rot} B - j \quad \Big\| \quad \operatorname{div} E = \rho \quad \Big\| \quad j = \dot{q}(t)\, e\varphi(x - q(t))$$

Lorentz force

$$\underbrace{m_0}_{\text{bare}} \ddot{q} = e \int dx\, \varphi(x-q) \Big(E(x,t) + \dot{q} \wedge B(x,t) + E_{ex}(x,t) + \dot{q} \wedge B_{ex}(x,t) \Big)$$

dynamic external

Dirac 1938

point charge limit

$$\rho(x) \rightsquigarrow R^{-3}\,\rho(x/R) \qquad R \to 0$$

$$\rightsquigarrow \quad \left(m_0 + \frac{e^2}{6\pi c^2 R}\right)\ddot{q} = \text{forces}$$

$m_0 + \dfrac{e^2}{6\pi c^2 R}$ fixe $\qquad m_0 \to -\infty \qquad$ motion is unstable

runaway

HERE external fields vary slowly on scale R, R/c

electron $R = 10^{-14}$ m , $R/c = 10^{-23}$ sec

ρ fixed , $m_0 > 0$ stable motion

modified energy-momentum

initial conditions $\quad q, \dot{q} = v, \quad E, B$ adjusted local Coulomb

TRUE SOLUTION $\quad q_\varepsilon(t) \quad v_\varepsilon(t)$ $\quad \varepsilon$ parameter for slow variation

$E_{ex}(\varepsilon x, \varepsilon t), B_{ex}(\varepsilon x, \varepsilon t)$

$$q_\varepsilon(t) \cong \varepsilon^{-1} r(\varepsilon t)$$

$$v_\varepsilon(t) \cong u(\varepsilon t) \leftarrow \mathcal{O}(1)$$

a priori bounds

rescale $\quad t = \varepsilon^{-1} \tau \leftarrow \mathcal{O}(1)$

to order ε^0: $\quad$ effective hamiltonian $\quad E(p)$

to order ε^1: $\quad$ friction $\quad$ error bound ε^2

$\|$ intial slip $\|$

6. Sommerfeld - Page

$|v| \ll 1$, linearized memory term

$$\varrho(x) = \frac{1}{4\pi R^2}\, \delta(|x| - R)$$

$E^{ex} = 0$, $B^{ex} = (0, 0, B_0)$ in plane motion

$$\Rightarrow \quad m_0 \dot{v} = e B_0 v^{\perp} + \frac{e^2}{12\pi R^2} \big(v(\cdot - 2R) - v \big) \qquad (v_1, v_2)^{\perp} = (-v_2, v_1)$$

delay

gyration frequency $\omega = \frac{eB_0}{m_0}$, delay $\tau = \frac{2R}{c}$, friction $\alpha = \frac{e^2}{3\pi m_0 \tau^2}$

$$\Rightarrow \quad \dot{v} = \omega v^{\perp} + \alpha \big(v(\cdot - \tau) - v \big)$$

electron , $B_0 = 10$ Tesla , $R = 10^{-13}$ cm

$\alpha\tau = 10^{-5}$

$\omega\tau = 10^{-8}$

10 cm $\rightsquigarrow$ 1 μm

10^{14} revolutions , 1 sec

given

$v(t) = u(t)$, $0 \le t \le \tau$

Taylor expansion in R

$$\left(m_0 + \frac{e^2}{6\pi c^2 R}\right) \dot{w} = \frac{e B_0}{c} w^{\perp} + \frac{e^2}{6\pi c^3} \ddot{w}$$

Dirac: $R \to 0$ $\qquad m_{eff} = m_0 + \frac{e^2}{6\pi c^2 R} > 0$

limit has run away, except for constrained initial conditions

slow variation (small B_0) $\qquad$ time $\varepsilon^{-1} t$

$$\varepsilon \frac{d}{dt} v(t) = \varepsilon \omega v(t)^{\perp} + \alpha\left(v(t-\varepsilon\tau) - v(t)\right)$$

$$\rightsquigarrow \quad (1+\alpha\tau)\frac{d}{dt} v(t) = \omega v(t)^{\perp} + \frac{1}{2}\alpha\tau^2 \varepsilon \frac{d^2}{dt^2} v(t)$$

comparison dynamics

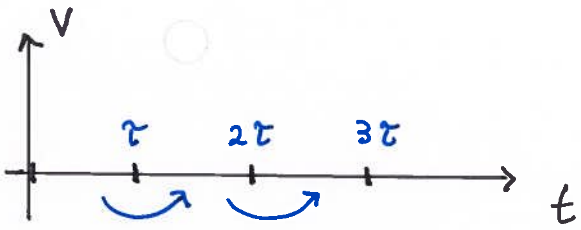
center manifold

LL

$$\left[\!\left[(1+\alpha\tau)\dot{W} = \omega W^{\perp} - \frac{1}{2}\varepsilon\alpha\left(\frac{\omega\tau}{1+\omega\tau}\right)^2 W \right]\!\right]$$

solution map

linear



spectral analysis

initial slip, $B_0 = 0$, $v(t) \to \bar{v}$ exponentially fast

comparison dynamics $w(t)$ with $w(0) = \bar{v}$, $\varepsilon = 1$,

$\rightsquigarrow$ $|v(t) - w(t)| \leq O((\omega\tau)^2)$ for $t/\tau \geq \log(\omega\tau)^2$

$\rightsquigarrow$ LL units $|v(t) - w_\varepsilon(t)| \leq c_0 \, \varepsilon^2$ uniform in t.

Conclusions

- radiative friction

  Landau-Lifshitz equations

  2nd order in t stable

  simple numerics

- microscopic model

  - smeared charge
  - slow variation of external fields

- Sommerfeld-Page ✓

  Abraham model error bounds too weak