Probing quantum nonlocality of bipartite entangled qutrits by generalising Wigner's argument

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Background

• A seminal contribution showing an incompatibility between QM and the notion of local realism is *Bell's inequality*.

• Bell's inequality is a testable algebraic consequence of the combination of notions of *realism* and *locality*.

• Soon after the discovery of Bell's inequality, a different formulation of local realist inequality was provided by Wigner. However, Wigner's original formulation was restricted in showing the QM incompatibility with local realism for the bipartite qubit *singlet states*.

Wigner's original derivation [E. P. Wigner, *Am. J. Phys.* **38**, *1005 (1970)*]

• Let two spin-1/2 particles are prepared in a singlet state and are then spatially separated. The spin components of the particles, respectively, are measured along three directions, say, a, b and c.

• Reality + Locality $\Rightarrow$ the existence of overall joint probabilities for the individual outcomes of measuring the pertinent observables specified in the underlying stochastic hidden variable (HV) space and the observable marginal probabilities can be obtained from those overall joint probabilities.

• Example: corresponding to an underlying stochastic HV, say λ, one can define $p_\lambda(v_1(a), v_1(b), v_1(c); v_2(a), v_2(b), v_2(c))$ as the overall joint probability of occurrence of the outcomes, where $v_1(a)$ represents an outcome (± 1) of the measurement of the observable a for the first particle, and so on.

The observable joint probability of getting $+1$ for both the outcomes if the observables a and b are measured on the first and the second particle respectively can be written, using the *perfect anti-correlation property* of the singlet state, as
$p_\lambda(a+, b+) = p_\lambda(+, -, +; -, +, -) + p_\lambda(+, -, -; -, +, +)$.

Wigner's original derivation [E. P. Wigner, *Am. J. Phys.* **38**, *1005 (1970)*]

• Similarly, writing $p_\lambda(c+, b+)$ and $p_\lambda(a+, c+)$ as marginals, assuming non-negativity of the overall joint probability distributions in the HV space, and by integrating over the HV space for an arbitrary distribution, one can obtain the original form of Wigner's inequality

$$p(a+, b+) \leq p(a+, c+) + p(c+, b+)$$

• If the respective angles between a and b, a and c, b and c are θ_{12}, θ_{13} and θ_{23}, then substituting the QM expressions for the relevant joint probabilities in the inequality, one obtains

$$\frac{1}{2}\sin^2(\theta_{12}/2) \leq \frac{1}{2}\sin^2(\theta_{13}/2) + \frac{1}{2}\sin^2(\theta_{23}/2)$$

$\rightarrow$ not valid for arbitrary values of $\theta_{12}, \theta_{13}, \theta_{23}$.

$\Rightarrow$ incompatibility between QM and Wigner's inequality, restricted for the singlet state in the bipartite case.

• The above argument is within the framework of stochastic HV theory, subject to the locality condition $\rightarrow$ the notion of determinism has not been used here.

Generalized Wigner inequality (GWI) for any N-partite qubit system [D. Home, D. Saha, S. Das, *Phys. Rev. A* **91**, *012102 (2015)*]

• "Reality" & "Locality" $\rightarrow$ the existence of overall joint probabilities for the individual outcomes of measuring the pertinent observables in the underlying stochastic HV space. The observable marginal probabilities can be obtained from those overall joint probabilities.

• Perfect anti-correlation property of the singlet state has not been used here.

• Pair of dichotomic observables measured on each of the spatially separated systems are a_i or $a_i^{'}$ $(i = 1, 2, ..., N)$.

• Assuming non negativity of the overall joint probability distributions in the HV space, one can get the following form GWI for N-partite qubit system:

$$p(a_1+, a_2+, a_3+, ..., a_N+) - p(a_1'+, a_2+, a_3+, ..., a_N+) - p(a_1+, a_2'+, a_3+, ..., a_N+) - ... - p(a_1+, a_2+, a_3+, ..., a_N'+) - p(a_1'-, a_2'-, a_3'-, ..., a_N'-) \leq 0$$

• The efficacy of GWI is probed, comparing with the Seevinck-Svletlichny multipartite Bell-type inequality [*Phys. Rev. Lett. 89, 060401 (2002)*], by calculating threshold visibilities for the quadripartite GHZ, Cluster, and W states that determine their respective robustness with respect to the QM violation of GWI in the presence of white noise.

Generalized Wigner inequality (GWI) for any bipartite qutrit system [D. Das, S. Datta, S. Goswami, A. S. Majumdar, D. Home, *arXiv: 1605.08913 (2016)*]

Motivation:

• Qutrit system not only has fundamental relevance in laser physics, but also has generated much interest from the perspective of information processing.

• Use of qutrit systems makes the quantum cryptography protocols robust against eavesdropping attack compared to that of qubit systems.

[*Bruss, Macchiavello, Phys. Rev. Lett.* **88**, *127901 (2002); Cerf, Bourennane, Karlson, Gisin, Phys. Rev. Lett.* **88**, *127902 (2002); Durt, Cerf, Gisin, Zukowski, Phys. Rev. A* **67**, *012311 (2003)*]

• Security of quantum bit commitment and coin flipping protocols are higher with entangled qutrits.

[*Spekkens, Rudolph, Phys. Rev. A* **65**, *012310 (2002)*]

• A qutrit system is of special interest because information processing appears to have great potential in a three-level system as it best fits into dimensionality aspect of Hilbert space. The Hilbert space dimensionality is maximized for $d = 3$, and hence the computing power.

[*Greentree, Schirmer, Green, Hollenberg, Hamilton, Clark, Phys. Rev. Lett.* **92**, *097901 (2004)*]

Generalized Wigner inequality (GWI) for any bipartite qutrit system [D. Das, S. Datta, S. Goswami, A. S. Majumdar, D. Home, *arXiv: 1605.08913 (2016)*]

• Let us consider that pairs of trichotomic observables a or a' and b or b' are measured on the first and the second particle respectively. Possible outcomes of measurement of each of the observables are denoted by $+1$, 0 and -1.

• Assuming reality and locality condition as stated before, and assuming non negativity of the overall joint probability distributions in the stochastic HV space, one gets the following form of GWI for bipartite qutrit system

$$K_{GWI} = p(a+, b0) - p(a'+, b0) - p(a+, b'0) - p(a'-, b0)$$

$$-p(a-, b'0) - p(a'0, b'+) - p(a'0, b'-) + p(a-, b0) \leq 0$$

Similarly, other following forms of GWI for bipartite qutrit systems can be obtained using different combinations of the observable joint probabilities.

• The magnitude of the QM violation of GWI is denoted by the positive value of K_{GWI}.

Other local realist inequalities for probing nonlocality of bipartite qutrits

(i) WZPW inequality derived by Wu et. al. [Phys. Lett. A **281**, 203 (2001)] :

$$P(a+,b+)-P(a+,b'+)+P(a'+,b'+)+P(a'0,b0)+P(a'0,b-)+P(a'-,b0)+P(a'-,b-)\leq 1$$

$\rightarrow$ derived based on the assumption of local HV model satisfying the factorizability condition and using a few algebraic theorems.

(ii) CGLMP inequality derived by Collins et. al. [Phys. Rev. Lett. **88**, 040404 (2002)] :

$$[P(a=b)+P(b=a'+1)+P(a'=b')+P(b'=a)]-[P(a=b-1)+P(b=a')+P(a'=b'-1)+P(b'=a-1)]\leq 2$$

$\rightarrow$ derived based on a constraint that the correlations exhibited by a local realist theory must satisfy.

• None of these two inequalities has been derived from the assumption of existence of a joint probability distribution in the HV space.

QM violation of GWI for bipartite qutrit singlet and isotropic states

- We have considered two types of maximally entangled qutrit states:

(i) Isotropic states: $|\psi_i\rangle = \frac{|00\rangle+|11\rangle+|22\rangle}{\sqrt{3}}$.

(ii) Singlet states: $|\psi_s\rangle = \frac{|02\rangle-|11\rangle+|20\rangle}{\sqrt{3}}$,

where $|0\rangle$, $|1\rangle$ and $|2\rangle$ are the eigenstates of spin angular momentum operator along z-direction corresponding to the eigenvalues $+1$,0 and -1 respectively (assuming $\hbar = 1$).

- a, a', b, b' $\Rightarrow$ spin-1 component observables in arbitrary directions $(sin\theta_i cos\phi_i, sin\theta_i sin\phi_i, cos\theta_i)$ $(i = a, a', b, b')$. Here θ_i and ϕ_i are the polar and azimuthal angle respectively.

- The magnitude of maximum QM violation of GWI for bipartite qutrit isotropic and singlet state using spin-1 component observables are the same, equal to 0.13807 for the appropriate choices of the measurement settings.

Efficacy of GWI for bipartite qutrit systems

• GWI is compared with other local realist inequalities for bipartite qutrits in terms of the robustness of their QM violation against

(i) unsharpness or fuzziness of the measurements

$\rightarrow$ common in practical scenario $\rightarrow$ due to imperfections in the measuring apparatus, which leads to inaccuracies in outcome readings.

(ii) white noise incorporated in the pure states.

$\rightarrow$ common in practical scenario $\rightarrow$ due to environmental decoherence effects.

• In the case of unsharp measurement, the probability of an outcome is determined by the effect operator, which is defined as

$$F_{(\pm,0)} = \lambda P_{(\pm,0)} + (1-\lambda)\frac{\mathbb{I}_3}{3}$$

where, $\lambda \rightarrow$ sharpness parameter $(0 < \lambda \leq 1)$.

$(1-\lambda) \rightarrow$ amount of fuzziness or unsharpness present in the measurement.

P_+, P_0 and $P_- \rightarrow$ projectors onto the Eigenstates of spin-1 component observable in arbitrary direction with Eigenvalues $+1$, 0 and -1 respectively.

$\mathbb{I}_3 \rightarrow$ 3-dimensional identity operator.

• Given the above specification of the effect operator, the probability of an outcome, say $+1$, is given by $Tr(\rho F_+)$, where ρ is the state of the system on which measurement is done.

• We have compared the ranges of λ for which different local realist inequalities are violated using spin-1 component observables.

Efficacy of GWI for bipartite qutrit systems (Incorporating white noise in the states)

• Let us introduce the visibility parameter pertaining to a state. Consider the bipartite qutrit mixed state given by,

$$\rho = p|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{I}_3 \otimes \mathbb{I}_3}{3^2}$$

where, $p \rightarrow$ visibility parameter which changes the pure state into a mixed state.

$(1-p) \rightarrow$ amount of white noise present in the state $|\psi\rangle$ (Here we take $|\psi\rangle$ to be either the isotropic state or the singlet state).

$p = 0 \rightarrow$ the maximally mixed separable state.

• The minimum value of p for which QM violates local realist inequality determines the maximum amount of white noise that can be present in the given state for the persistence of the QM violation of the relevant local realist inequality, and this value of p is known as the threshold visibility pertaining to the given local realist inequality.

• Threshold visibility of GWI has been compared with that of other local realist inequalities for bipartite qutrits.

Summary of the key results

• GWI is the most robust among the other aforementioned local realist inequalities in terms of their QM violations against unsharp measurement of spin-1 component observables.

$\Rightarrow$ there is a range of the sharpness parameter for which the QM violation of local realism can be probed using the GWI, but *not* using the CGLMP inequality, or the WZPW inequality for both bipartite qutrit isotropic and singlet states.

Local realist inequality	**The range of sharpness parameter (λ) for the persistence of the QM violation of the local realist inequality**	
	for isotropic states	**for singlet states**
GWI	$(0.873, 1]$	$(0.873, 1]$
CGLMP	$(0.889, 1]$	$(0.889, 1]$
WZPW	$(0.887, 1]$	$(0.887, 1]$

Summary of the key results

• GWI is the most robust among the considered local realist inequalities for the persistence of the QM violation in the presence of white noise incorporated in both the qutrit isotropic and singlet states using spin-1 component observables.

Local realist inequality	Threshold visibilities for	
	isotropic states	singlet states
GWI	0.763	0.763
CGLMP	0.791	0.791
WZPW	0.786	0.786

Conclusion and outlook

• In this work we have extended Wigner's approach by deriving generalized Wigner type local realist inequalities (GWI) for probing nonlocality of bipartite qutrit entangled systems.

• Efficacy of the derived GWI for the bipartite qutrit systems has been probed by analysing the robustness of its QM violation against unsharp measurements as well as with respect to white noise incorporated in the singlet state and the isotropic state using spin-1 component observables.

• Probing the efficacy of GWI using six port beam splitter is one of the future areas of study.

• It requires to be studied whether the comparison between the three different types of local realist inequalities treated in this work can be extended in the context of nonmaximally entangled bipartite qutrit states.

• Finally, it should be worth probing the possibility of any information theoretic application of GWI, for example, in the context of device independent quantum key generation.