Relativistic QFT

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Relativistic QFT

Overview of relativistic quantum field theory predictions

Light in moving media

QFT examples in moving media

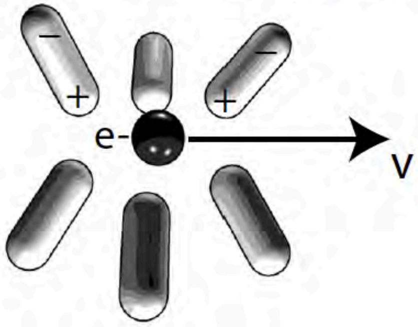
Why a time varying medium implies photon pair creation

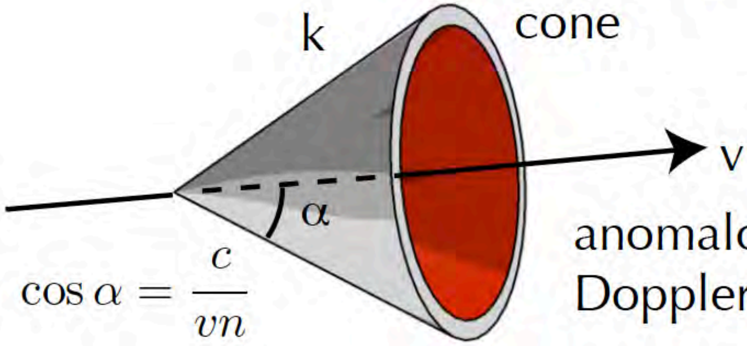
Time-varying physics in new materials,
e.g. epsilon-near-zero media
-> detailed analysis

Relativistic Quantum effects

Superluminal, constant v

> Cerenkov effect
  -> superluminal charge
  -> induces charge displacement





Relativistic Quantum effects

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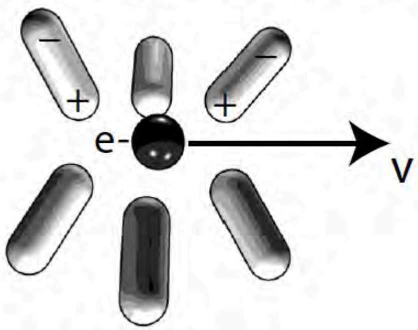
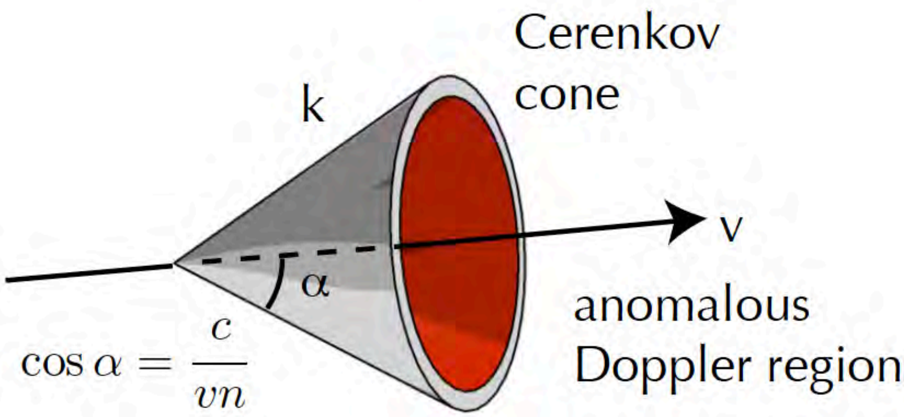
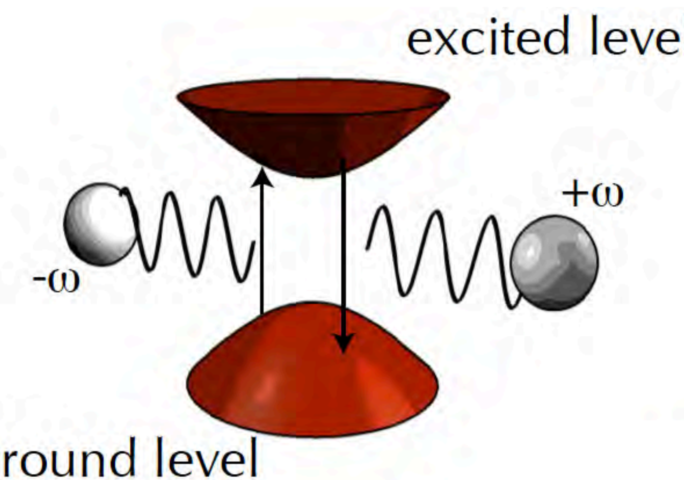




> Anomalous Doppler effect
  -> superluminal 2-level atom
  -> transition to excited state + emission

Refs.: Ginzburg, Frolov et al.

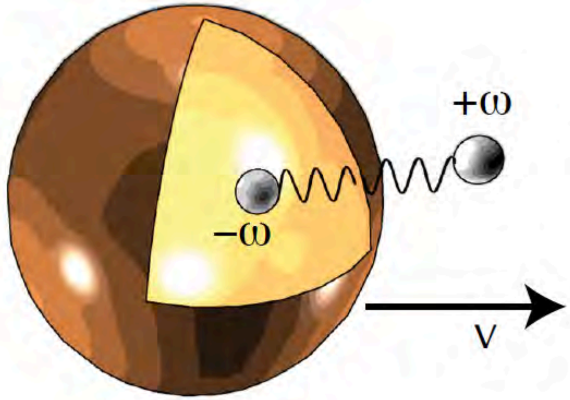


Relativistic Quantum effects

Superluminal, constant v

> Hawking effect
    -> v gradient: sub->superluminal
    -> event horizon -> Hawking radiation

$$kT = \frac{\hbar g_s}{2\pi c}$$

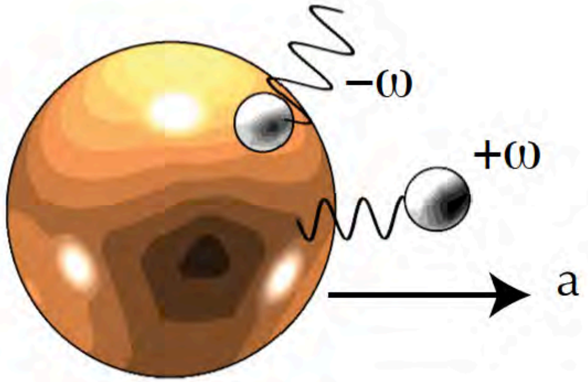


Relativistic Quantum effects

uniform acceleration

> Fulling-Davies-Unruh effect
    -> heated vacuum

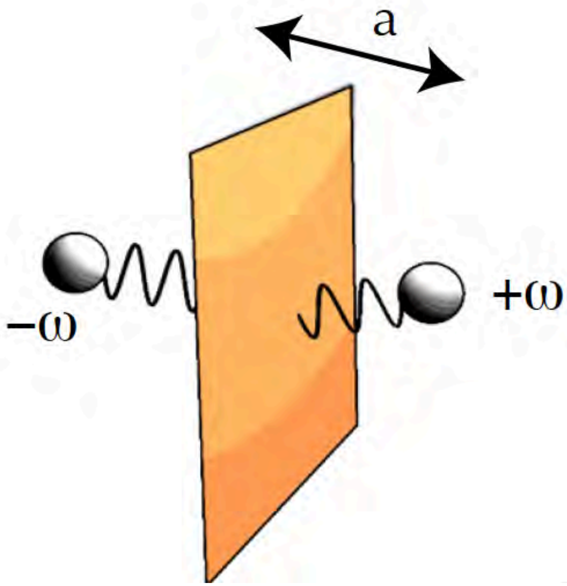
$$kT = \frac{\hbar a}{2\pi c}$$



non-uniform acceleration

> Dynamical Casimir effect
    -> oscillating mirror or cavity

Refs.: S.A. Fulling, P.C.W. Davies, Proc. R. Soc. Lond. 348, 393 (1976);



Relativistic Quantum effects

	v	threshold	T
Anomalous Doppler	v=const	v$\geq c/n$	
Hawking	v=const	v$\geq c/n$	kT $= \hbar g_s/2\pi c$
Unruh	a =const	a$\geq 10^{22}g$	kT $= \hbar a/2\pi c$
Dynamical Casimir	a $\neq$ const		

Light in moving media

The main idea is that non-stationary materials can be used to quantum field theory in curved spacetime backgrounds.

Some history…

1818 – Fresnel -> Moving medium drags light

$$v = \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)$$

1923 – Gordon metric -> first analogy between GR and light in moving medium

1950's - Pham Mau Quam -> ray optics in moving media

Landau, Lifshitz -> light in moving media, but 'neglect light effects due to the possibility of a velocity gradient'

Light in moving media

How does one move a medium at $v \to c$?

Wave eq. in 1D $$\partial_t^2 \Phi - c^2 \partial_x^2 \Phi = 0$$

Include a space-time varying n $$\frac{n^2(x - vt)}{c^2} \partial_t^2 \Phi - \partial_x^2 \Phi = 0.$$

Can derive a metric $$ds^2 = \frac{c^2}{n^2(x - vt)} dt^2 - dx^2$$

The properties of the spacetime metric 'seen' by ϕ are fully determined by

$$n^2(x - vt)$$

Light in moving media

The problem of creating a desired spacetime metric or Space-time varying background for an optical waves is reduced to the problem of creating the desired refractive index profile.

One way to do this is to act on the material polarisation

We can use nonlinear optical effects to modify the refractive index in space and time

Light in moving media

$$P = \varepsilon_0(\chi^{(1)} + \chi^{(3)}E^2)E$$

$$E = |E|\cos(\omega t)$$

$$P_{NL} = \frac{1}{4}\chi^{(3)}|E|^3\cos(3\omega t) + \frac{3}{4}\chi^{(3)}|E|^3\cos(\omega t)$$

$$P = \varepsilon_0\left(\chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2\right)E$$

$$n = \sqrt{1 + \chi^{(1)} + \frac{3}{4}\chi^{(3)}|E|^2}$$

$$n = n_0 + \delta n$$

$$\delta n = n_2 I(x - vt) \qquad n_2 = \frac{3}{8}\frac{\chi^{(3)}}{n_0}$$

Light in moving media

Use nonlinear effects of any order...

$$P = \varepsilon_0(\chi^{(1)} + \chi^{(3)} E^2)E$$

$$P = \varepsilon_0(\chi^{(1)} + \chi^{(2)} E)E$$

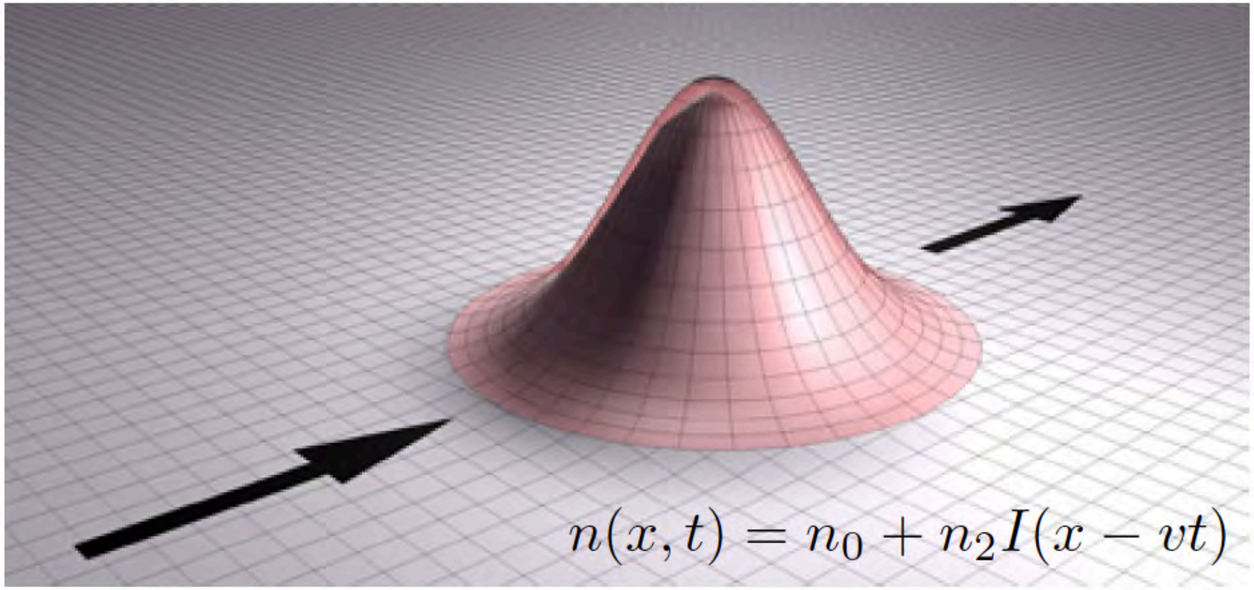
$$\chi^{(2)} : \chi^{(2)} \to \chi^{(3)}$$

Common feature is the effective medium is controlled by a “pump” pulse, E

Light in moving media

Some examples:

Spacetime metrics with horizons:



Light in moving media

Some examples:

Spacetime metrics with horizons:

$$ds^2 = \frac{c^2}{n^2(x-vt)}dt^2 - dx^2.$$

$$t' = \gamma(t - \tfrac{v}{c^2}x),\ x' = \gamma(x - vt)$$

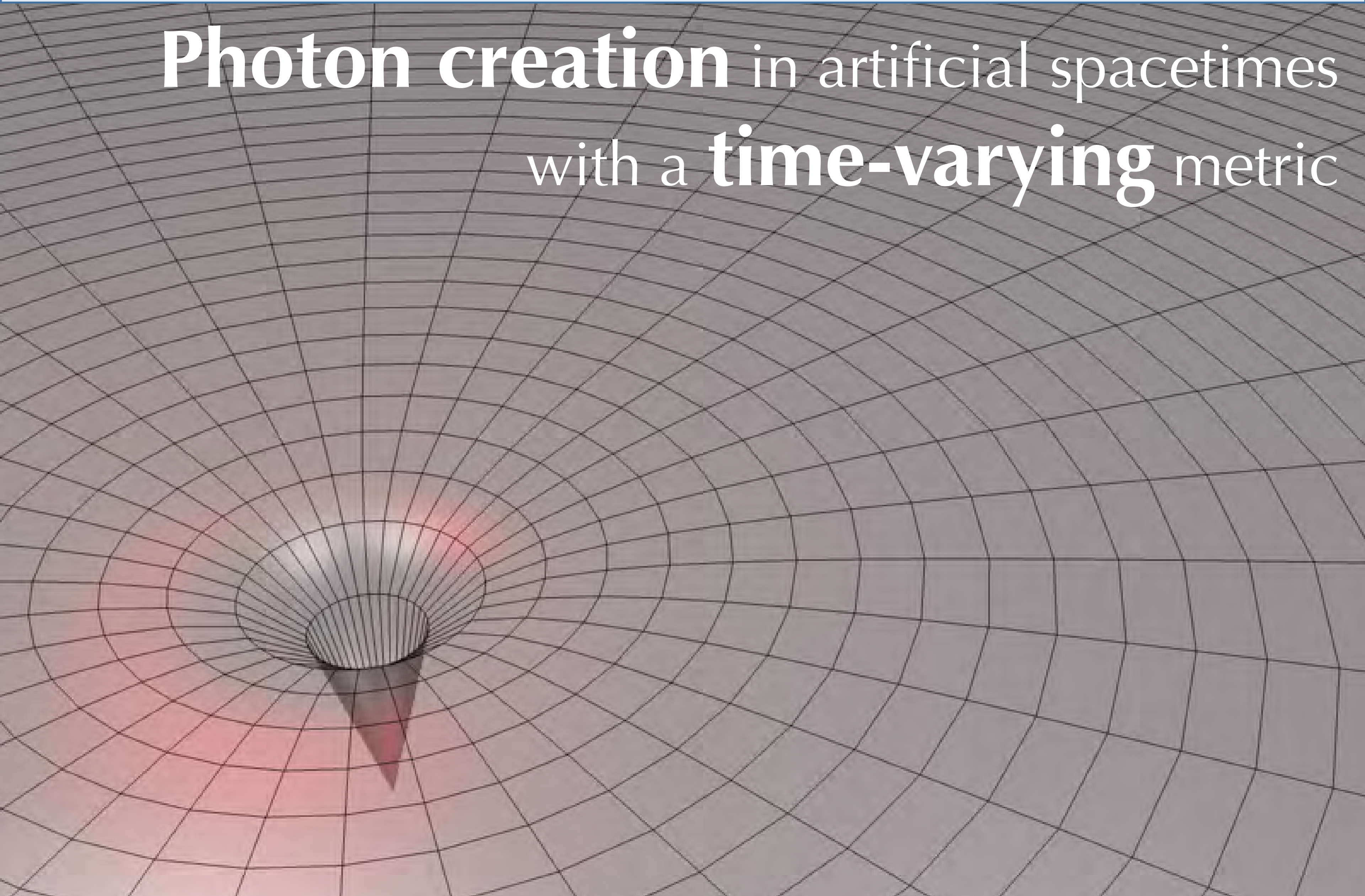
$$ds^2 = c^2\gamma^2\frac{1}{n^2}(1+\frac{nv}{c})(1-\frac{nv}{c})dt'^2 + 2\gamma^2\frac{v}{n^2}(1-n^2)dt'dx' - \gamma^2(1+\frac{v}{nc})(1-\frac{v}{nc})dx'^2,$$

Have an ergosphere at $g_{00} = 0$

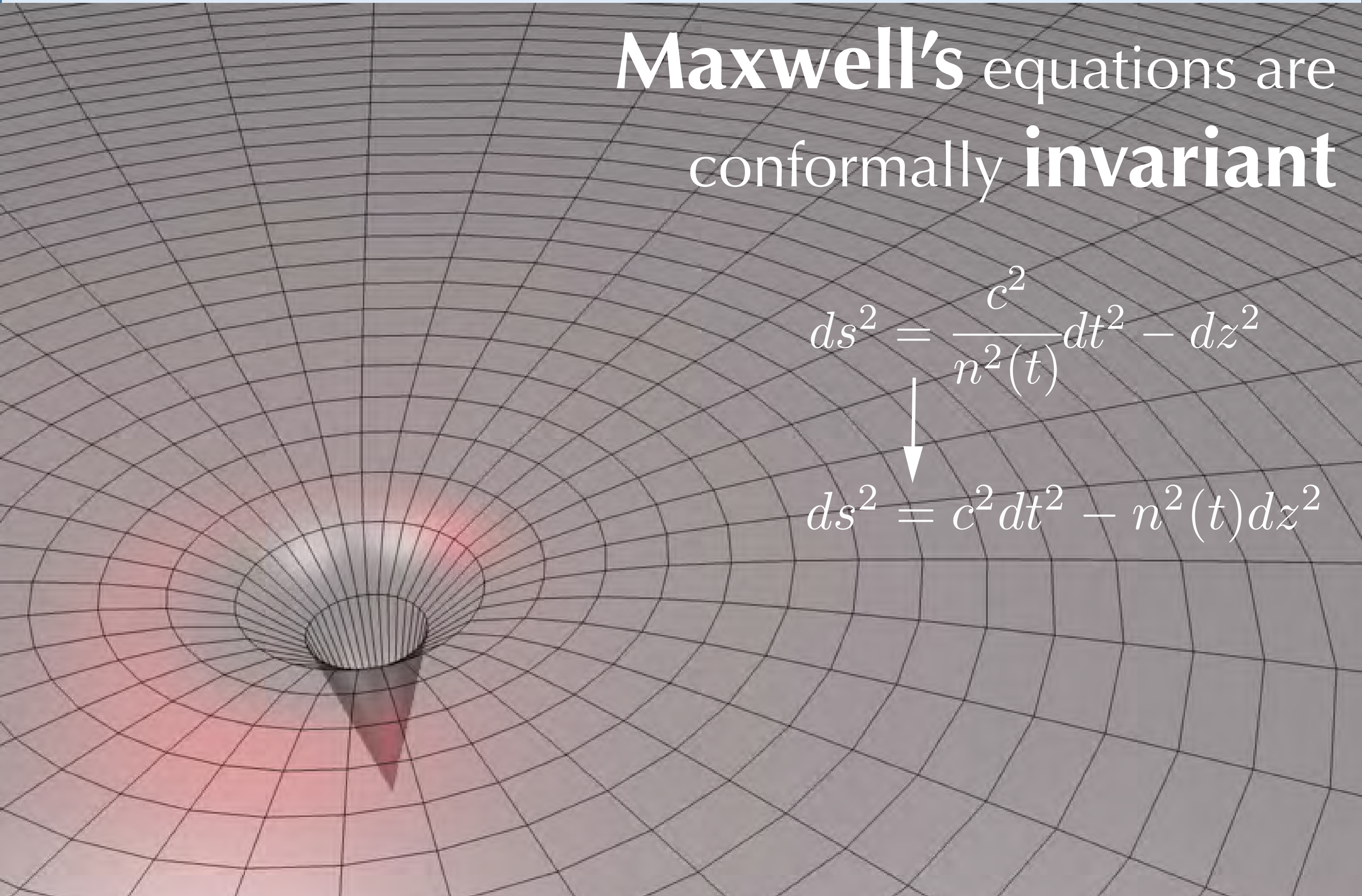
$$1 - n(u)\tfrac{v}{c} = 0$$

Actually, an horizon for $\dfrac{1}{n_0+\delta n} < \dfrac{v}{c} < \dfrac{1}{n_0}$

Light in time varying media



Light in time varying media



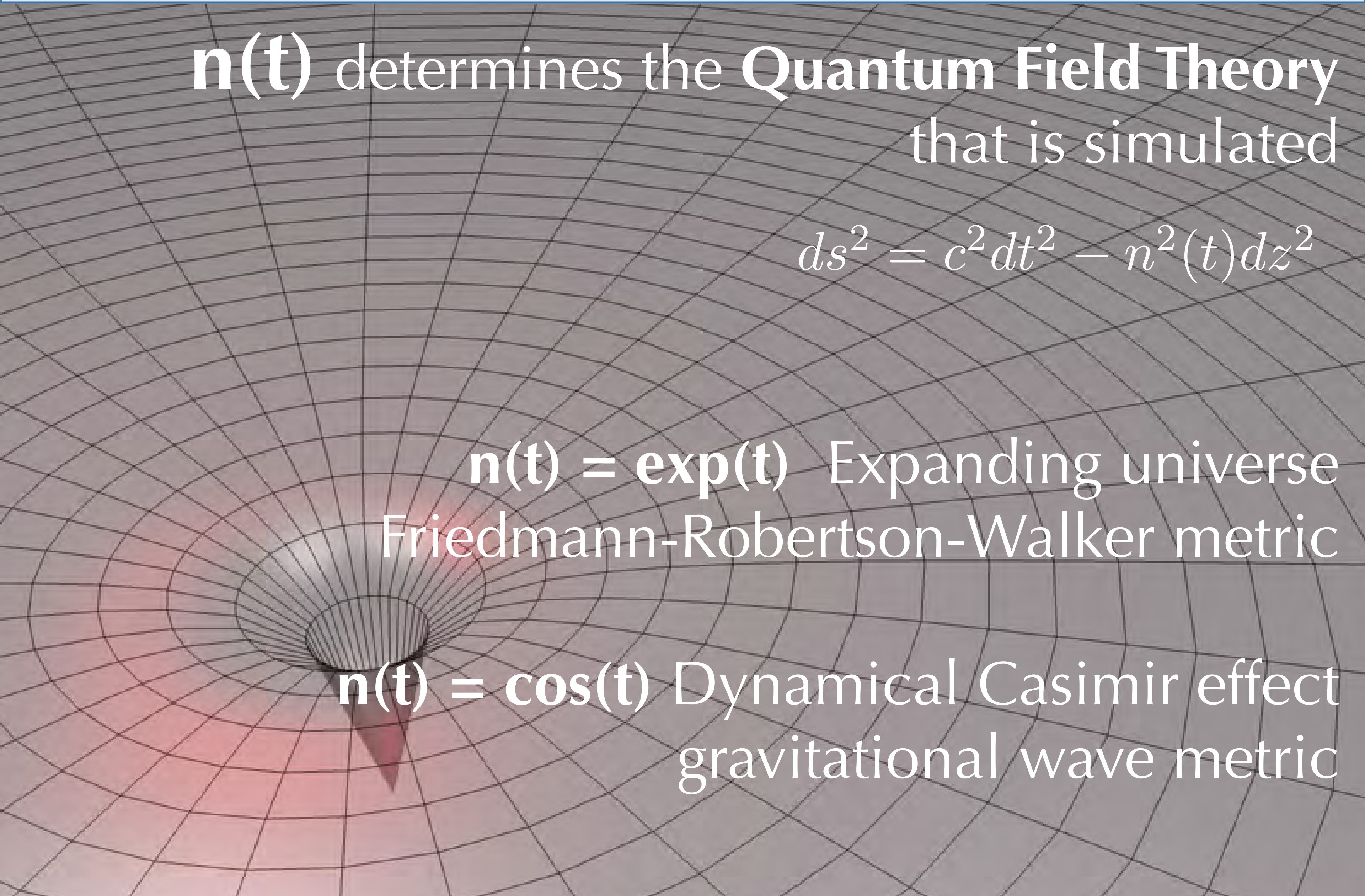
Maxwell's equations are conformally **invariant**

$$ds^2 = \frac{c^2}{n^2(t)}dt^2 - dz^2$$

$$\downarrow$$

$$ds^2 = c^2 dt^2 - n^2(t) dz^2$$

Light in time varying media



n(t) determines the **Quantum Field Theory** that is simulated

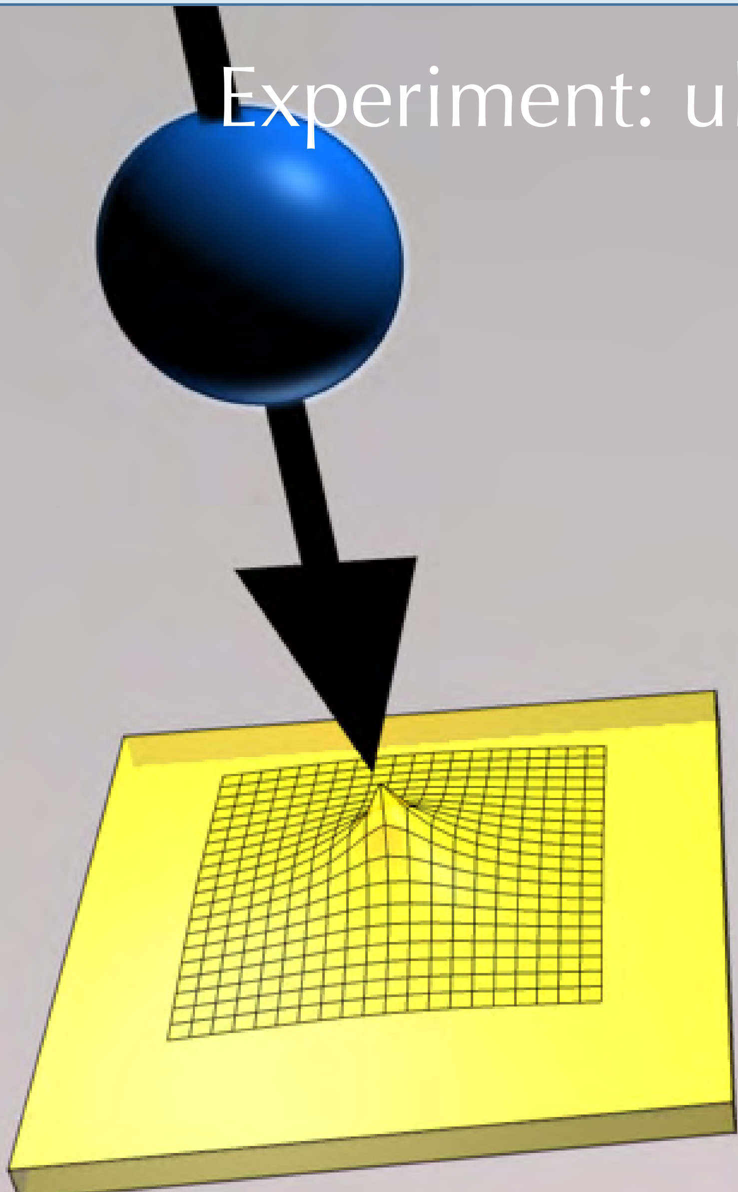
$$ds^2 = c^2 dt^2 - n^2(t) dz^2$$

n(t) = exp(t) Expanding universe
Friedmann-Robertson-Walker metric

n(t) = cos(t) Dynamical Casimir effect
gravitational wave metric

Light in time varying media

Experiment: ultrashort **laser pulse** hits a **thin** nonlinear **film**



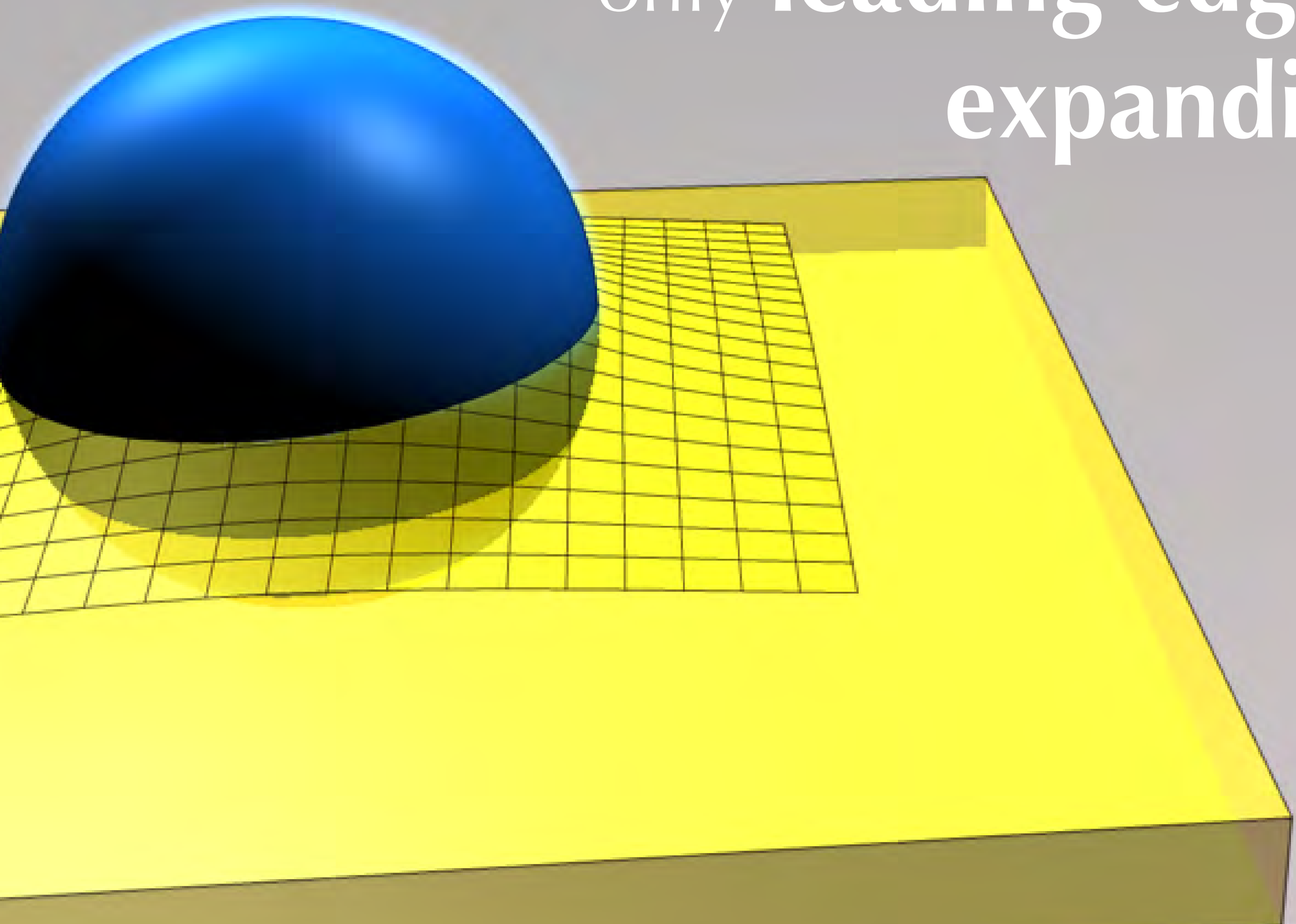
n(t) variation created by nonlinear Kerr effect

Dynamical Casimir emission

film is much thinner than laser pulse

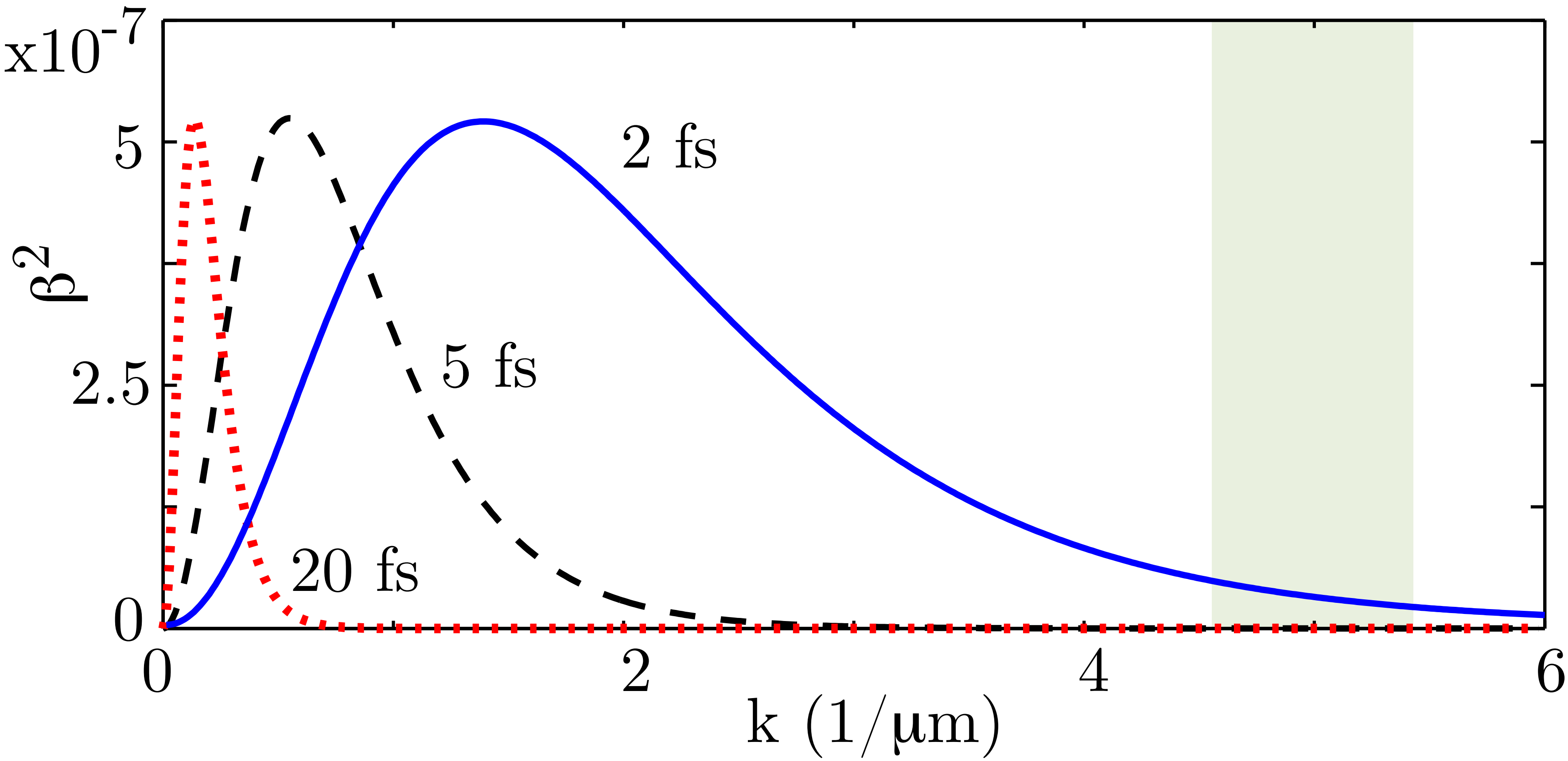
only **leading edge** overlaps =

expanding universe

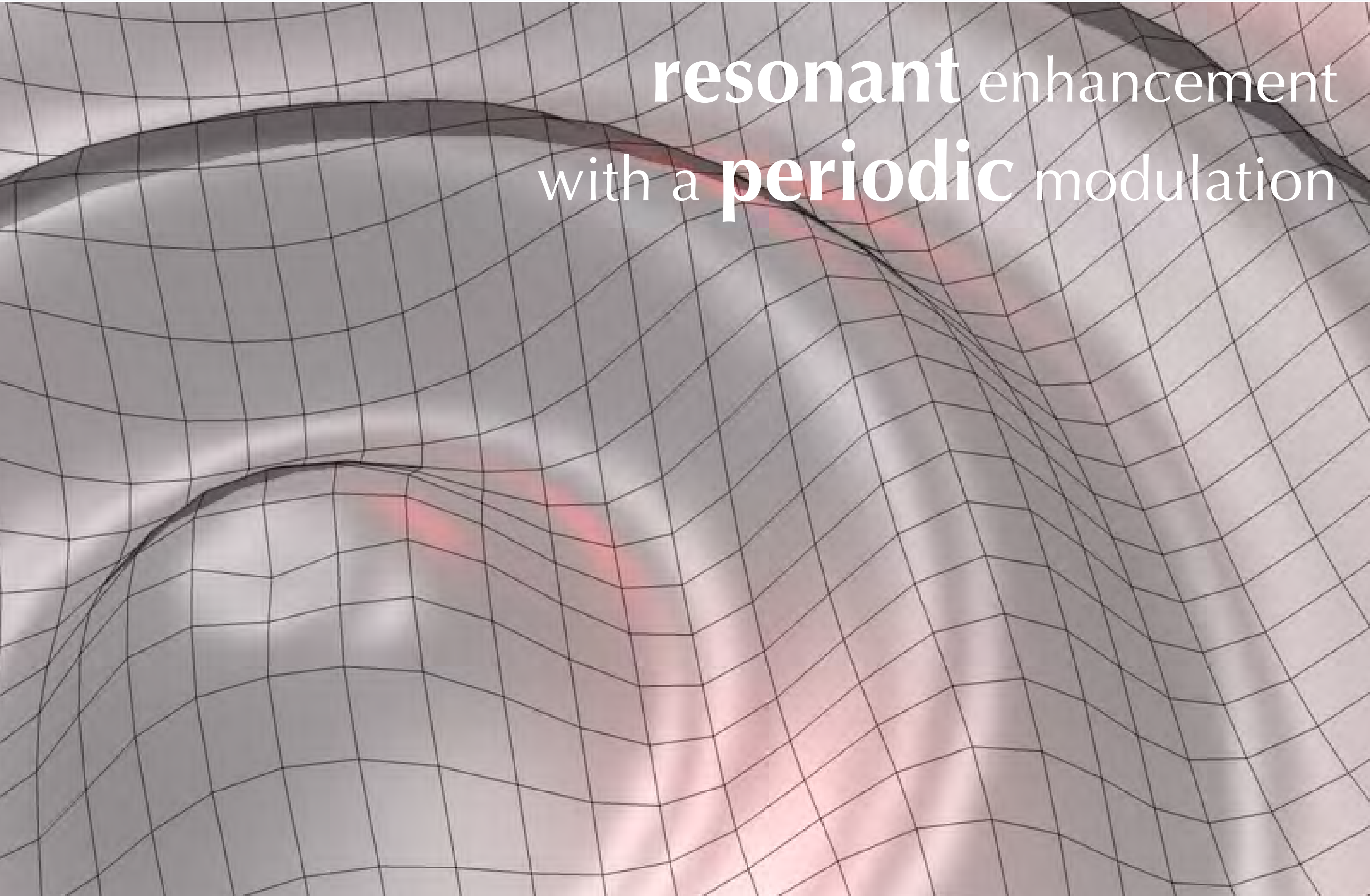


Dynamical Casimir emission

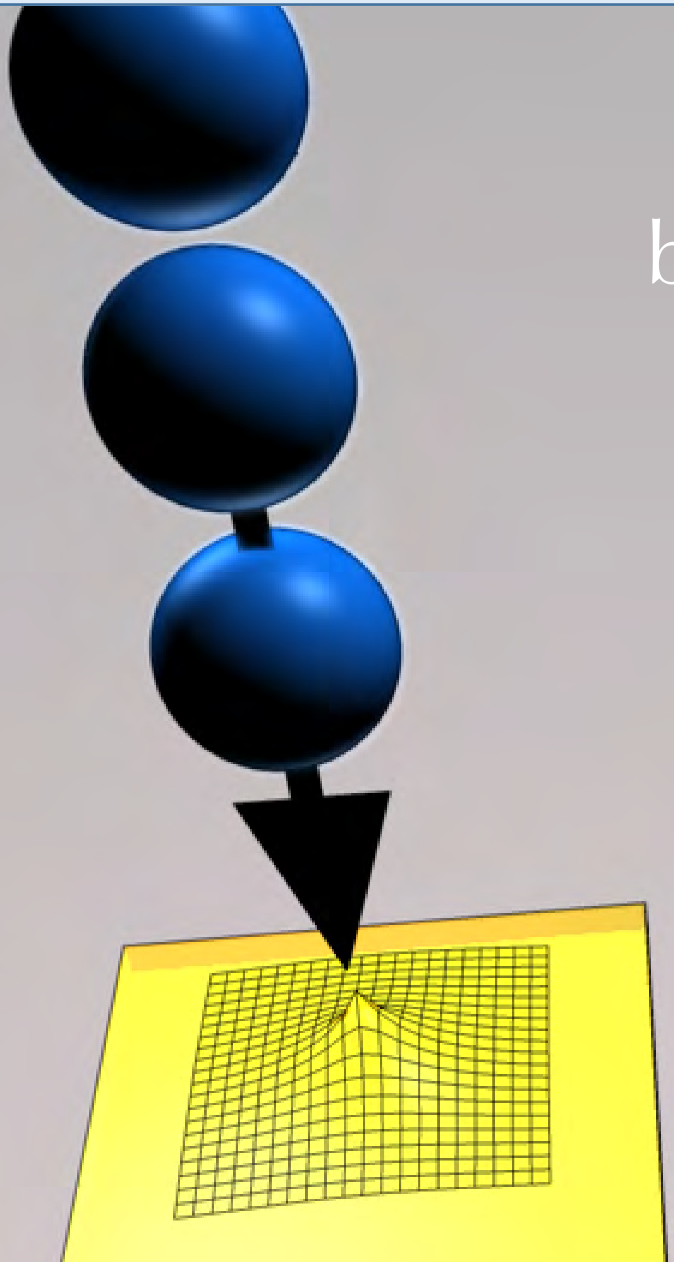
Emission is extremely weak and typically
at very long wavelengths



Resonantly enhanced DCE



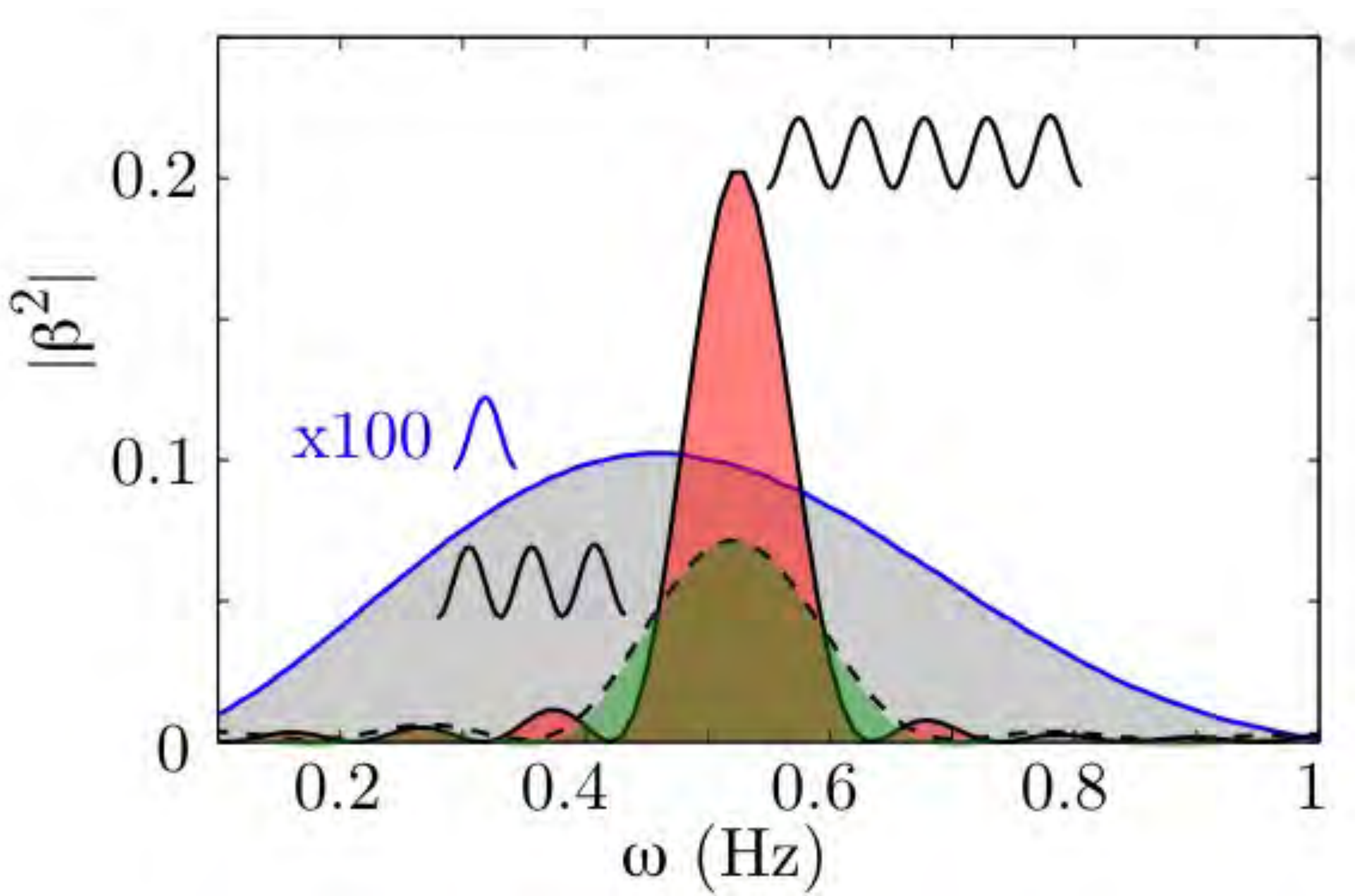
Resonantly enhanced DCE



periodicity is obtained
by using a **train** of laser pulses

Fourier synthesis in the
optical domain allows **PHz**
pulse trains

Resonantly enhanced DCE



Light in moving media

Oscillating spacetime metric acts like a **gravitational** wave

NLO metric

$$ds^2 = c^2 dt^2 - dx^2[n_0 + \delta n \cos(\omega t)]^2$$

$$\frac{dx}{dt} \simeq \frac{c}{n_0}\left[1 - \frac{\delta n}{n_0}\cos(\omega t)\right]$$

GW metric

$$ds^2 = c^2 dt^2 - dx^2[1 + h_+ \cos(\omega t)]$$

$$\frac{dx}{dt} = \frac{c}{\sqrt{1 + h_+ \cos(\omega t)}} \simeq c\left[1 - \frac{h_+}{2}\cos(\omega t)\right]$$

QFT calculations in time-varying media

End of examples.

Lets try and be a bit more quantitative.

Take the purely time-varying case as our study case

QFT calculations in time-varying media

The main claim of this section is:

A time varying medium leads to a time-dependent frequency of any propagating waves.

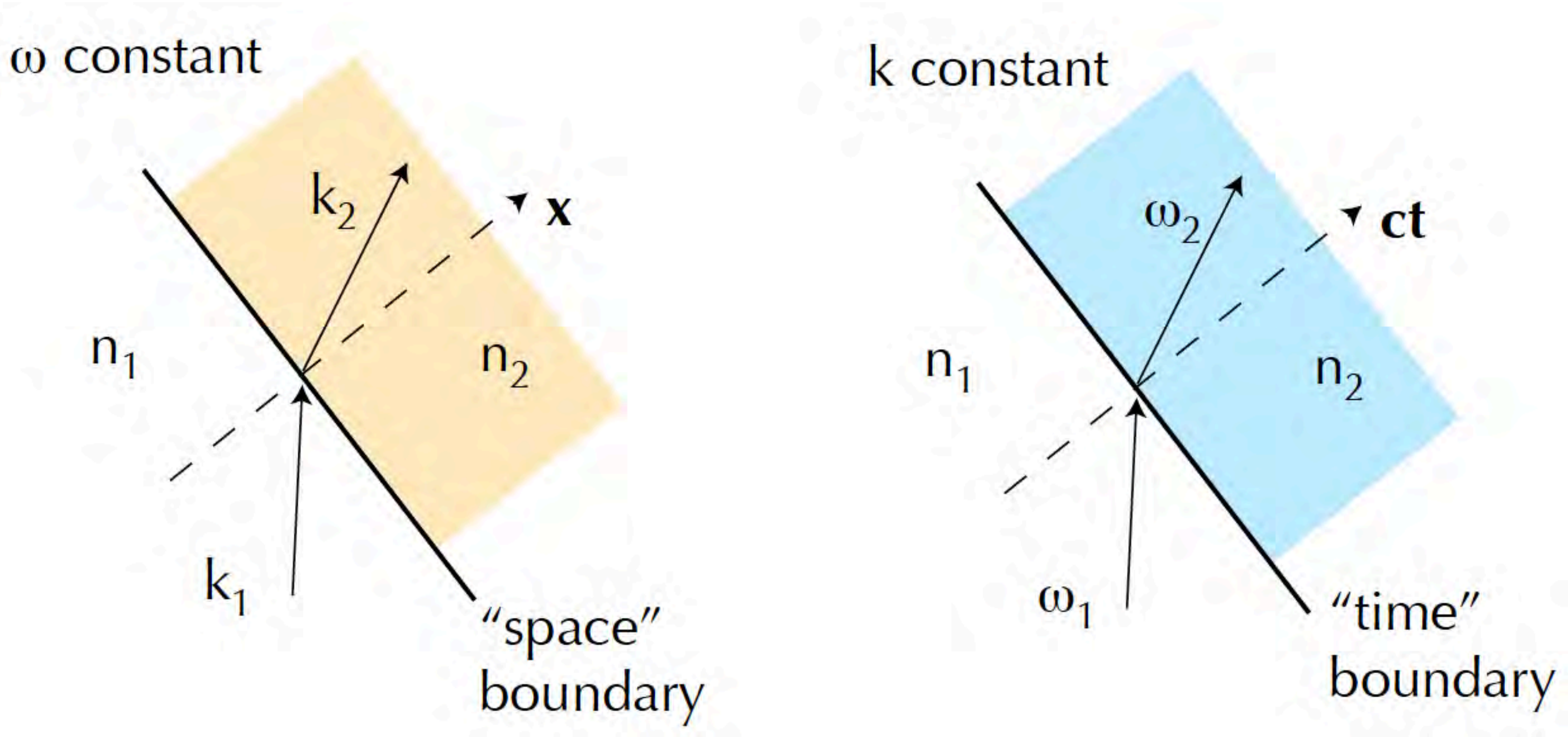
The corollary to this is:

A time-dependent frequency implies particle (photon) production. Always.

QFT calculations in time-varying media

A time varying medium leads to a time-dependent frequency of any propagating waves.

See papers by Mendonca et al., keyword: `time refraction'



QFT calculations in time-varying media

A spatially uniform, time-varying medium leads to a time-dependent frequency of any propagating waves.

$$E = |E| e^{i\phi(z,t)} \qquad \phi = kz - \omega t$$

$$k = \frac{\partial \phi}{\partial z} \qquad \omega = -\frac{\partial \phi}{\partial t}$$

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \phi}{\partial z} = -\frac{\partial \omega}{\partial z} \qquad \omega = \frac{kc}{n(z,t)}$$

$$\frac{\partial k}{\partial t} = -\frac{\partial \omega}{\partial z} = \frac{\omega}{n}\frac{\partial n}{\partial z} = 0 \qquad \frac{\partial \omega}{\partial t} = -\frac{\omega}{n}\frac{\partial n}{\partial t} - \frac{\partial \omega}{\partial r} = -\frac{\omega}{n}\frac{\partial n}{\partial t}$$

Spatial uniformity

k does not change

Time dependent freq.

ω does change

QFT calculations in time-varying media

A time-dependent frequency implies particle (photon) production. Always.

Rephrased: a quantum harmonic oscillator with a time-dependent frequency necessarily produces particles

Very standard result of QFT

- Cosmological particle production
- Dynamical Casimir effect
- Hawking radiation
- Unruh radiation
- Etc. etc.

QFT calculations in time-varying media

Start from harmonic oscillator Eq. with time-dep. freq.

$$\frac{d^2 f}{dt^2} + \omega^2(t) f = 0$$

Or...even simpler, lets use the text-book space-dep. Eq.

$$\frac{d^2 f}{dx^2} + (E - V(x))\, f = 0$$

Well understood problem. We assume $E > V(x)$

With V decaying for large x $\quad 0 \xleftarrow{x \to -\infty} V(x) \xrightarrow{x \to +\infty} 0$

For large x we have $\quad \dfrac{d^2 f}{dx^2} + E\, f = 0, \qquad (\text{only at } x \to \pm\infty)$

QFT calculations in time-varying media

With solutions $e^{+i\sqrt{E}x}$, and $e^{-i\sqrt{E}x}$

At large distances, the solution is therefore given by

$$Ae^{i\sqrt{E}x} + Be^{-i\sqrt{E}x} \leftarrow f(x) \rightarrow Ce^{i\sqrt{E}x} + De^{-i\sqrt{E}x}$$

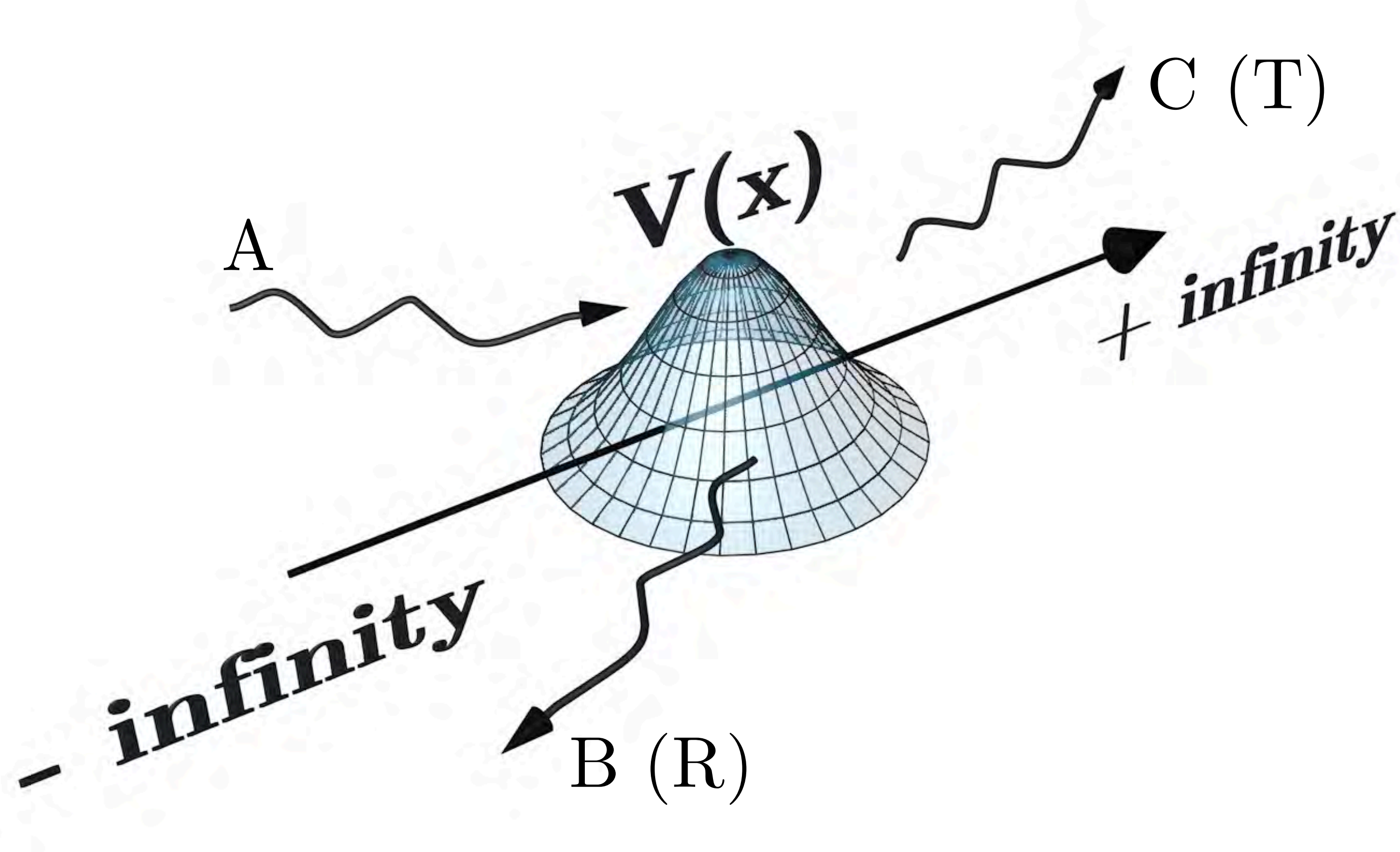
-> combinations of solutions moving to the right (+i) and left (-i)
But we expect only a right-going mode at the output: $D=0$

$$Ae^{i\sqrt{E}x} + Be^{-i\sqrt{E}x} \longleftarrow f(x) \longrightarrow Ce^{i\sqrt{E}x}$$

Rename $B/A=R$ and $C/A=T$, i.e. standard reflectivity and transmissivity

$$e^{i\sqrt{E}x} + Re^{-i\sqrt{E}x} \xleftarrow{x\to-\infty} f(x) \xrightarrow{x\to+\infty} Te^{i\sqrt{E}x}$$

QFT calculations in time-varying media



QFT calculations in time-varying media

Lets use $\omega^2(x) = E - V(x)$

Then our scattering problem is written as $(E => \omega_0^2)$

$$\frac{d^2 f}{dx^2} + \omega^2(x) f = 0$$

$$\omega_0 \xleftarrow{x \to -\infty} \omega(x) \xrightarrow{x \to +\infty} \omega_0$$

Lets take things one step further and actually refer to the full time-dependent problem, by simply exchanging x with -t in the Harmonic oscillator eq.

$$e^{-i\omega t} \leftrightarrow e^{ikx}$$

Note: positive (right-moving) freq. modes are $e^{-i\omega t}$

QFT calculations in time-varying media

$$\frac{d^2 f}{dt^2} + \omega^2(t) f = 0$$

With solutions

$$Be^{+i\omega t} \leftarrow f(t) \rightarrow Ce^{-i\omega t} + De^{+i\omega t}$$

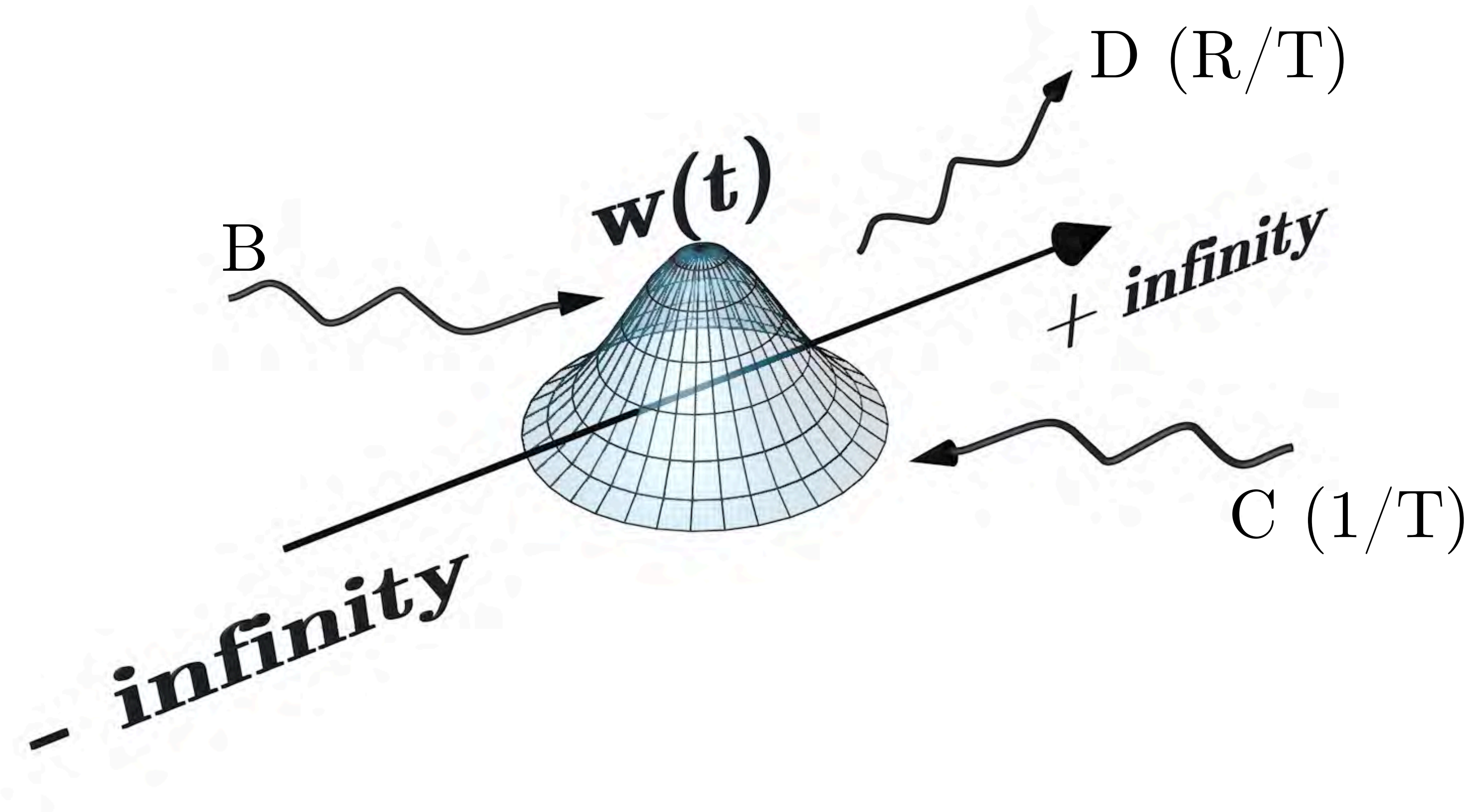
Note that I now put $A=0$ and kept C and D as we cannot expect solutions that propagate bwds in time at the input. <u>This is the key difference when passing from x to t.</u>

We can normalise the input condition, $B=1$ and rename things

$$e^{-i\omega t} \xleftarrow{t\rightarrow-\infty} f(t) \xrightarrow{t\rightarrow+\infty} \alpha e^{-i\omega t} + \beta e^{+i\omega t}$$

α and β are the so-called Bogoliubov coeffients.

QFT calculations in time-varying media



QFT calculations in time-varying media

We see that the Bogo coefficients are related to R and T

$$\alpha := \frac{1}{T}, \quad \text{and } \beta := \frac{R}{T}$$

Note that $E > V(x)$ implies $\omega^2(t) > 0$

Also, the standard scattering relation $|R|^2 + |T|^2 = 1$ implies

$$|\beta|^2 = \frac{|R|^2}{|T|^2} = \frac{1}{|T|^2} - 1$$

Or, also

$$|\alpha|^2 - |\beta|^2 = 1$$

In other words, $|\beta| > 0$ is identical to saying that $|T| < 1$ or $|R| > 0$ for the standard scattering problem.

QFT calculations in time-varying media

Summarising:
reflection $R>0$ occurs at a space boundary and, the temporal analogue of this is $\beta>0$.
A key relation is

$$|\alpha|^2 - |\beta|^2 = 1$$

In terms of reflection and transmission coefficients...

$$\frac{1}{T^2} - \frac{R^2}{T^2} = 1$$

Note that as in the space-scattering problem, we have $R^2 < 1$

QFT calculations in time-varying media

Summarising:
Classical solution contains negative freq. term due to $\omega = \omega(t)$

But why does this imply particle creation?

Hamiltonian for classical time-dep. Oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)q^2$$

Heisenberg picture, introduce operators for fixed freq. oscillator

$$a(t) := \sqrt{\frac{\omega}{2}}\left(q(t) + \frac{\mathrm{i}}{\omega}p(t)\right), \quad a^\dagger := (a)^\dagger$$

$$H = \frac{1}{2}\omega\, a(t)a^\dagger(t) \qquad\qquad a(t) = \mathbf{a}\mathrm{e}^{-\mathrm{i}\omega t}, \quad a^\dagger(t) = \mathbf{a}^\dagger\mathrm{e}^{\mathrm{i}\omega t}$$

QFT calculations in time-varying media

Number operator and Hamiltonian are both indep. of time

as a result of $a(t)a^\dagger(t) = \mathbf{a}\mathbf{a}^\dagger$

$$N := a(t)a^\dagger(t)$$

Total energy is given by $\langle H \rangle = \left(\langle N \rangle + \frac{1}{2} \right) \omega = \left(n + \frac{1}{2} \right) \omega$

In the time-dependent case this cannot be substituted by

$$\langle H \rangle = \left(n + \frac{1}{2} \right) \omega(t)$$

And the ground state is not

$$\langle H \rangle = \frac{1}{2} \omega(t)$$

QFT calculations in time-varying media

Dirac approach to time-independent case:

Split the position operator in two

$$q = \mathbf{a}v(t) + \mathbf{a}^\dagger v^*(t)$$

And impose

$$[q,p] = \mathrm{i} \qquad [\mathbf{a},\mathbf{a}^\dagger] = 1$$

Then solve evolution $\ddot{q} + \omega^2 q = 0$

by imposing $\ddot{v} + \omega^2 v = 0$

Easy solution with constant ω $\qquad v := \dfrac{\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{2\omega}}$

QFT calculations in time-varying media

Time dependent `quantisation' following same approach:

We assume that at early and late times $\omega(t) = \omega_0$
Time variation occurs in the middle, $[-T, T]$

Assume $q = \mathbf{a}v(t) + \mathbf{a}^\dagger v^*(t)$

The equation of motion is now $\ddot{q} + \omega(t)q = 0$

For early times $t < -T$ same sol. as before

$$v := \frac{e^{-i\omega t}}{\sqrt{2\omega}}, \quad t < -T$$

QFT calculations in time-varying media

And we have already solved the problem for $t > T$

$$v = \alpha \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{2\omega}} + \beta \frac{\mathrm{e}^{+\mathrm{i}\omega t}}{\sqrt{2\omega}}, \quad t > T$$

We can now calculate the expectation value for the hamiltonian for the early time vacuum state ($\mathbf{a}|\Omega\rangle = 0$)

$$\begin{aligned}
\langle H \rangle &= \langle \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)q^2 \rangle \\
&= \left(\langle \mathbf{a}\mathbf{a}^\dagger \rangle + \langle \mathbf{a}^\dagger \mathbf{a} \rangle\right) \left(\frac{1}{2}|v'|^2 + \frac{1}{2}\omega^2(t)|v|^2 \right) \\
&= (2n+1)\,\omega(t)\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \\
&= \left(|\beta|^2 + \frac{1}{2}\right)\omega(t).
\end{aligned}$$

QFT calculations in time-varying media

Comparing the two results, i.e. in-vacuum with out-vacuum:

$$\left(n + \frac{1}{2}\right)\omega \quad (n{=}0) \qquad \qquad \left(|\beta|^2 + \frac{1}{2}\right)\omega(t)$$

We see that the out vacuum state is populated with $n = |\beta|^2$

Summarising:

A time dependent freq. transforms $\mathrm{e}^{-\mathrm{i}\omega t}$ into $\alpha\mathrm{e}^{-\mathrm{i}\omega t} + \beta\mathrm{e}^{\mathrm{i}\omega t}$

always with a non-zero β

As a result of this, an input vacuum state is populated @ output

QFT calculations in time-varying media

Recipe for calculating photon production from time-varying background:

1) Define the system by an evolution equation $F(\psi)$

2) Take an input condition for $\psi = e^{i\omega t}$
and then evolve it through F

3) Express the output as $\alpha e^{-i\omega t} + \beta e^{+i\omega t}$

4) Use β to calculate actual photon number

$$\frac{dN}{d\mathrm{Vol}} = \int 2\pi k \, |\beta_k|^2 d^2 k$$