Quantum Trajectory formalism for Weak Measurements

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N. Gisin, Phys. Rev. Lett. 52 (1984) 1657
Apoorva Patel and Parveen Kumar, arXiv:1509.08253

Abstract

Projective measurement is used as a fundamental axiom in quantum mechanics, even though it is discontinuous and cannot predict which measured operator eigenstate will be observed in which experimental run. The probabilistic Born rule gives it an ensemble interpretation, predicting proportions of various outcomes over many experimental runs. Understanding gradual weak measurements requires replacing this scenario with a dynamical evolution equation for the collapse of the quantum state in individual experimental runs. We revisit the quantum trajectory framework that models quantum measurement as a continuous nonlinear stochastic process. We describe the ensemble of quantum trajectories as noise fluctuations on top of geodesics that attract the quantum state towards the measured operator eigenstates. Investigation of the restrictions needed on the ensemble of quantum trajectories, so as to reproduce projective measurement in the appropriate limit, shows that the Born rule follows when the magnitudes of the noise and the attraction are precisely related, in a manner reminiscent of the fluctuation-dissipation relation. That implies that the noise and the attraction have a common origin in the measurement interaction between the system and the apparatus. We analyse the quantum trajectory ensemble for the dynamics of quantum diffusion and quantum jump, and show that the ensemble distribution is completely determined in terms of a single evolution parameter, which can be tested in weak measurement experiments. We comment on how the specific noise may arise in the measuring apparatus.

Axioms of Quantum Dynamics

(1) Unitary evolution (Schrödinger):

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle \ , \quad i\frac{d}{dt}\rho = [H,\rho] \ .$$

Continuous, Reversible, Deterministic.

Pure state evolves to pure state.

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(2) Projective measurement (von Neumann):

$$|\psi\rangle \longrightarrow P_i|\psi\rangle/|P_i|\psi\rangle|,\ P_i = P_i^\dagger,\ P_iP_j = P_i\delta_{ij},\ \sum_i P_i = I.$$

Discontinuous, Irreversible, Probabilistic choice of "i".

Pure state evolves to pure state. Consistent on repetition.

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Instead, with Born rule and ensemble interpretation,

$$prob(i) = \langle\psi|P_i|\psi\rangle = Tr(P_i\rho) \ , \quad \rho \longrightarrow \sum_i P_i\rho P_i \ .$$

Pure state evolves to mixed state. Predicted expectation values are averages over many experimental runs.

Quantum Measurement Terminology

$$\rho_i = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \xrightarrow{\text{Decoherence}} \rho_r = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Collapse ($\rho_i \to \rho_f$) Quantum jump ($\rho_r \to \rho_f$)

$$\rho_f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The evolution steps involved in the quantum measurement process for a qubit (ρ_i and ρ_f are pure states, while $\rho_{\rm r}$ is obtained from an entangled state):

(a) Decoherence deterministically entangles the system with its environment, and drives the off-diagonal reduced density matrix components to zero. Magnitudes of the off-diagonal components are not changed, but their phases are randomised by environmental scattering.

(b) Quantum jump removes the system-apparatus entanglement, and probabilistically converts the diagonal reduced density matrix into a measurement eigenstate.

(c) Collapse is the overall process that yields measurement eigenstates probabilistically. It describes individual experimental outcomes, and may or may not go through decoherence.

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von Neumann "non-demolition" interaction is a particular aspect of the decoherence paradigm. It creates perfect entanglement between the measured eigenstates of the system and the pointer basis states of the apparatus.

$$H_{\mathrm{vN}} = g\ x_S \otimes p_A \ : \quad |x\rangle_S |0\rangle_A \longrightarrow |x\rangle_S |x\rangle_A$$

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Unraveling of quantum collapse:

(a) Quantum jump is discontinuous, probabilistic and irreversible.

(b) Quantum trajectories are continuous, stochastic and tractable.

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New questions:

- Can all measurements be made continuous? What about decays?
- How is the projection replaced by a continuous evolution?
- What is the local evolution rule during measurement?
- What is the state if the measurement is left incomplete?
- How is the ensemble to be interpreted?
- How should multipartite measurements be described?

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The answers are important for increasing accuracy of quantum control and feedback. Knowledge of what happens in a particular experimental run (and not just the ensemble average) can improve efficiency and stability.

The projective measurement axiom needs to be replaced by a different continuous stochastic dynamics.

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Such a dynamical process exists! Gisin (1984)

Salient Features

A precise ratio of evolution towards the measurement eigenstates and unbiased white noise is needed to reproduce the Born rule as a constant of evolution.

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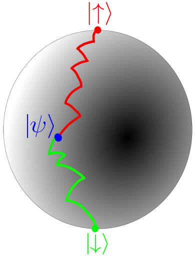
Technological advances allow us to monitor the quantum evolution during weak measurements. That can test the validity of the stochastic measurement formalism, and then help us figure out what may lie beyond.



Measurement $\equiv$ An effective process of a more fundamental theory.

Beyond Quantum Mechanics

Physical:

(1) Hidden variables with novel dynamics may produce quantum mechanics as an effective theory, with extra rules supplementing Schrödinger's equation.

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Bypass:

Many worlds interpretation—each evolutionary branch is a different world, and we only observe the measurement outcome corresponding to the world we live in (anthropic principle).

Uncountable proliferation of evolutionary branches is highly ungainly.

Quantum Geodesic Trajectory

Let the projective measurement arise from a continuous geodesic evolution, with parameter $s \in [0,1]$:

$$|\psi\rangle \longrightarrow Q_i(s)|\psi\rangle / |Q_i(s)|\psi\rangle| \ , \quad Q_i(s) = (1-s)I + sP_i \ .$$

Then the quantum trajectory evolves as

$$\rho \longrightarrow \frac{(1-s)^2\rho + s(1-s)(\rho P_i + P_i\rho) + s^2 P_i\rho P_i}{(1-s)^2 + (2s-s^2)\,Tr(P_i\rho)} \ , \quad Tr(\rho) = 1 \ .$$

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Expansion around $s = 0$ gives the geodesic evolution equation:

$$\frac{d}{dt}\rho = g[\rho P_i + P_i\rho - 2\rho \ Tr(P_i\rho)] \ .$$

$s \to gt$ in terms of the system-apparatus coupling g, and the "measurement time" t.

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- This nonlinear evolution preserves pure states,

$$\rho^2 = \rho \Longrightarrow \frac{d}{dt}(\rho^2 - \rho) = \rho\frac{d}{dt}\rho + (\frac{d}{dt}\rho)\rho - \frac{d}{dt}\rho = 0 \ ,$$

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in addition to maintaining $Tr(\rho) = 1$.

- Projective measurement is the fixed point of this equation:

$$\frac{d}{dt}\rho = 0 \quad \text{at} \quad \rho^* = P_i\rho P_i / Tr(P_i\rho) \ .$$

Convergence to fixed point makes the measurement consistent on repetition.

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- In a bipartite setting, $\{P_i\} = \{P_{i_1} \otimes P_{i_2}\}$. The evolution is linear in the projection operators, and $\sum_i P_i = I$. So partial trace over the unobserved environment gives the same equation for the reduced density matrix for the system, as long as g is independent of the environment.

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- For pure states, the equation can be written as:

$$\frac{d}{dt}\rho = -2g\mathcal{L}[\rho]P_i$$

This structure (involving the Lindblad operator) hints at an action-reaction relation between the processes of decoherence and collapse, possibly following from a conservation law.

Interpretation: When $\mathcal{L}[\rho]P_i$ decoheres the apparatus pointer state P_i (it cannot remain in superposition by definition), there is an equal and opposite effect $-\mathcal{L}[\rho]P_i$ on the system state ρ leading to its collapse.

Ensemble of Quantum Geodesic Trajectories

The prefered basis $\{P_i\}$ is fixed by the system-apparatus interaction, but there are many fixed points, requiring a separate criterion to determine which P_i will occur in a particular experimental run.

Quantum jump: The geodesic trajectory is chosen at some point during the measurement and remains unaltered thereafter.

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For describing evolution during weak measurements, a local dynamical rule governing quantum trajectories is desirable.

Assign time-dependent real weights $w_i(t)$ to the evolution trajectories for P_i, which depend only on the measured degrees of freedom:

$$\frac{d}{dt}\rho = \sum_i w_i \; g[\rho P_i + P_i \rho - 2\rho Tr(P_i \rho)] \; , \quad \sum_i w_i = 1 \; .$$

Evolution still preserves $\rho^2 = \rho$. Every $\rho = P_i$ becomes a fixed point.

Ensemble Evolution

The weighted trajectory evolution is:

$$\frac{d}{dt}(P_j\rho P_k) = P_j\rho P_k \; g[w_j + w_k - 2\sum_i w_i Tr(P_i\rho)] \; .$$

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$$\frac{2}{P_j\rho P_k}\frac{d}{dt}(P_j\rho P_k) = \frac{1}{P_j\rho P_j}\frac{d}{dt}(P_j\rho P_j) + \frac{1}{P_k\rho P_k}\frac{d}{dt}(P_k\rho P_k)$$

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For one-dimensional projections, $P_j\rho(t)P_j = d_j(t)P_j$,

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Phases of the off-diagonal projections $P_j\rho P_k$ do not change.

Also, off-diagonal $P_j\rho P_k$ may not vanish asymptotically.

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The diagonal projections evolve according to:

$$\frac{d}{dt} d_j = 2g \; d_j (w_j - w_{\rm av}) \; , \quad w_{\rm av} \equiv \sum_i w_i d_i \; .$$

Evolution is restricted to the subspace spanned by all the $d_j(t=0) \neq 0$.

Diagonal elements with $w_j > w_{\rm av}$ grow; those with $w_j < w_{\rm av}$ decay.

This behaviour is stable under small perturbations of ρ.

Choice of Trajectory Weights

Instantaneous Born rule: $w_j = w_j^{IB} \equiv Tr(\rho(t)P_j)$

This is a local and appealing choice for the trajectory weights throughout the measurement process. Then

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Noise can be added to the geodesic trajectory weights w_i, in a structure similar to the variational calculus. Possibilities include:

(a) White noise (quantum diffusion), (b) Shot noise (quantum jump).

Quantum Diffusion: Single Qubit Measurement

The evolution equations simplify considerably for a qubit.

Let $|0\rangle$ and $|1\rangle$ be the measurement eigenstates.

$$\frac{d}{dt}\rho_{00} = 2g\ (w_0 - w_1)\rho_{00}\rho_{11}\ ,$$

$$\rho_{01}(t) = \rho_{01}(0)\left[\frac{\rho_{00}(t)\rho_{11}(t)}{\rho_{00}(0)\rho_{11}(0)}\right]^{1/2} .$$

With $\rho_{11}(t) = 1 - \rho_{00}(t)$ and $w_1(t) = 1 - w_0(t)$, only one independent variable describes evolution of the system.

Quantum Diffusion: Single Qubit Measurement

The evolution equations simplify considerably for a qubit.
Let $|0\rangle$ and $|1\rangle$ be the measurement eigenstates.

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Evolution obeys Langevin dynamics, when unbiased white noise with spectral density S_ξ is added to w_i^{IB}. The trajectory weights become:

$$w_0 - w_1 = \rho_{00} - \rho_{11} + \sqrt{S_\xi}\ \xi\ .$$
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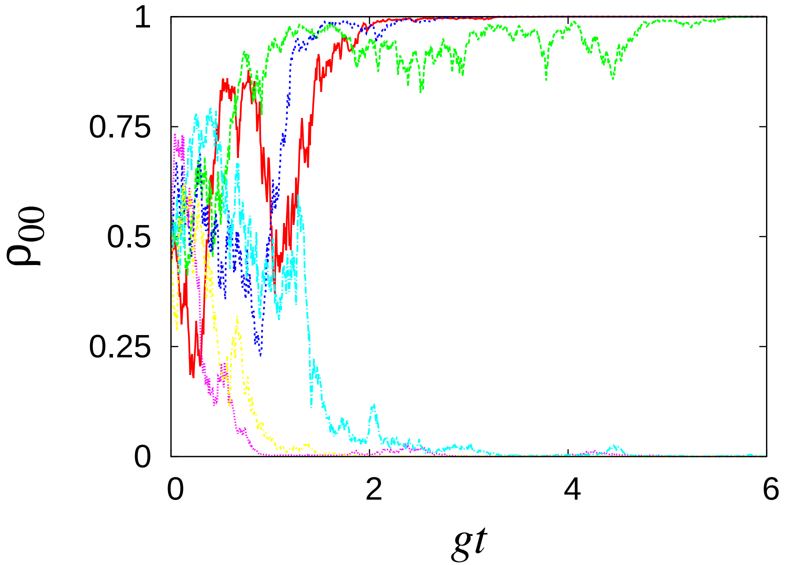
This is a stochastic differential process on the interval $[0,1]$.
The fixed points at $\rho_{00} = 0, 1$ are perfectly absorbing boundaries.
A quantum trajectory would zig-zag through the interval before ending at one of the two boundary points.



Individual quantum evolution trajectories for the initial state $\rho_{00} = 0.5$, with measurement eigenstates $\rho_{00} = 0, 1$, and in presence of measurement noise satisfying $gS_\xi = 1$.

Single Qubit Measurement (contd.)

Let $P(x)$ be the probability that the initial state with $\rho_{00} = x$ evolves to the fixed point at $\rho_{00} = 1$. Then by symmetry,

$$P(0) = 0,\ P(0.5) = 0.5,\ P(1) = 1\ .$$

No noise : $S_\xi = 0 \implies P(x) = \theta(x - 0.5)$.

Only noise : $S_\xi \to \infty \implies P(x) = 0.5$.

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It is instructive to convert the stochastic evolution equation from the differential Stratonovich form to the Itô form that specifies forward evolutionary increments:

$$d\rho_{00} = 2g \; \rho_{00}\rho_{11}(\rho_{00} - \rho_{11})(1 - gS_\xi)dt + 2g\sqrt{S_\xi} \; \rho_{00}\rho_{11} \; dW \; ,$$

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The Wiener increment dW can be modeled as a random walk.

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The Wiener increment dW can be modeled as a random walk.

The first term produces drift in the evolution, while the second gives rise to diffusion. The evolution with no drift, i.e. the pure Wiener process with $gS_\xi = 1$, is rather special:

$\langle d\rho_{00}\rangle = 0 \iff$ Born rule is a constant of evolution.

Single Qubit Measurement (contd.)

In absence of drift, starting at x, one moves to $x + \epsilon$ with some probability, moves to $x - \epsilon$ with the same probability, and stays put otherwise. Balancing the probabilities,

$$P(x) = \alpha(P(x+\epsilon) + P(x-\epsilon)) + (1-2\alpha)P(x) \; .$$

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The general solution, independent of the choice of α and ϵ, is that $P(x)$ is a linear function of x, which is the Born rule:

$$\boxed{gS_\xi = 1,\ \ P(0) = 0, P(1) = 1 \implies P(x) = x}$$

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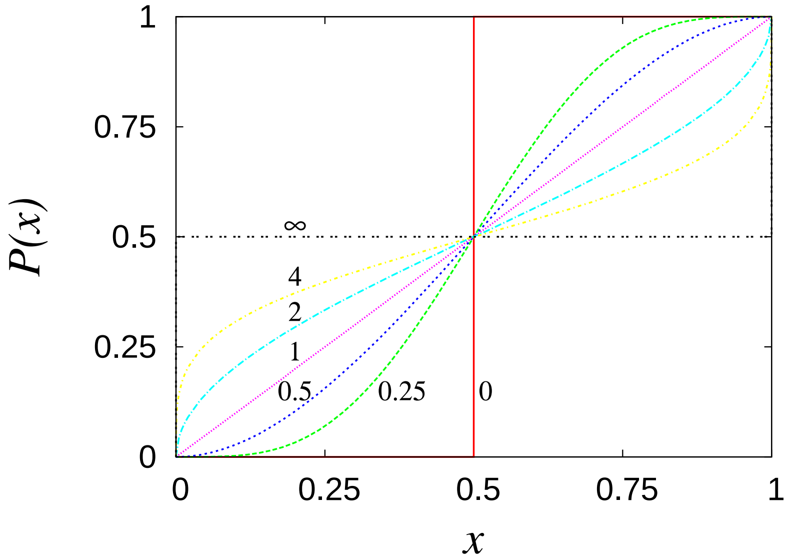
Numerical tests were performed for different values of gS_ξ.

$$\frac{\rho_{00}(t+\tau)}{\rho_{11}(t+\tau)} = \frac{\rho_{00}(t)}{\rho_{11}(t)} e^{2g\tau\overline{w}} \ , \quad \overline{w} = \frac{1}{\tau}\int_t^{t+\tau}(w_0 - w_1)dt \ .$$

With $g\tau \ll 1$, $\overline{w}$ was generated as a Gaussian random number with mean $\rho_{00}(t) - \rho_{11}(t)$ and variance S_ξ/τ.

The data clearly show the special status of $gS_\xi = 1$.



Probability that the initial qubit state $\rho_{00} = x$ evolves to the measurement eigenstate $\rho_{00} = 1$, for different values of the measurement noise. The gS_ξ values label the curves.

Ensemble Evolution Dynamics

During measurement, the probability distribution $p(\rho_{00}, t)$ of the set of quantum trajectories evolves according to the Fokker-Planck equation (with $gS_\xi = 1$):

$$\frac{\partial p(\rho_{00},t)}{\partial t} = 2g \frac{\partial^2}{\partial^2 \rho_{00}} \left(\rho_{00}^2 (1-\rho_{00})^2 p(\rho_{00}, t) \right) \ .$$

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Let $\tanh(z) = 2\rho_{00} - 1 = \rho_{00} - \rho_{11}$ map $\rho_{00} \in [0,1]$ to $z \in (-\infty, \infty)$.

$$\frac{dz}{dt} = g \tanh(z) + \sqrt{g}\ \xi \ , \quad dz = g \tanh(z)\ dt + \sqrt{g}\ dW \ ,$$

$$\frac{\partial p(z,t)}{\partial t} = -g \frac{\partial}{\partial z} \left(\tanh(z)\ p(z,t)\right) + \frac{g}{2} \frac{\partial^2}{\partial^2 z} p(z,t) \ .$$

Then the two peaks are diffusing Gaussians, with their centres at $z_\pm(t) = \tanh^{-1}(2x-1) \pm gt$ and common variance gt.

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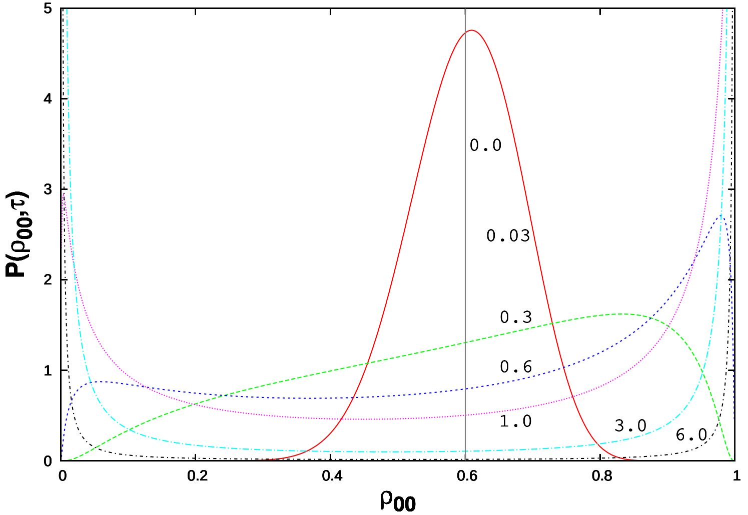
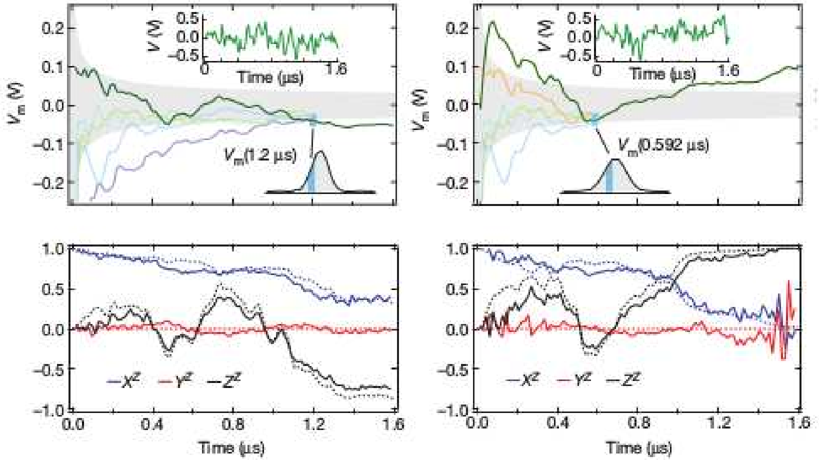
The peaks reach the boundaries only asymptotically, as $t \to \infty$.
The precise nature of this distribution is experimentally testable.



Distribution of the quantum measurement trajectories for quantum diffusion evolution of a qubit state. The initial state is $\rho_{00}(\tau = 0) = 0.6$, and the curves are labeled by the values of the evolution parameter $\tau \equiv \int_0^t g(t)dt$. The narrow initial distribution splits into two non-interfering components that converge to the measurement eigenstates at $\rho_{00} = 1, 0$ as $\tau \to \infty$.



Observed quantum trajectories for weak Z-measurement of a superconducting qubit. The initial state is polarised along the X-axis. The top panels show the measured voltage distribution as a function of time, together with a few individual contributions. The lower panels display quantum trajectories obtained from the measured signal (dotted lines), and those reconstructed using tomography (solid lines).

Murch et al. (2013)

Ensemble Evolution Dynamics (contd.)

The evolving probability distribution is:

$$p(z,t) = \frac{1}{\sqrt{2\pi g t}} \left(x \exp\left[-\frac{(z-z_+)^2}{2gt} \right] + (1-x) \exp\left[-\frac{(z-z_-)^2}{2gt} \right] \right) .$$

Upon taking the ensemble average,

$$\int_{\infty}^{\infty} \tanh(z(t))\ p(z,t)\ dz = 2x - 1 \ ,$$

$$\int_{\infty}^{\infty} \mathrm{sech}(z(t))\ p(z,t)\ dz = e^{-gt/2} \mathrm{sech}(z(0)) \ .$$

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The resultant expectation value of the density matrix is:

$$\rho(0) = \begin{pmatrix} x & \rho_{01}(0) \\ \rho_{10}(0) & 1-x \end{pmatrix} \implies \langle \rho(t) \rangle = \begin{pmatrix} x & e^{-gt/2}\rho_{01}(0) \\ e^{-gt/2}\rho_{10}(0) & 1-x \end{pmatrix} .$$

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It is identical to the solution of the Lindblad master equation for the same system, with the single decoherence operator $L_\mu = \sqrt{\gamma}\sigma_3$, $\gamma = g/4$:

$$\frac{d}{dt}\rho = \gamma(\sigma_3 \rho \sigma_3 - \rho) \ .$$

The details of averaging differ: Here the off-diagonal elements are driven to zero, without any change in phase, by the dynamics of the diagonal elements. In decoherence, the off-diagonal elements are driven to zero, by an average over the fluctuating scattering phases.

Fluctuation-Dissipation Relation

The size of the fluctuations is:

$$\langle (d\rho_{00})^2 \rangle = 4g^2 S_\xi \ \rho_{00}^2 \rho_{11}^2 \ dt \ .$$

The geodesic evolution term is:

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Dropping the subleading $o(dt)$ terms, $gS_\xi = 1$ gives the coupling-free relation:

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In general stochastic processes, vanishing drift and fluctuation-dissipation relation are quite unrelated properties. The fact that both lead to the Born rule is a remarkable feature of quantum trajectory dynamics.

Larger Quantum Systems

Preceding results are valid for binary orthogonal measurements on any quantum system, with the replacement $\rho_{ii} \to Tr(\rho P_i)$.

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Another option for n-dimensional quantum measurements is to use the orthonormal set of weights in the convention of $SU(n)$ Cartan generators $(k = 1, \ldots, n-1)$:

$$\sum_{i=0}^{k-1} w_i - k w_k = \sum_{i=0}^{k-1} \rho_{ii} - k \rho_{kk} + \sqrt{\frac{k(k+1)S_\xi}{2}}\, \xi_k ,$$

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where ξ_k are independent white noise terms.

The trajectory weights w_i depend only on the measured degrees of freedom, and not on the unobserved environmental degrees of freedom.

The condition for the evolution to be a pure Wiener process, and hence satisfy the Born rule, remains $gS_\xi = 1$.

Parametric Freedom

With the Born rule as a constant of evolution, the formal "measurement duration" can be made finite by making g time-dependent, and replacing gt by $\int_0^t g(t)dt$. (Detectors generically have nonlinear amplifiers.)

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The white noise distribution remains unspecified beyond the mean and the variance. Appropriate choice can be made.

Gaussian noise is generic as per the central limit theorem.

$$P(dW) = \frac{1}{\sqrt{2\pi\ dt}} \exp(-(dW)^2/(2\ dt)) \ .$$

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The nonlinear stochastic evolution, after averaging over noise, becomes a linear evolution described by a completely positive trace-preserving map. It can be written in a Kraus decomposed form in several ways.

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• Measurement outcomes are independent of $\rho_{i\neq j}$, and so are unaffected by decoherence. A different noise can be added to the phases of $\rho_{i\neq j}$ without spoiling the described evolution of ρ_{ii}. The Born rule imposes no constraint on that off-diagonal noise.

Quantum Jump: Single Qubit Measurement

We can construct a binary measurement scenario, where one eigenstate is reached by continuous geodesic evolution, while the other eigenstate is reached by sudden, infrequent but large jumps.

With trajectory weights $w_i = \delta_{i0}$, and shot noise $dN \in \{0, 1\}$, we have

$$d\rho = g[\rho P_0 + P_0\rho - 2\rho Tr(P_0\rho)]dt + (P_1 - \rho)dN \ .$$

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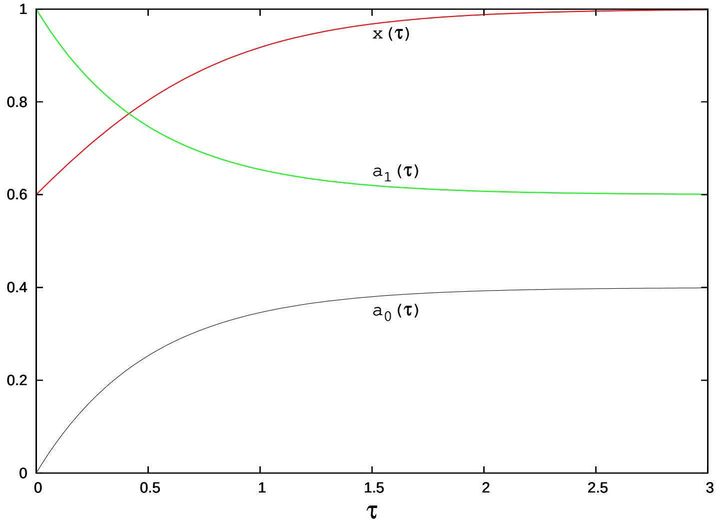
For initial $p(\rho_{00}, 0) = \delta(x)$, this biased random walk has the evolution:
$p(\rho_{00}, t) = (x + (1-x)e^{-2gt}) \ \delta(\frac{x}{x+(1-x)e^{-2gt}}) + (1-x)(1-e^{-2gt})\delta(0)$.
Both the components remain local. The one moving to $\rho_{00} = 1$ steadily reduces in magnitude, while the other fixed at $\rho_{00} = 0$ grows.



Properties of the quantum measurement trajectories for quantum jump evolution of a qubit state. The initial state is $\rho_{00}(\tau = 0) = 0.6$, and the evolution parameter is $\tau \equiv \int_0^t g(t)dt$. The initial distribution splits into a monotonically moving component $a_1(\tau)\delta(x(\tau))$ and a stationary component $a_0(\tau)\delta(0)$, which respectively travel to the measurement eigenstates $\rho_{00} = 1$ and $\rho_{00} = 0$ as $\tau \to \infty$.

Ensemble Evolution Dynamics

The distribution for the off-diagonal elements is also a similar sum of two local components:

$$p(\rho_{01},t) = (x + (1-x)e^{-2gt})\ \delta\left(\frac{\rho_{01}(0)}{xe^{gt}+(1-x)e^{-gt}}\right) + (1-x)(1-e^{-2gt})\delta(0).$$

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Upon taking the ensemble average over the noise,

$$\rho(0) = \begin{pmatrix} x & \rho_{01}(0) \\ \rho_{10}(0) & 1-x \end{pmatrix} \implies \langle \rho(t) \rangle = \begin{pmatrix} x & e^{-gt}\rho_{01}(0) \\ e^{-gt}\rho_{10}(0) & 1-x \end{pmatrix}.$$

This is again the solution of the Lindblad master equation for the same system, with the single decoherence operator $L_\mu = \sqrt{\gamma}(P_0 - P_1)$, $\gamma = g/2$.

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With $(dN)^2 = dN$, the size of the fluctuations is:

$$\langle(d\rho_{00})^2\rangle = \rho_{00}^2\langle dN\rangle\ .$$

The geodesic evolution term is:

$$(d\rho_{00})_{\rm geo} = 2g\rho_{00}\rho_{11}\ dt\ .$$

Dropping the subleading $o(dt)$ terms, $\langle dN\rangle = 2g\rho_{11}dt$ gives the relation:

$$\langle(d\rho_{00})^2\rangle = \rho_{00}(d\rho_{00})_{\rm geo}\ ,$$

which is again independent of $g\ dt$.

Origin of Noise

The quadratically nonlinear quantum measurement equation for state collapse supplements the Schrödinger evolution:

$$d\rho = \sum_i w_i \; g[\rho P_i + P_i \rho - 2\rho Tr(\rho P_i)] \; dt + noise \; .$$

The underlying dynamics is the system-apparatus measurement interaction, and the nature of the noise depends on it.

What mechanism can simultaneously produce attraction towards the measurement eigenstates (geodesic evolution) and irreducible noise (stochastic fluctuations), with precisely related magnitudes?

Such features appear in variational principles and the path integral framework.

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A model for the measurement apparatus is needed to understand where the noise comes from. The observed signal is amplified, usually nonlinearly, from the quantum to the classical regime.

Coherent states that continuously interpolate between quantum and classical regimes are a convenient choice for the apparatus pointer states.

$$|\alpha\rangle \equiv e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \; .$$

Work in Progress

Coherent states are the minimum uncertainty states in the Fock space.
The von Neumann interaction can amplify α and separate the pointer states. For measurement of a qubit using the electromagnetic field in a cavity, the von Neumann interaction gives:

$$H_{\rm int} = ig\ |1\rangle\langle 1| \otimes (a^\dagger - a)\ ,$$
$$|0\rangle_S|0\rangle_A \longrightarrow |0\rangle_S|0\rangle_A\ ,\quad |1\rangle_S|0\rangle_A \longrightarrow |1\rangle_S|\alpha = gt\rangle_A\ .$$

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Amplification produces quantum noise when the extracted information is not allowed to return (e.g. spontaneous vs. stimulated emission).

Does chaotic delocalisation of the entangled degree of freedom inside the apparatus give rise to irreversibility? (Generally apparatus $\gg$ system)

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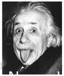
Einstein strikes back!



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