FORMAL METHODS FOR DIFFERENTIAL EQUATIONS

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Contents

1. Introduction

In this set of notes, I want to introduce a set of formal methods that are useful in analysing nonlinear partial differential equations. Typically, these methods are called "perturbation methods" or "the method of multiple scales" and are part of the broader subject of Asymptotic analysis. I prefer the term Formal Methods to emphasise that much of material presented here is of a formal nature, mathematically speaking. These notes do not involve proofs nor will we be concerned about convergence issues. That said, there is considerable literature attempting to make rigorous the results obtained via these formal methods. [1]

The main goal of these notes is to introduce the unified manner in which one can analyse weakly nonlinear differential equations. Such equations are typically of the

[1] It seems to me that mathematicians and non-mathematicians use the word formal in two completely opposite senses. To the working scientist, a formal argument is a mathematical argument whereas for a mathematician, a formal argument is precisely one that is not substantiated or is lacking a degree of rigour.

form

$$Lq + \epsilon N(q) = 0,$$

where L is some linear operator, ϵ is a small real parameter and N is a nonlinear function of q. We will see many examples of such equations that arise in a number of applications. In some applications, ϵ is a genuinely small physical parameter and in other cases it is a formal parameter (which may or may not be small).

In all cases, we wish to find 'approximate' solutions to these equations. The word approximate is placed in quotation marks since we will not define any norm nor quantify how accurate our approximations are. Furthermore, the emphasis is not on deriving the approximate solution, but equations satisfied by approximate solutions. As it turns out, often the approximate equations are of a universal character: the same equation appears in multiple contexts. Also, often times, the equation satisfied by the approximate solution is simpler either by involving fewer variables or having a very specific structure or just simpler to interpret the physics content.

2. Linear equations

The main mathematical result we require is the Fredholm alternative. Sticking to the formal nature of these notes, I won't give a precise statement. Rather I will give a more informal description. Firstly we state the problem. We wish to solve

$$Ax = b,$$

where $x, b \in \mathbb{R}^n$ are vectors and A is a (real-valued, say) matrix. If $\det A \neq 0$ (equivalently, A has no zero eigenvalue), then A is an invertible matrix. In such a case, we can solve the above equation for x to obtain

$$x = A^{-1}b$$

Moreover, we can find x no matter what b is. Said another way, when $\det A \neq 0$, for *every* vector $b \in \mathbb{R}^n$ there is a vector x such that $Ax = b$.

A more interesting scenario, and relevant for our purposes, is when $\det A = 0$. Clearly, now A is not invertible and hence we can't solve for x. The vanishing determinant is equivalent to imposing the linear dependence of the columns of A. In general, the range of A is given by the span of the columns of A.

Definition 2.1. *The span of a set of vectors $\{v_i\}_{i=1}^n$ is the set of all vectors of the form $\sum_i \alpha_i v_i$ where α_i are any real scalars. In other words, the span consists of all possible linear combinations of v_i.*

When the columns of A are all linearly independent, then the span of A is the whole space $\mathbb{R}^n$. This is precisely why we can solve $Ax = b$ for any b. Any vector b can be placed on the right-hand side of the equality. On the other hand, when the columns of A are linearly dependent, then the span of A is *not* the full space.

Naively this implies we cannot solve $Ax = b$ for any b but only some special b which lie in the range of A.

The above argument can be made a bit more precise. Consider the equation $Ax = b$ where $\det A = 0$ and take an inner product with a vector y. This leads to

$$y^T Ax = y^T b \Rightarrow (A^T y)x = y^T b. \tag{1}$$

If we chose y so that $A^T y = 0$ then clearly $y^T b = 0$ and hence b must be orthogonal to all such y. Since we assumed $\det A = 0$, there certainly exist such vectors y (both A and A^T have the same eigenvalues and thus A^T has a null-space).

To summarise, given a square matrix A

- Either $\det A \neq 0$ and then we can solve $Ax = b$ given any vector b,
- Or $\det A = 0$ and then we can solve $Ax = b$ only if $y^T b = 0$ for all y such that $A^T y = 0$. In such a case, clearly the solution is not unique (we can always add any vector v, such that $Av = 0$, to the solution).

The above is an informal statement of the Fredholm alternative. I will leave it as a simple exercise to see how things change if the vectors and matrices we considered were complex valued. Our main interest in discussing the Fredholm alternative is to establish the necessary condition for a solution, namely $y^T b = 0$ for all y such that $A^T y = 0$, when $\det A = 0$.

The Fredholm alternative is actually true in a much wider context and indeed we will frequently make use of it to find necessary conditions to solve linear systems of differential equations.

3. Some algebraic examples

Though most of the examples in these notes are related to differential equations (specifically partial differential equations), we start with a couple of examples related to algebraic equations. The hope is to investigate the role of the Fredholm alternative in solving nonlinear equations in a slightly more familiar context. Consider the following system of equations

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_2^2 + x_1 x_2 \\ x_1^2 + x_1 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

We wish to solve the above equations for the vector $x = [x_1 \ x_2]^T$. The above can easily be placed in the generic form $Lx + \epsilon N(x) = 0$ by simply rescaling the unknown $x \to \epsilon x$, for $\epsilon \in \mathbb{R}$. We then consider solutions x which are given by a power series in ϵ. Since N, the nonlinear operator, is also polynomial in x, we expect the entire left-hand side $Lx + \epsilon N(x)$ to be a power series in ϵ. Hence we assume

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2),$$

substitute the above into $Lx + \epsilon N(x) = 0$ and gather terms of like powers in ϵ. This leads to

$$\mathcal{O}(1): \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3}$$

since the matrix here is invertible. This then leads to the following equation for $[x_1^{(1)} \; x_2^{(2)}]^T$

$$\mathcal{O}(\epsilon): \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} (x_2^{(0)})^2 + x_1^{(0)} x_2^{(0)} \\ (x_1^{(0)})^2 + x_1^{(0)} x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4}$$

where we have used the now determined value of $x_1^{(0)}, x_2^{(0)}$. Evidently, we observe that even $x_1^{(1)}, x_2^{(1)}$ are identically zero. Repeating for further (higher order in ϵ) corrections for x_1, x_2 leads to the inevitable conclusion that $x_1 = x_2 = 0$ is the only solution for all values of ϵ in a neighborhood of the origin in the $x_1 - x_2$ plane. Of course to make the previous sentence a mathematical theorem one has to do a bit more work and invoke the implicit function theorem, but morally the above statement is true (and is consistent with our formal calculation).

Exercise 3.1. *Indeed the student is encouraged to verify that the second order correction $x_1^{(2)}, x_2^{(2)}$ also vanishes when we consider the equation (2) at order $\mathcal{O}(\epsilon^2)$. Remember to rescale $x \to \epsilon x$ before expanding in powers of ϵ!*

The above example lead to the trivial solution precisely because the linear problem for $x_1^{(0)}, x_2^{(2)}$ was associated with an invertible matrix. Let us change the problem slightly to violate this condition. Consider the following algebraic equation

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \epsilon \begin{pmatrix} x_2^2 + x_1 x_2 + 3x_1 \\ x_1^2 + x_1 x_2 + 6x_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

Once again, expanding the unknowns in a power series of ϵ

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2),$$

and collecting terms of the same power in ϵ we have

$$\mathcal{O}(1): \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \tag{6}$$

where α is any real number (at this stage undetermined). Notice the difference from the previous example: since the matrix is now non-invertible, with an eigenvector $[2, -1]^T$ corresponding to the zero eigenvalue, we have a non-trivial solution to the lowest order problem.

The equations at order ϵ are now

$$\mathcal{O}(\epsilon): \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} -\alpha^2 + 6\alpha \\ 2\alpha^2 - 6\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7}$$

Clearly, the solution at order ϵ depends on the solution at the previous order (this is the magic of power series). However, at this stage we recall that α is actually an unknown parameter. For the $\mathcal{O}(1)$ problem, it could be any real number. We now see that the Fredholm alternative imposes conditions on α.

Indeed, since the matrix for the order ϵ problem is non-invertible (notice it is the same matrix in the $\mathcal{O}(1)$ problem), the Fredholm alternative says the vector

$$\begin{pmatrix} -\alpha^2 + 6\alpha \\ 2\alpha^2 - 6\alpha \end{pmatrix}$$

must be orthogonal to all vectors in the null-space of

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix};$$

the transpose of the original matrix. The null-space of the above matrix is given by all scalar multiples of $[1 \;\; -1]^T$ and hence the condition to solve (7) is

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -\alpha^2 + 6\alpha \\ 2\alpha^2 - 6\alpha \end{pmatrix} = 0 \quad \Rightarrow -3\alpha^2 + 12\alpha = 0 \Rightarrow \alpha = 0, 4. \tag{8}$$

We see we obtain two possible values for the real number α as a consequence of the Fredholm alternative. We will disregard $\alpha = 0$ since this implies $x_1^{(0)} = x_2^{(0)} = 0$ and then $[x_1^{(1)} \; x_2^{(1)}]^T$ satisfies the original order 1 problem. Not much progress has been made if $\alpha = 0$!

On the other hand, $\alpha = 4$ is much more interesting. We obtain a non-trivial solution for $[x_1^{(0)} \; x_2^{(0)}]^T$ and the following equation at order ϵ

$$\mathcal{O}(\epsilon): \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = -\begin{pmatrix} 8 \\ 8 \end{pmatrix}. \tag{9}$$

Note we still have to invert an non-invertible matrix to solve for $[x_1^{(1)}\ x_2^{(1)}]^T$. The Fredholm alternative gave us only a necessary condition for solvability: it did not eliminate the non-invertibility of the original problem! The standard approach to solving equations such as those above, is to assume the unknown $[x_1^{(1)}\ x_2^{(1)}]^T$ lies in a space orthogonal to the null-space. In other words we assume

$$[2\ -1]\begin{pmatrix} x_1^{(1)} \\ x_2^{(2)} \end{pmatrix} = 0 \Rightarrow x_2^{(1)} = 2x_1^{(1)}$$

Combining this with (9) we have

$$\mathcal{O}(\epsilon): \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ 2x_1^{(1)} \end{pmatrix} = -\begin{pmatrix} 8 \\ 8 \end{pmatrix} \Rightarrow x_1^{(1)}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = -\begin{pmatrix} 8 \\ 8 \end{pmatrix} \tag{10}$$

and hence we have the following solution to (5)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4\begin{pmatrix} 2 \\ -1 \end{pmatrix} - \frac{8\epsilon}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{11}$$

And that's the basic idea.

> Whenever we have a non-invertible matrix, the Fredholm alternative enforces a solvability condition that fixes the unknown parameter of the lower order problem.

Let's summarise what we have done, while simultaneously generalising the procedure to see how higher order corrections (higher powers of ϵ) may be encorporated. We wish to solve the problem of type $Lx + \epsilon N(x) = 0$ given by

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \epsilon\begin{pmatrix} x_2^2 + x_1x_2 + 3x_1 \\ x_1^2 + x_1x_2 + 6x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12}$$

To do so, we employ a series expansion in ϵ of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha(\epsilon)\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2), \tag{13}$$

where

$$\alpha(\epsilon) = 1 + \epsilon\alpha_1 + \epsilon\alpha_2 + \mathcal{O}(\epsilon^3).$$

This leads to the following equations (upon collecting terms of different powers of ϵ)

$$\text{(14)} \qquad \mathcal{O}(1): \qquad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{(15)} \qquad \mathcal{O}(\epsilon): \qquad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} (x_2^{(0)})^2 + x_1^{(0)}x_2^{(0)} + 3x_1^{(0)} \\ (x_1^{(0)})^2 + x_1^{(0)}x_2^{(0)} + 6x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Note there ought to be a term

$$\alpha_1 \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix},$$

in the $\mathcal{O}(\epsilon)$ equation above. However, we have employed the $\mathcal{O}(1)$ equation to eliminate this term. A nontrivial solution to the $\mathcal{O}(1)$ equation leads to

$$\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \alpha_0 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

where α_0 is as yet undetermined. Substituting this into the $\mathcal{O}(\epsilon)$ equation we obtain

$$\text{(16)} \qquad \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} -\alpha_0^2 + 6\alpha_0 \\ 2\alpha_0^2 - 6\alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

has a transpose with

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as the eigenvector with zero eigenvalue. Hence the Fredholm alternative requires

$$(1 \ -1) \begin{pmatrix} -\alpha_0^2 + 6\alpha_0 \\ 2\alpha_0^2 - 6\alpha_0 \end{pmatrix} = 0 \Rightarrow \alpha_0 = 0, 4$$

We can then solve for $[x_1^{(1)} \ x_2^{(1)}]^T$ if we further demand that this vector be orthogonal to the null space of

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Exercise 3.2. *Having found $[x_1^{(1)} \ x_2^{(1)}]^T$, the student is encouraged to write out the $\mathcal{O}(\epsilon^2)$ equations and determine α_1 from the Fredholm alternative solvability condition. Continue to determine higher order corrections in α and x_1 x_2.*

Remark 1. *Note how you determine x_1, x_2 at one order and then determine α from the next higher order calculation. This is a generic feature and will remain even when we consider partial differential equations.*

Remark 2. *In general the calculations become quite tedious very quickly. For most problems we will only determine the equivalent of α_0 and not even bother computing $x_1^{(1)}, x_2^{(1)}$.*

4. A nonlinear oscillator

Consider the ordinary differential equation

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0. \tag{17}$$

Our goal is to find solutions to the above ordinary differential equation. In general one should prescribe some initial conditions, however we will not prescribe any such condition. Instead we demand the solution be periodic. We also will not prescribe the period. Indeed, the actual period however will be one of the unknowns and is obtained by the solution.

To begin with, we rewrite the equation in the form $Lx + \epsilon N(x) = 0$. To this end, let $[x_1\ x_2]^T$ be the vector $[y(t)\ y'(t)]^T$. Then we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ x_1^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{18}$$

In contrast to the previous algebraic problem, here the linear 'matrix' L is actually the operator

$$L = \frac{d}{dt} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Looking ahead we will be interested in the null space of L and it's 'transpose' L^T. Finding the null space of L is a routine calculation involving eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Don't worry, we will go through this calculation in just a while. Our first question is how to define the transpose of the operator L. The main device we need is the inner product for vector valued functions. Indeed let $f = [f_1(t)\ f_2(t)]^T$ and $g = [g_1(t)\ g_2(t)]^T$ be two real vector-valued functions of t such that all functions f_1, f_2, g_1, g_2 have the same period τ. We then define the inner product as

$$(f, g) = \int_0^\tau \big(f_1(t)g_1(t) + f_2(t)g_2(t)\big)dt$$

The transpose of an operator L is defined as the operator $L^\dagger$ such that

$$(f, Lg) = (L^\dagger f, g)$$

for all vector valued functions f, g. The usual way to move the operator from g to f is by integration by parts. Let's see how this works for our example.

$$\begin{aligned}
(19) \qquad (f, Lg) &= \int_0^\tau (f_1(t)\ f_2(t)) \left(\frac{d}{dt} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \right) dt \\
&= \int_0^\tau \left(-\frac{d}{dt} \begin{pmatrix} f_1(t) & f_2(t) \end{pmatrix} + \begin{pmatrix} f_1(t) & f_2(t) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} dt, \\
&= (L^\dagger f, g)
\end{aligned}$$

where we have neglected terms arising from the boundary due to the assumption of periodicity of all functions. Hence we have

$$L^\dagger = -\frac{d}{dt} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that $L^\dagger$ now acts on column vectors.

4.1. **Short detour on solving linear systems.** Let $u \in \mathbb{C}^n$ and $A \in \mathbb{C}^n \times \mathbb{C}^n$ be a matrix. We wish to solve equations of the form

$$\frac{d}{dt} u = Au, \ u(0) = u_0,$$

where u_0 is some given initial condition. Supposing we have solutions of the form $u = v \exp(\lambda t)$ for some time-independent vector v, we readily see that v is an eigenvector of A with eigenvalue λ. Indeed we have

$$\lambda v = Av$$

If A is diagonalisable, then there exist n linearly independent eigenvectors v_i that span the whole space $\mathbb{C}^n$. As a result, the initial condition u_0 can be expressed as a linear combination of these v_i: $u_0 = \sum_i \alpha_i v_i$ where α_i are the components of the vector u_0 in the direction v_i. Hence the solution is given by $u(t) = \sum_i \alpha_i v_i \exp(\lambda_i t)$, where λ_i is the eigenvalue associated with vector v_i. It is straightforward to verify the solution given actually satisfies the equation. That it is unique follows from the usual Gron onwall's argument or by noting that the difference between two soutions (for the same initial condition) satisfies the same equation but with zero initial conditions. Since v_i are linearly independent, $u_0 = 0$ implies $\alpha_i = 0$ for all i. Consequently, the solution is identically zero for all time.

Remark 3. *If we desired real-valued solutions, one can simply take the real or imaginary part of the solution obtained above. Since the equation is linear, $u(t) + c.c.$ is evidently also a solution. Likewise for the imaginary part.*

When the matrix A is not diagonalisable, namely when the algebraic multiplicity of at least one eigenvalue is larger than the associated geometric multiplicity, the set of eigenvectors v_i do not span the entire space $\mathbb{C}^n$. In such a case, one requires generalised eigenvectors in addition to the eigenvectors to obtain a basis of vectors for the entire space $\mathbb{C}^n$. The general solution also accordingly changes. In these notes, we will not encounter the non-diagonalisable case and hence we won't pursue the matter any further.

Coming back to equation (18), we first assume an expansion of the solution in powers of ϵ

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2).$$

Substituting into (18) and collecting terms of powers of ϵ we obtain

$$\mathcal{O}(1): \qquad \frac{d}{dt}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{20}$$

$$\mathcal{O}(\epsilon): \qquad \frac{d}{dt}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 \\ (x_1^{(0)})^3 \end{pmatrix}. \tag{21}$$

The solution to the $\mathcal{O}(1)$ equation is

$$\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + \text{c.c.}, \tag{22}$$

where c.c. denotes complex conjugate (so that we get real solutions). We will not impose any initial conditions since we're looking for periodic solutions and hence we simply note at this stage that the solution $x^{(0)}(t)$ is 2π periodic in t.

Exercise 4.1. *Show this!*

Note here α may be a complex constant in general. The $\mathcal{O}(\epsilon)$ equations are (in symbolic form)

$$Lx^{(1)} = -N(x^{(0)}),$$

where we now know the vector $x^{(0)}$ from the solution at $\mathcal{O}(1)$. From the Fredholm alternative, the $\mathcal{O}(\epsilon)$ equation can be solved for the vector $x^{(1)}$ only if the right-hand side is orthogonal to the null-space of $L^\dagger$, the adjoint of L. From our definition of $L^\dagger$ above, we find solutions to $L^\dagger y = 0$ to be given by linear combinations of

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}, \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}.$$

Exercise 4.2. *Show this too!*
Notice that here the null-space of the L and $L^\dagger$ happen to be spanned by the same set of vectors. This should not be too surprising once you realise that $L^\dagger = -L$. So putting things together, we require

$$N(x^{(0)}) = \begin{pmatrix} 0 \\ (x_1^{(0)})^3 \end{pmatrix} = \begin{pmatrix} 0 \\ (\alpha e^{it} + \text{c.c})^3 \end{pmatrix},$$

to be orthogonal to *both*

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}, \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}.$$

This amounts to requiring

$$\int_0^{2\pi} ie^{-it}(\alpha e^{it} + \text{c.c})^3 dt = 0, \text{ and } \int_0^{2\pi} ie^{it}(\alpha e^{it} + \text{c.c})^3 dt = 0.$$

The above integrals are straightforward to do and lead to demanding

$$|\alpha|^2\alpha = |\alpha|^2\bar{\alpha} = 0$$

which implies $\alpha = 0$. This is a *huge* problem since it implies the $\mathcal{O}(1)$ solution vanishes! As a result, the $\mathcal{O}(\epsilon)$ problem is now the same as the original $\mathcal{O}(1)$ problem. Just like our finite dimensional example, something wrong has happened. We have only found the trivial solution and we need a way out.

Let's take a step back and see if there is a way out of our problem. Although the original equation (18) is nonlinear, the $\mathcal{O}(\epsilon)$ equation is linear. Indeed it is an inhomogeneous problem since we know the term on the right-hand side; it is given in terms of the solution to the $\mathcal{O}(1)$ equation.

You may recall an old trick from your ordinary differential equation course by the name of **variation of paramters**. The basic idea was to replace the constants of a problem into functions of the independent variable. We will adopt the same principle here. Which constant you say? well α of course (just like you do for variation of parameters). Now suppose $\alpha \to \alpha(t)$. Clearly this can't be right. For one, our $\mathcal{O}(1)$ problem changes dramatically and we know solutions to that problem have exponential solutions; $\alpha(t)e^{it}$ type!

Upon some further reflection, you may note that α was set to zero by the Fredholm alternative requirement in the $\mathcal{O}(\epsilon)$ equation. Hence, to prevent this, we need the time derivative of α to appear in this equation and not the $\mathcal{O}(1)$ equation. How can this be assured? The most straightforward way is to assume that $\alpha = \alpha(\epsilon t)$. We denote by τ the new variable ϵt. Hence α is a function of new 'slow' time scale $\tau = \epsilon t$.

So, we assume the solution to the $\mathcal{O}(1)$ equation is

$$\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \alpha(\epsilon t)\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + \text{c.c.}, \tag{23}$$

Now the $\mathcal{O}(\epsilon)$ equations become

$$\frac{d}{dt}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 \\ (\alpha e^{it} + \text{c.c.})^3 \end{pmatrix} - \frac{d\alpha}{d\tau}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + \text{c.c.} \tag{24}$$

Note here c.c. only refers to the immediately preceeding term. The Fredholm alternative once again requires the right-hand side of the above equation be orthogonal to both

$$\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}, \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}.$$

This leads to

$$2\frac{d\alpha}{d\tau} = 3i|\alpha|^2\alpha, \tag{25}$$

and the equation obtained by taking the complex conjugate of this one. In obtaining this equation, we assumed on the time scale $[0, 2\pi]$, the function $\alpha(\epsilon t)$ is roughly a constant and hences comes out of the integral.

For the purposes of integration, we treat t and ϵt as though they were independent variables.

Multiplying equation (25) by $\bar{\alpha}$ and adding the complex conjugate of the resulting equation, we obtain

$$\frac{d|\alpha|^2}{d\tau} = 0 \Rightarrow |\alpha|^2 = \text{constant} = |\alpha_0|^2, \tag{26}$$

where α_0 is the initial value of α. Hence

$$2\frac{d\alpha}{d\tau} = 3i|\alpha_0|^2\alpha \Rightarrow \alpha = \alpha_0 \exp(3|\alpha_0|^2 i\tau/2). \tag{27}$$

Here α_0 is an arbitrary constant. Our $\mathcal{O}(1)$ solution now stands as

$$\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \end{pmatrix} = \alpha_0\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i(1+3\epsilon|\alpha_0|^2/2)t} + \text{c.c.} \tag{28}$$

Notice, how the period of the solution has changed by a small amount.

4.2. **Solving for $x^{(1)}$.** Now that we have $x^{(0)}$ as a function of t which satisfies the orthogonality condition, we can attempt to solve the linear equation for $x^{(1)}$

$$\frac{d}{dt}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 \\ (\alpha e^{it} + \text{c.c.})^3 \end{pmatrix} - \frac{d\alpha}{d\tau}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} + \text{c.c.} \tag{29}$$

However, the operator on the left-hand side is still non-invertible (remember, we found non-trivial solutions to the $\mathcal{O}(1)$ equation) and so in general the above equation does not have unique solutions; we can always add $x^{(0)}(t)$ to any given solution. To get around this point, we demand that solutions to the above equation for $x^{(1)}(t)$ themselves be orthogonal to $x^{(0)}(t)$. This is an important point: in addition to demanding the right-hand side be orthogonal to the null-space of the adjoint $L^\dagger$, we also demand $x^{(1)}(t)$ be orthogonal to the null-space of L. In practice, demanding solution be orthogonal to the null-space of L simply means neglecting the contribution of the homogeneous solution: one only requires the particular solution due to the right-hand side functions. Enforcing the relevant orthogonality condition on the right-hand side is equivalent to simply neglecting any term of the form

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} \text{ or } \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$$

Note any function of ϵt that multiplies these terms is considered a constant. Nonlinear terms such as

$$\begin{pmatrix} 0 \\ (\alpha e^{it} + \text{c.c.})^3 \end{pmatrix}$$

should be projected onto the orthogonal complement of the above vectors. For partial differential equations, we will not solve for the associated $x^{(1)}(t)$ term explicitly. As a consequence, we avoid writing out $x^{(1)}(t)$ here. Although, it is perhaps a worthwhile endeavour for you.

5. Nonlinear wave equation

Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + u + u^3 = 0 \tag{30}$$

Once again we will assume periodic boundary conditions in x. Let's also rescale $u \to \epsilon u$ and rewrite the equation as the system

$$\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 - \partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \epsilon^2\begin{pmatrix} 0 \\ q^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{31}$$

where q, p correspond to u, u_t of the original PDE. Note the equation is now of the generic form $Lz + \epsilon^2 N(z) = 0$ for the vector-valued function $z = [q(x,t) \;\; p(x,t)]^T$.

5.1. **The linear operator.** The associated linear operator for this problem is

$$L = \frac{\partial}{\partial t} + \begin{pmatrix} 0 & -1 \\ 1-\partial_x^2 & 0 \end{pmatrix}.$$

We look for solutions periodic in x with period 2π, say. The periodic boundary conditions in x suggests we employ a Fourier basis to represent the functions $q(x,t), p(x,t)$. Consider solutions given by the form

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{ikx-i\omega t},$$

then evidently we have

$$\begin{pmatrix} 0 & -1 \\ 1+k^2 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = i\omega \begin{pmatrix} A \\ B \end{pmatrix},$$

and thus $i\omega$ is an eigenvalue for the spatial part of the linear operator written in Fourier space. It is then not too hard to see that, *for fixed* k, the eigenvalues of the matrix is given by $\pm i\omega(k)$ where $\omega(k) = +\sqrt{1+k^2}$. The associated eigenvectors are $[1 \;\; \mp i\omega(k)]^T$. Thus the following 2π-periodic-in-x vector-valued function

$$\begin{pmatrix} q \\ p \end{pmatrix} = \alpha e^{ikx-i\omega(k)t} \begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + \text{c.c} + \beta e^{ikx+i\omega(k)t} \begin{pmatrix} 1 \\ i\omega(k) \end{pmatrix} + \text{c.c.} \tag{32}$$

where α, β are complex scalars, satisfies

$$Lz = \frac{\partial}{\partial t} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1-\partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

5.2. **The adjoint operator.** Once k is fixed, the quantity $2\pi/\omega(k)$ specifies the period of (32) in the $t-$variable. For any two vector-valued functions $\tilde{z} = [\tilde{q} \;\; \tilde{p}]^T$ and $z = [q \;\; p]^T$, we define the inner product as

$$(\tilde{z}, z) = \int_0^{2\pi/\omega(k)} \int_0^{2\pi} (\tilde{q}q + \tilde{p}p)\; dxdt. \tag{33}$$

This inner product allows us to define the adjoint operator $L^\dagger$ from the following relation

$$(\tilde{z}, Lz) = (L^\dagger \tilde{z}, z), \quad \text{for all } \tilde{z}, z. \tag{34}$$

Essentially, we perform integration by parts to deduce $L^\dagger$. The boundary terms provide no contribution since we assume all functions are periodic in x and t. A straightforward calculation shows

$$L^\dagger = -\frac{\partial}{\partial t} + \begin{pmatrix} 0 & 1-\partial_x^2 \\ -1 & 0 \end{pmatrix}$$

and then it follows that the functions

$$e^{ikx+i\omega(k)t}\begin{pmatrix} i\omega(k) \\ -1 \end{pmatrix}, \quad e^{ikx-i\omega(k)t}\begin{pmatrix} i\omega(k) \\ 1 \end{pmatrix}, \tag{35}$$

and their complex conjugates are in the null-space of $L^\dagger$.

5.3. **The perturbation series.** Substituting

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q^{(0)} \\ p^{(0)} \end{pmatrix} + \mathcal{O}(\epsilon)$$

into equation (31) we obtain at $\mathcal{O}(1)$ the following equation

$$\frac{\partial}{\partial t}\begin{pmatrix} q^{(0)} \\ p^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1-\partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} q^{(0)} \\ p^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{36}$$

We take

$$\begin{pmatrix} q^{(0)} \\ p^{(0)} \end{pmatrix} = \alpha e^{ikx-i\omega(k)t}\begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + \text{c.c.} \tag{37}$$

which we know to be one particular solution to the equation $Lz = 0$. We could have taken a more general solution however these simply lead to more tedious calculation without changing the essential character of the final result. At this stage, one may ask, what precisely is the goal of these calculations? Recall that in our previous example of the ordinary differential equation, the goal was to find periodic solutions. The calculation there showed one could start with a function which was 2π periodic in t and a solution to the linear equation (the $\mathcal{O}(1)$ problem). The impact of the nonlinear term was to introduce a correction to the frequency (and hence period): the additional $3|\alpha_0|^2/2$ factor in the exponential in (28). In quite the similar fashion, we intend to see how the nonlinear term of (31) introduces corrections to both the spatial and temporal periods of the base solution given in (37). As we will see, in the case of partial differential equations, the situation will be a bit more subtle.

As the reader may expect at this stage, we add a correction to our perturbation series for $[q \quad p]^T$ and allow the scalar constant α to depend "slowly" on x and t: $\alpha = \alpha(X,T)$ where $X = \epsilon x$ and $T = \epsilon t$. This slow dependence will be the cause of the modulation of the base solution (37). Hence substituting

$$\begin{pmatrix} q \\ p \end{pmatrix} = \alpha(X,T)e^{ikx-i\omega(k)t}\begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + \text{c.c.} + \epsilon\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2)$$

into equation (31) we obtain at $\mathcal{O}(\epsilon)$ the following equation

$$\frac{\partial}{\partial t}\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1-\partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} = -\alpha_T\, e^{ikx-i\omega(k)t}\begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + \text{c.c.} + 2ik\alpha_X\, e^{ikx-i\omega(k)t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \text{c.c.} \tag{38}$$

Once again, from the Fredholm alternative we note that the right-hand side must be orthgonal to elements in the null-space of $L^\dagger$. Here orthogonal means the inner product of the right-hand side with all elements of the null-space of $L^\dagger$ must vanish identically. We have already determined the functions that lie in the null-space of $L^\dagger$ and hence from the orthogonality condition we have

$$\omega(k)\alpha_T + k\alpha_X = 0 \Rightarrow \alpha(X,T) = \psi(X-\omega'(k)T), \tag{39}$$

for any function $\psi(\xi)$. Note $\omega'(k) = k/\omega(k)$. In deducing the above condition from the orthogonality condition remember to consider α (and it's derivatives) as constants for the purposes of integration over the interval $[0,2\pi]\times[0,2\pi/\omega(k)]$. In other words, we treat X and T as variables independent of x and t. The $\mathcal{O}(\epsilon)$ equations now become

$$\frac{\partial}{\partial t}\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1-\partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} = \omega'(k)\psi_\xi\, e^{ikx-i\omega(k)t}\begin{pmatrix} 1 \\ i\omega(k) \end{pmatrix} + \text{c.c.}$$

Since an exponential appears on the right-hand side, we look for solutions of the form

$$\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} = e^{ikx-i\omega(k)t}\begin{pmatrix} A \\ B \end{pmatrix} + \text{c.c.}$$

where A, B are possibly functions of X, T, *i.e.* they have slow dependence on x, t. Thus at $\mathcal{O}(\epsilon)$, A, B are constants and the derivatives in space and time only affect the exponential. We now have

$$\begin{pmatrix} -i\omega(k) & -1 \\ 1+k^2 & -i\omega(k) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + \text{c.c.} = \omega'(k)\psi_\xi\begin{pmatrix} 1 \\ i\omega(k) \end{pmatrix} + \text{c.c.}$$

Using the fact $\omega(k)^2 = 1+k^2$ we obtain

$$Ai\omega(k) + B = -\omega'(k)\psi_\xi \Rightarrow \begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} = e^{ikx-i\omega(k)t}\begin{pmatrix} A \\ -i\omega(k)A - \omega'(k)\psi_\xi \end{pmatrix} + \text{c.c.},$$

or in other words

$$\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} = e^{ikx-i\omega(k)t}A\begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + \text{c.c.} - \omega'(k)\psi_\xi\, e^{ikx-i\omega(k)t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \text{c.c.},$$

where A is as yet arbitrary. Note however that the first term on the right-hand side is indeed a solution to the $\mathcal{O}(1)$ equation and hence is in the null-space of L. To define the $\mathcal{O}(\epsilon)$ correction uniquely we must impose an additional condition. Typically one would require that the solution be orthogonal to the null-space of L. To this end, we set $A = 0$ and then arrive at

$$\begin{pmatrix} q^{(1)} \\ p^{(1)} \end{pmatrix} = -\omega'(k)\psi_\xi \; e^{ikx-i\omega(k)t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \text{c.c.}.$$

The careful reader will note that setting $A = 0$ is *not* equivalent to requiring $[q^{(1)} \;\; p^{(1)}]^T$ be orthogonal to the null-space of L since the vector $[0 \;\; 1]^T$ has a component along $[1 \quad -i\omega]^T$. However, looking ahead, this particular choice is more convenient for calculations at the next order. In any case, one can employ any condition that uniquely specifies a solution to the equation.

We are not finished with our perturbation calculations and hence we march on to the next order. So combining all we know till now we assume

$$\begin{aligned} \begin{pmatrix} q \\ p \end{pmatrix} &= \psi(\xi,\tau)e^{ikx-i\omega(k)t} \begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + \text{c.c.} - \epsilon\, \omega'(k)\psi_\xi(\xi,\tau)\; e^{ikx-i\omega(k)t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \text{c.c.} \\ &\quad + \epsilon^2 \begin{pmatrix} q^{(2)} \\ p^{(2)} \end{pmatrix} + \mathcal{O}(\epsilon^3), \end{aligned} \tag{40}$$

where $\xi = \epsilon x - \epsilon\omega'(k)t$ and $\tau = \epsilon^2 t$. Substituting this expression into (31) we have at $\mathcal{O}(\epsilon^2)$

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} q^{(2)} \\ p^{(2)} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1-\partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} q^{(2)} \\ p^{(2)} \end{pmatrix} &= -\, e^{ikx-i\omega(k)t}\begin{pmatrix} 0 \\ 3|\psi|^2\psi \end{pmatrix} + \text{c.c} - e^{3ikx-3i\omega(k)t}\begin{pmatrix} 0 \\ \psi^3 \end{pmatrix} + \text{c.c} \\ &\quad -\, e^{ikx-i\omega(k)t}\left[\psi_\tau \begin{pmatrix} 1 \\ -i\omega(k) \end{pmatrix} + (1-\omega'(k)^2)\psi_{\xi\xi}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] + \text{c.c.} \end{aligned} \tag{41}$$

We observe the right-hand side contains exponential functions associated with solutions of the linear equation, namely $\exp ikx - i\omega(k)t$. Invoking one final time the Fredholm alternative, imposing the right-hand side be orthogonal to all elements of the null-space of the adjoint, we obtain the following condition

$$2i\omega(k)\psi_\tau + (1-\omega'(k)^2)\psi_{\xi\xi} + 3|\psi|^2\psi = 0, \tag{42}$$

and the equation obtained by taking the complex conjugate. Thus the amplitude of the solution to the $\mathcal{O}(1)$ equation, namely the solution to the linear problem, satisfies the above partial differential equation. Often times, the above equation is called the modulation equation, since it represents the slow modulation of the periodic wave $\exp ikx - i\omega(k)t$. Noting that $\omega''(k)\omega(k) = 1 - \omega'(k)^2$ we may rewrite the equation in a more standard form

$$i\psi_\tau + \frac{\omega''(k)}{2}\psi_{\xi\xi} + \frac{3}{2\omega(k)}|\psi|^2\psi = 0,$$

where $\xi = \epsilon x - \epsilon\omega'(k)t$ and $\tau = \epsilon^2 t$. The above equation is known as Nonlinear Schrödinger equation. The physical meaning is that the amplitude of a periodic plane wave $\exp ikx - i\omega(k)t$, when viewed in a frame of reference traveling at the group speed ($\xi = \epsilon(x-\omega'(k)t)$) varies slowly in time ($\tau = \epsilon^2 t$) according to the above equation. Notice the similarity between this equation and the one obtained for the ordinary differential equation. There we were able to directly solve the equation in terms of an initial condition. Here the situation is a bit more complicated due to spatial variation of the amplitude, as well as temporal variation. As it happens, the above equation is well studied and we know a great many details on solutions and their behaviour. However, we will have to leave those discussions for another time.

6. Korteweg-de Vries⟶Nonlinear Schrödinger

The Nonlinear Schrödinger equation arises as the weakly nonlinear limit of a number of other equations. Here we will start with an equally well known equation,

$$u_t + u_{xxx} + uu_x = 0, \tag{43}$$

known as the Korteweg-de Vries equation. In subsequent sections we show how this equation itself arises as a weakly nonlinear model for other equations. However, for now we just assume Korteweg-de Vries (or KdV) is given to us. As always we will rescale the unknown function $u \to \epsilon u$ to obtain

$$u_t + u_{xxx} + \epsilon uu_x = 0. \tag{44}$$

6.1. **The linear operator.** We begin by neglecting the nonlinear term and focussing solely on the linear differential equation

$$u_t + u_{xxx} = 0. \tag{45}$$

This equation is readily solved using Fourier series for 2π periodic functions in x. Once again, we fix k and let

$$u = Ae^{ikx - i\omega(k)t} + \text{c.c.},$$

from which we readily see $\omega(k) = -k^3$. Equivalently, functions proportional to $\exp(ikx - i\omega(k)t)$ for $\omega(k) = -k^3$ (as well as their complex conjugates and all linear combinations thereof) are in the null-space of the operator $L = \partial_t + \partial_x^3$.

6.2. **The adjoint operator.** Once k is fixed, then the period in t is uniquely fixed. Indeed it is $2\pi/\omega(k)$. For functions $u, \tilde{u}$ periodic in $[0, 2\pi] \times [0, 2\pi/\omega(k)]$ we define the inner product as

$$(\tilde{u}, u) = \int_0^{2\pi/\omega(k)} \int_0^{2\pi} \tilde{u}\; u\, dxdt$$

and then the adjoint $L^\dagger$ is given by

$$(\tilde{u}, Lu) = (L^\dagger \tilde{u}, u), \quad \text{for all } \tilde{u}, u.$$

This then leads to

$$L^\dagger = -\partial_t - \partial_x^3 = -L,$$

and so L and $L^\dagger$ have the same null-space.

6.3. **The catch.** The Korteweg-de Vries equation distinguishes itself from the other equations considered so far. Specifically, it is unlike the nonlinear wave equation considered in the previous section. For fixed spatial and temporal periods, there is a unique wavenumber k in the case of the nonlinear wave equation (check this!) whereas for KdV, there are two wavenumbers. Indeed, $\exp(ikx - i\omega(k)t)$ for both $k = k_0$ and $k = 0$ correspond to solutions to the linear equation (45) and have the share a common period in both x and t: 2π and $2\pi/k_0^3$. Note, constants are periodic with every period.

Spatial and temporal periods define the inner product. Moreover, the inner product was our main tool to distinguish the element of the null-space of L and $L^\dagger$ in all previous exampls. As a result, we conclude that the inner product in this case, distinguishes only the whole null-space and not individual elements. We will see the consequences of this when we proceed with the perturbative expansion.

So what makes KdV so different and could we have foreseen this? The answer to both questions lies in the dispersion relation $\omega(k) = -k^3$ which vanishes at $k = 0$. Thus the zero-mode, *i.e.* any non-zero constant, is a legitimate solution to the linear problem whereas it is not a solution to the nonlinear wave equation (31).

A caveat to the catch: KdV has a quadratic nonlinearity unlike the cubic nonlinearity in (31). This means the $\mathcal{O}(\epsilon)$ equations will give rise to a contribution from the nonlinear term, unlike the situation for (31). Summary, the existence of a mean-mode depends on whether $\omega(0) = 0$. The precise role of the mean-mode in the analysis depends on the nonlinear term.

6.4. **The perturbation series.** Assume

$$u = u^{(0)} + \epsilon u^{(1)} + \epsilon^2 u^{(2)}$$

and substitute into (44). We obtain the equations

$$\mathcal{O}(1): \quad u_t^{(0)} + u_{xxx}^{(0)} = 0. \tag{46}$$

Note, for fixed spatial and temporal periods, we noted above that there were two solutions to the above equation. We will, as always, begin with $u^{(0)}$ set to only one of those solutions, specifically $u^{(0)} = A\exp(ikx - i\omega(k)t)$ where k is fixed and non-zero.

Once again, to proceed to the next order, we let $A = A(X,T)$ where $X = \epsilon x$ and $T = \epsilon t$ introducing a slow variation of the constant in x, t. This leads to

$$\begin{aligned}\mathcal{O}(\epsilon): \quad u_t^{(1)} + u_{xxx}^{(1)} &= -e^{ikx-i\omega(k)t}(A_T - 3k^2 A_X) + \text{c.c.}\\ &\quad - (Ae^{ikx-i\omega(k)t} + \text{c.c.})(ikAe^{ikx-i\omega(k)t} + \text{c.c.})\\ &= -e^{ikx-i\omega(k)t}(A_T - 3k^2 A_X) - \text{c.c.} + A^2 e^{2ik-2i\omega(k)t} + \text{c.c.} \qquad (47)\end{aligned}$$

Observe that we have a term proportional to $e^{ikx-i\omega(k)t}$ on the right-hand side. This exponential lies in the null-space of $L^\dagger$ and thus the Fredholm alternative requires the coefficient $A_T - 3k^2 A_X = 0$.

Remark 4. *In the above, I simply recognised a solution to the problem $L^\dagger z = 0$ present in the right-hand side and set the coefficient to zero to enforce the orthogonality condition. Alternatively, I could have computed the inner product of the right-hand side with respect to each linearly independent function in the null-space of $L^\dagger$ to arrive at the same conclusion. The careful reader should convince themselves that the two procedures are indeed equivalent and both lead to the same conclusion for $A(X,T)$.*

Notice in obtaining the second equality of (47, the term proportional to $ik|A|^2$ canceled with its complex conjugate. Had it not, a term proportional to e^0, namely the mean-mode of the $\mathcal{O}(1)$ equations, would have been present on the right-hand side. In such a case, we would have had to update our original $\mathcal{O}(1)$ solution to be $u^{(0)} = A(X,T)e^{ikx-i\omega(k)t} + \text{c.c.} + m(X,T)$ where $m(X,T)$ is a real valued function of the slow variables. However, we see there is no mean-mode due to the nonlinearity at $\mathcal{O}(\epsilon)$ and hence our assumption of a single frequency solution for $u^{(0)}$ is a consistent choice.

The $\mathcal{O}(\epsilon)$ orthogonality condition

$$A_T - 3k^2 A_X = 0,$$

is readily satisfied if we take $A(X,T) = \psi(X + 3k^2 T)$ for some function $\psi(\xi)$. Consequently, the equations satisfied by $u^{(1)}$ are

$$u_t^{(1)} + u_{xxx}^{(1)} = -\psi^2 e^{2ik-2i\omega(k)t} + \text{c.c.} \qquad (48)$$

A particular solution to this equation is of the form $Be^{2ikx-2i\omega(k)t}$ from which we conclude $B = \psi^2/(6k^2)$. To this particular solution we add a solution to the homogeneous problem

$$u_t^{(1)} + u_{xxx}^{(1)} = 0,$$

periodic on $[0, 2\pi] \times [0, 2\pi/\omega(k)]$ and which is also orthogonal to $e^{ikx-i\omega(k)t}$ (*i.e.* $u^{(0)}$). In this case, that function is precisely the mean-mode $m(\xi)$, a real-valued function that depends only on the slow variables. Hence

$$u^{(1)} = \psi^2/(6k^2)e^{2ikx-2i\omega(k)t} + \text{c.c.} + m(\xi).$$

Let us know push further to the next order in our perturbation series. Again, remember to let all previous 'constants' depend on an even slower time-scale, $\tau = \epsilon^2 t$. Thus we take

$$u(x,t) = \psi(\xi,\tau)e^{ikx-i\omega(k)t} + \text{c.c.} + \epsilon\psi^2/(6k^2)e^{2ikx-2i\omega(k)t} + \text{c.c.} + \epsilon m(\xi,\tau) + \epsilon^2 u^{(2)}.$$

Substitution into (44) leads to

$$\begin{aligned}\mathcal{O}(\epsilon^2): \quad u_t^{(2)} + u_{xxx}^{(2)} = \text{Something } e^{2ikx-2i\omega(k)t} + \text{c.c.} + \text{Stuff } e^{ikx-i\omega(k)t} + \text{c.c.}\\ + \text{Other stuff } e^0\end{aligned} \tag{49}$$

Notice that functions in the null-space of $L^\dagger$ appear in the right-hand side of the above equation. In particular, *both* linearly independent functions, $e^{ikx-i\omega(k)t}$ and e^0, are present. From the Fredholm alternative, the coefficient multiplying those terms must vanish (that's what being orthogonal means!). Recall, for the purposes of integration the coefficients mulitplying the exponential are treated as constants. Thus we must impose

$$\text{Stuff } = 0, \quad \text{Other stuff } = 0.$$

Working out what the above means, leads to

$$m_T + \psi\bar{\psi}_\xi + \bar{\psi}\psi_\xi = 0, \tag{50}$$

$$\psi_\tau + 3ik\psi_{\xi\xi} + \frac{i}{3k}|\psi|^2\psi - \frac{i}{6k}|\psi|^2\psi + mik\psi = 0. \tag{51}$$

Note $m_T = -\omega'(k)m_\xi$ since $\xi = X + 3k^2T$ and $\omega = -k^3$. The first equation can then be integrated to get $m = |\psi|^2/\omega'(k)$. Substituting this into the second equation above leads to

$$i\psi_\tau + \frac{\omega''(k)}{2}\psi_{\xi\xi} + \frac{1}{6k}|\psi|^2\psi = 0, \tag{52}$$

after a bit of manipulation.

Thus our approximate solution to (44) is characterised as

$$u = \psi e^{ikx-i\omega(k)t} + \text{c.c.} + \epsilon\left(\frac{\psi^2}{6k^2}e^{2ikx-2i\omega(k)t} + \text{c.c.} + \frac{|\psi|^2}{\omega'(k)}\right) + \mathcal{O}(\epsilon^2)$$

where $\omega = -k^3$ and ψ satisfies (52) with $\xi = \epsilon(x - \omega'(k)t)$ and $\tau = \epsilon^2 t$.

7. Weakly-nonlinear shallow water-waves

A model for weakly-nonlinear water waves in the shallow regime is given by the following PDE.

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ u\end{pmatrix}+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ u\end{pmatrix}+\epsilon\begin{pmatrix}u_{xxx}/3+\partial_x(u\eta)\\ uu_x\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{53}$$

where u, η represent the depth-averaged horizontal velocity of the water wave and η the deviation of the fluid surface from a constant level. The non-dimensional parameter represents an aspect ratio: $\epsilon = (h/L)^2$ where h is the depth of the fluid when there is no disturbance and L is a typical length scale. ϵ small represents then shallow water. Note unlike other models we have considered so far, here we have a linear term multiplying ϵ. Physically this refers to considering solutions which are smooth since the high derivative is multiplied by a small parameter.

We will not present details regarding how one obtains the above equation from the full set of equations governing fluid motion. Though straightforward, it would take us considerably further away from the main goal of these notes.

7.1. The linear operator.

Setting $\epsilon = 0$ we obtain the linear operator

$$Lz=\frac{\partial}{\partial t}\begin{pmatrix}\eta\\ u\end{pmatrix}+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ u\end{pmatrix}, \quad \text{for } z=\begin{pmatrix}\eta\\ u\end{pmatrix}.$$

The 2×2 matrix in the definition of L has eigenvalues ± 1 and the eigenvalue decomposition

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ \pm 1\end{pmatrix}=\pm 1\begin{pmatrix}1\\ \pm 1\end{pmatrix}$$

and hence we can write

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{or } AV=V\Lambda.$$

Then $Lz=0$ is equivalent to

$$\left(\frac{\partial}{\partial t}+\Lambda\frac{\partial}{\partial x}\right)V^{-1}\begin{pmatrix}\eta\\ u\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Notice these equations are now decoupled and readily solved to obtain

$$V^{-1}\begin{pmatrix}\eta\\ u\end{pmatrix}=f(x-t)\begin{pmatrix}1\\ 0\end{pmatrix}+g(x+t)\begin{pmatrix}0\\ 1\end{pmatrix}$$

where f, g are any functions of a single variable. We denote $\xi = x - t$ and $\zeta = x + t$. Hence f, g represent rightward and leftward traveling waves of a given profile. The solution to the linear equation $Lz = 0$ is then

$$\begin{pmatrix} \eta \\ u \end{pmatrix} = f(x-t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + g(x+t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{54}$$

7.2. **The adjoint operator.** We first define an inner product. Let $z = [\eta \quad u]^T$ and $\tilde{z} = [\tilde{\eta} \quad \tilde{u}]^T$ be two vector-valued functions of x and t. The model equations are derived from the full set of equations for fluid motion under the assumption of periodicity in the x variable. Hence we assume all functions are periodic with period 2π. Notice, from our analysis of the linear equation above, solutions are given in terms of non-dispersive waves traveling at unit speed in positive and negative x directions. Thus the solution will be time periodic with period 2π also. Hence we consider functions which are 2π periodic in space and time t. This motivates the inner product of $z, \tilde{z}$ as

$$(z, \tilde{z}) = \int_0^{2\pi} \int_0^{2\pi} (\tilde{\eta}\eta + \tilde{u}u)\, dxdt.$$

The adjoint $L^\dagger$ of L is then given by $(\tilde{z}, Lz) = (L^\dagger \tilde{z}, z)$ for all $z, \tilde{z}$. We then have

$$L^\dagger = -\frac{\partial}{\partial t} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} = -L$$

since the terms arising from the boundary (upon integrating by parts) vanish. We also conclude that the null-space of $L^\dagger$ is the same as that of L.

7.3. **Perturbation series.** As always, assume

$$\begin{pmatrix} \eta \\ u \end{pmatrix} = \begin{pmatrix} \eta^{(0)} \\ u^{(0)} \end{pmatrix} + \epsilon \begin{pmatrix} \eta^{(1)} \\ u^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2)$$

and substitute into (53). The equations at $\mathcal{O}(1)$ are

$$\frac{\partial}{\partial t}\begin{pmatrix} \eta^{(0)} \\ u^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \eta^{(0)} \\ u^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have already determined the general solution to these equations above. Let us assume waves only travel to the right. The entire analysis can be repeated for waves traveling to the left. Hence we take

$$\begin{pmatrix} \eta^{(0)} \\ u^{(0)} \end{pmatrix} = f(\xi)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi = x - t.$$

To proceed to the next order, we assume the scalar $f(\xi)$ depends slowly on t. Hence we take $f = f(\xi, \tau)$ where $\tau = \epsilon t$. Hence we have

$$\begin{pmatrix} \eta \\ u \end{pmatrix} = f(\xi, \tau)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \epsilon \begin{pmatrix} \eta^{(1)} \\ u^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2).$$

Substitution of the above into (53) leads to

$$\frac{\partial}{\partial t}\begin{pmatrix}\eta^{(1)}\\ u^{(1)}\end{pmatrix}+\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\eta^{(1)}\\ u^{(1)}\end{pmatrix}+\begin{pmatrix}f_\tau\\ f_\tau\end{pmatrix}+\begin{pmatrix}f_{\xi\xi\xi}/3+\partial_\xi(f^2)\\ ff_\xi\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{55}$$

at $\mathcal{O}(\epsilon)$.

Once again, the Fredholm alternative requires that sum of all the terms arising from $\eta^{(0)}, u^{(0)}$ should be orthogonal to the null-space of $L^\dagger$. Taking the inner product with respect to $\psi(x-t)[1 \;\; 1]^T$ for an arbitrary function $\psi(\xi)$ and setting it to zero, we have

$$\int_0^{2\pi}\int_0^{2\pi}\psi(x-t)\left(2f_\tau+\frac{1}{3}f_{\xi\xi\xi}+3ff_\xi\right)\,dxdt=0. \tag{56}$$

Since ψ is arbitrary, the only possible way the integrand sums to zero is if

$$2f_\tau+\frac{1}{3}f_{\xi\xi\xi}+3ff_\xi=0, \tag{57}$$

which is an equation for the slow modulation of the right-ward traveling wave.

It's worth commenting that just as the modulation equation for the nonlinear oscillator or nonlinear wave, lead to a modulation of a periodic signal into a signal with a slightly different period, here too a similar effect may be observed. Our initial assumption was the profile f was 2π periodic in space. Our reduced equation above says the spatial period need not change, however, the temporal period is no longer 2π but is modulated according to the dynamics of KdV.

Exercise 7.1. *Repeat the analysis for leftward traveling waves. What if we take*

$$\begin{pmatrix}\eta\\ u\end{pmatrix}=f(x-t)\begin{pmatrix}1\\ 1\end{pmatrix}+g(x+t)\begin{pmatrix}1\\ -1\end{pmatrix}+\epsilon\begin{pmatrix}\eta^{(1)}\\ u^{(1)}\end{pmatrix}+\mathcal{O}(\epsilon^2),$$

instead?

8. Quasi-geostrophic theory

The equations for a fluid on a rotating sphere, under the shallow water approximation

9. A coupled system of Bose-Einstein condensates

We wish to analyse the long-time small-amplitde dynamics of the following equation

$$i\hbar\frac{\partial\psi_j}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi_j}{\partial x^2}+\sum_{k=1}^{N}\alpha_{jk}|\psi_k|^2\psi_j, \tag{58}$$

that models the behaviour of a coupled system of Bose-Einstein condensates. Here ψ_j is the field associated with the $j-$th species, α is the coupling matrix and $\hbar, m$ are some physical constants (taken to be positive real numbers).

In order to derive the associated KdV model we first perform the usual Madelung transform [?] to obtain a set of hydrodynamic equations for the density and velocity,

$$\psi_j(x,t) = \sqrt{\rho_j(x,t)}e^{i(m/\hbar)\int_0^x v_j(x',t)dx'} \tag{59}$$

where ψ_j is the macroscopic wavefunction of the j^{th} condensate, $\rho_j(x,t)$ is the corresponding density field and $v_j(x,t)$, the velocity field where $j = 1, N$. The resultant equations of motion are an equation of continuity (for the density)

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial}{\partial x}(\rho_j v_j) = 0, \tag{60}$$

and the Euler equation (for the velocity)

$$\frac{\partial v_j}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{v_j^2}{2} + \frac{1}{m}\sum_{k=1}^{N}\alpha_{jk}\rho_k - \left(\frac{\hbar^2}{2m}\right)\frac{\partial_x^2\sqrt{\rho_j}}{\sqrt{\rho_j}}\right]. \tag{61}$$

Note that the density of $j-$th component is uncoupled; the coupling matrix only appears in the momentum equations. It is worth emphasizing that the there is no approximation in obtaining the above equations. They are fully equivalent to the original multicomponent system.

A trivial solution to the hydrodynamic equations is given by setting the densities to non-negative constants ρ_{0j} and the velocities to zero. A natural question is how small perturbations to this background state evolve. The standard approach to this question is given by linearising the equations about the background state, namely setting $\rho_j = \rho_{0j} + \delta\rho_j$ and $v_j = \delta v_j$, to obtain the following linear evolution equations for the perturbations $\delta\rho_j, \delta v_j$

$$\frac{\partial}{\partial t}\delta\rho_j = -\rho_{0j}\frac{\partial}{\partial x}\delta v_j, \tag{62a}$$

$$\frac{\partial}{\partial t}\delta v_j = -\frac{1}{m}\frac{\partial}{\partial x}\sum_{k=1}^{n}\alpha_{jk}\delta\rho_k + \frac{\hbar^2}{4m\rho_{0j}}\frac{\partial^3}{\partial x^3}\delta\rho_j. \tag{62b}$$

These constant coefficient equations are readily solved using Fourier methods. Furthermore, for the simpler case of α equal to a diagonal matrix (the uncoupled case), the above equations have solutions of the form $\exp(ikx + i\omega t)$ with $\omega \approx c_1 k + c_2 k^3$ for some real constants c_1, c_2. This indicates waves travel at speed c_1 but also disperse due to the presence of the cubic term. This suggests that if we consider long-wavelength perturbations such that the dispersive term balances with the nonlinear corrections, we may arrive at a KdV-like model. This same argument applies also to the coupled case, *i.e.* for a generic matrix α.

Let's now proceed to the nonlinear equations for the perturbations

$$(\partial_t + \mathcal{A}\partial_x)\begin{pmatrix}\delta\rho\\ \delta v\end{pmatrix} = -\partial_x\begin{pmatrix}\mathcal{N}_1(\delta\rho,\delta v)\\ \mathcal{N}_2(\delta\rho,\delta v)\end{pmatrix}, \tag{63}$$

where

$$(\mathcal{N}_1)_j = \delta\rho_j\,\delta v_j,\ (\mathcal{N}_2)_j = \frac{\delta v_j^2}{2} - \left(\frac{\hbar^2}{2m}\right)\frac{2(\rho_{0j}+\delta\rho_j)\delta\rho_j'' - \delta\rho_j^2}{4(\rho_{0j}+\delta\rho_j)^2},$$

and

$$\mathcal{A} = \begin{pmatrix}\mathbf{0}_{n\times n} & \rho\\ \alpha/m & \mathbf{0}_{n\times n}\end{pmatrix},$$

with ρ an $n\times n$ diagonal matrix with elements $\rho_{0j} > 0$ and $\delta\rho, \delta v$ are the $n\times 1$ vectors for the perturbations in density and velocity. For simplicity, we will assume $\mathcal{A}$ is diagonalisable and has only real eigenvalues. Necessary and sufficient conditions are given in the paper [].

Let us rescale the variables so that $\partial_t \to \epsilon\partial_t$, $\partial_x \to \epsilon\partial_x$ and $\delta\rho,\delta v \to \epsilon^2\delta\rho, \epsilon^2\delta v$. This leads to

$$(\partial_t + \mathcal{A}\partial_x)\begin{pmatrix}\delta\rho\\ \delta v\end{pmatrix} = -\epsilon^2\partial_x\begin{pmatrix}\mathcal{N}_1(\delta\rho,\delta v)\\ \mathcal{N}_2(\delta\rho,\delta v,\epsilon)\end{pmatrix}, \tag{64}$$

where

$$\mathcal{N}_2(\delta\rho,\delta v,\epsilon^2) = \frac{\delta v_j^2}{2} - \left(\frac{\hbar^2}{2m}\right)\frac{2(\rho_{0j}+\epsilon^2\delta\rho_j)\delta\rho_j'' - \epsilon^4\delta\rho_j^2}{4(\rho_{0j}+\epsilon^2\delta\rho_j)^2}.$$

We now solve the above equation perturbatively. Assuming an expansion in ϵ^2 for the unknowns

$$\begin{pmatrix}\delta\rho\\ \delta v\end{pmatrix} = \begin{pmatrix}\delta\rho^{(0)}\\ \delta v^{(0)}\end{pmatrix} + \epsilon^2\begin{pmatrix}\delta\rho^{(1)}\\ \delta v^{(1)}\end{pmatrix},$$

and substituting into the above equation we obtain to lowest order

$$(\partial_t + \mathcal{A}\partial_x)\begin{pmatrix}\delta\rho^{(0)}\\ \delta v^{(0)}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{65}$$

Since $\mathcal{A} = V\tilde{\Lambda}V^{-1}$ is diagonalisable, this equation is equivalent to

$$\left(\partial_t + \tilde{\Lambda}\partial_x\right)V^{-1}\begin{pmatrix}\delta\rho^{(0)}\\ \delta v^{(0)}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{66}$$

which has a solution

$$V^{-1}\begin{pmatrix}\delta\rho^{(0)}\\ \delta v^{(0)}\end{pmatrix} = f_j^{(0)}(x - \tilde{\lambda}_j t)e_j,$$

or

$$\begin{pmatrix} \delta\rho^{(0)} \\ \delta v^{(0)} \end{pmatrix} = f_j^{(0)}(x - \tilde{\lambda}_j t) V e_j,$$

where $f_j^{(0)}(\xi)$ is any function, e_j is the unit vector in $\mathbb{R}^{2n}$ and $\tilde{\lambda}_j$ is any of the eigenvalues of $\mathcal{A}$. As common in the method of multiple scales, we will assume $f_j^{(0)}$ depends on $x - \tilde{\lambda} t$ as well as a new slow time scale $\tau = \epsilon^2 t$. Hence

$$\begin{pmatrix} \delta\rho^{(0)} \\ \delta v^{(0)} \end{pmatrix} = f_j^{(0)}(x - \tilde{\lambda}_j t, \epsilon^2 t) V e_j.$$

The equations at order ϵ^2 are then given by

$$(\partial_t + \mathcal{A}\partial_x) \begin{pmatrix} \delta\rho^{(1)} \\ \delta v^{(1)} \end{pmatrix} = -\partial_\tau \begin{pmatrix} \delta\rho^{(0)} \\ \delta v^{(0)} \end{pmatrix} - \partial_x \begin{pmatrix} \mathcal{N}_1(\delta\rho^{(0)}, \delta v^{(0)}) \\ \mathcal{N}_2(\delta\rho^{(0)}, \delta v^{(0)}, 0) \end{pmatrix}, \tag{67}$$

which is equivalent to

$$\left(\partial_t + \tilde{\Lambda}\partial_x\right) V^{-1} \begin{pmatrix} \delta\rho^{(1)} \\ \delta v^{(1)} \end{pmatrix} = -\partial_\tau V^{-1} \begin{pmatrix} \delta\rho^{(0)} \\ \delta v^{(0)} \end{pmatrix} - \partial_x V^{-1} \begin{pmatrix} \mathcal{N}_1(\delta\rho^{(0)}, \delta v^{(0)}) \\ \mathcal{N}_2(\delta\rho^{(0)}, \delta v^{(0)}, 0) \end{pmatrix}. \tag{68}$$

Notice this is a linear equation for the order ϵ^2 correction to $\delta\rho, \delta v$. The right-hand side is essentially a known function since every term on the right hand side can be written in terms of $f_j^{(0)}(x - \tilde{\lambda}_j t, \tau)$. Moreoever, the adjoint of the linear operator on the left-hand side has a null space: precisely those functions of the form $\psi(x - \tilde{\lambda}_j t) e_j$. From the Fredholm alternative, the right-hand side should be orthogonal to this null space. Notice all terms on the right are of the form $\psi(x - \tilde{\lambda}_j t)$. Hence we have the solvability condition

$$\left\langle e_j, -\partial_\tau V^{-1} \begin{pmatrix} \delta\rho^{(0)} \\ \delta v^{(0)} \end{pmatrix} - \partial_x V^{-1} \begin{pmatrix} \mathcal{N}_1(\delta\rho^{(0)}, \delta v^{(0)}) \\ \mathcal{N}_2(\delta\rho^{(0)}, \delta v^{(0)}, 0) \end{pmatrix} \right\rangle = 0,$$

which can be simplied using the expression for the zeroth order solution to

$$\left\langle e_j, -\partial_\tau f_j^{(0)} e_j - \partial_x V^{-1} \begin{pmatrix} \mathcal{N}_1(\delta\rho^{(0)}, \delta v^{(0)}) \\ \mathcal{N}_2(\delta\rho^{(0)}, \delta v^{(0)}, 0) \end{pmatrix} \right\rangle = 0,$$

or in other words

$$\partial_\tau f_j^{(0)} + \left\langle e_j, \partial_x V^{-1} \begin{pmatrix} \mathcal{N}_1(\delta\rho^{(0)}, \delta v^{(0)}) \\ \mathcal{N}_2(\delta\rho^{(0)}, \delta v^{(0)}, 0) \end{pmatrix} \right\rangle = 0.$$

Rewriting $\mathcal{N}_1, \mathcal{N}_2$ entirely in terms of $f_j^{(0)}$ we have the required KdV equation. This is true for each j and hence we have derived a system of uncoupled KdV equations. The system is uncoupled since the right-hand side terms $\mathcal{N}_1, \mathcal{N}_2$ are given entirely in terms of the profile $f_j^{(0)}$. Physically this corresponds to moving into different traveling frames centered around each pulse.

10. Damped Korteweg-de Vries

Consider the equation

$$u_t + 6uu_x + u_{xxx} = -\epsilon u,$$

which we wish to analyse using perturbation theory. Notice that the $\mathcal{O}(1)$ problem (setting $\epsilon = 0$) leads to a nonlinear PDE. Nonetheless, the basic idea of perturbation theory still works as we see below.

Although the $\epsilon = 0$ problem is nonlinear, we still can find special solutions which depend on some parameters. For instance, if we look for solutions of the form $u(x,t) = f(x - 4\kappa^2 t)$, for any real number κ to the $\epsilon = 0$ problem, we have

$$-4\kappa^2 f' + 6ff' + f''' = 0,$$

where primes indicate derivatives with respect to $\xi = x - 4\kappa^2 t$, the traveling frame coordinate. As it happens, one can exactly solve this ordinary differential equation assuming $f \to 0$ for $|\xi| \to \infty$ to obtain

$$f(\xi) = 2\kappa^2 \text{sech}^2(\kappa\xi),$$

which is valid for every real number κ. We will now construct an approximate solution to the original damped equation by perturbing the above traveling wave.

10.1. **Perturbation series.** Let

$$u = u^{(0)} + \epsilon u^{(1)} + \dots$$

Then we have the $\mathcal{O}(1)$ problem

$$u_t^{(0)} + 6u^{(0)}u_x^{(0)} + u_{xxx}^{(0)} = 0,$$

a solution to which is given by

$$u^{(0)} = 2\kappa^2 \text{sech}^2(\kappa\xi), \quad \xi = x - 4\kappa^2 t.$$

Before proceeding to the next order, we adopt the usual procedure of allowing slow variations of the order one problem. Thus we assume the parameters of the traveling wave, namely κ, which was heretofore an arbitrary constant, will now be a slow function of time. Hence $\kappa = \kappa(\epsilon t)$ and $\epsilon t = \tau$. This now leads to the $\mathcal{O}(1)$ equation

$$-4\kappa^2 u_\xi^{(1)} + 6\partial_\xi(u^{(0)}u^{(1)}) + u_{\xi\xi\xi}^{(1)} = -u^{(0)} - u_\tau^{(0)},$$

which we can symbolically write as

$$Lu^{(1)} = -u^{(0)} - u_\tau^{(0)},$$

where L is a variably coefficient linear differential operator acting on functions which vanish at infinity. We define an inner product as

$$(f,g) = \int_{-\infty}^{\infty} f(\xi)\ g(\xi)\ d\xi$$

and as a consequence the adjoint $L^\dagger$ is defined as

$$L^\dagger = 4\kappa^2\partial_\xi - 6u^{(0)}\partial_\xi - \partial_{\xi\xi\xi}.$$

Thus in order to solve $Lu^{(1)} = -u^{(0)} - u^{(0)}_\tau$ we apply the Fredholm alternative which demands that the right-hand side be orthogonal to the null-space of the adjoint $L^\dagger$. A proper function in the null-space is indeed the function $u^{(0)}$! Thus we impose the condition

$$\int_{-\infty}^{\infty} (u^{(0)} + u^{(0)}_\tau)u^{(0)}\, d\xi = 0.$$

This expression is equivalent to

$$\int_{-\infty}^{\infty} \left(u^{(0)} + \frac{\kappa_\tau}{\kappa}\left(2u^{(0)} + \xi u^{(0)}_\xi\right)\right) u^{(0)}\, d\xi = 0.$$

Upon integrating by parts we obtain

$$\frac{1}{\kappa}\kappa_\tau = -\frac{2}{3} \Rightarrow \kappa = \kappa_0 e^{-2/3\,\tau}.$$

Remark 5. *In case you find the integrals hard to do, here is a hint.*

$$\int_{-\infty}^{\infty} u^{(0)}\frac{\partial}{\partial\tau}u^{(0)}\, d\xi = \frac{\partial}{\partial\tau}\int_{-\infty}^{\infty}\left(\frac{u^{(0)}}{2}\right)^2 d\xi = \frac{\partial}{\partial\tau}\int_{-\infty}^{\infty} 2\kappa^4 \mathrm{sech}^4(\kappa\xi)d\xi = 6\kappa^2\frac{d\kappa}{d\tau}\int_{-\infty}^{\infty}\mathrm{sech}^4(y)dy$$

Then

$$\int_{-\infty}^{\infty} (u^{(0)} + u^{(0)}_\tau)u^{(0)}\, d\xi = 4\kappa^3\int_{-\infty}^{\infty}\mathrm{sech}^4(y)dy + 6\kappa^2\frac{d\kappa}{d\tau}\int_{-\infty}^{\infty}\mathrm{sech}^2(y)dy = 0$$

from which the ordinary differential equation for κ follows.

11. Spectrum of a real symmetric matrix

As another algebraic example, let's consider the problem of finding eigenvalues-eigenvectors of a symmetric matrix from the standpoint of perturbation theory. Let D be a real diagonal $\mathbb{R}^n \times \mathbb{R}^n$ matrix with distinct elements. Suppose we perturb this matrix by a small amount ϵ in the direction of a real symmetric matrix A (and so $A^T = A$). We wish to compute the eigenvalues-eigenvectors (the spectral data) of the perturbed matrix $D + \epsilon A$.

Notice the spectrum of D is easy to compute. The eigenvalues are simply the diagonal elements of D while the eigenvectors are just the columns of the $n \times n$ identity matrix. On the other hand, the eigenvectors of the perturbed matrix satisfy

$$(D + \epsilon A)v = \lambda v. \tag{69}$$

When $\epsilon = 0$, we have $\lambda = d_j, v = e_j$ where $d_j = D_{jj}$, the j-th term along the diagonal of D and e_j is the associated eigenvector: the j-th column of the identity matrix. Thus we suppose

$$\lambda = d_j + \epsilon\delta\lambda_j, \tag{70}$$

$$v = e_j + \epsilon\delta v_j. \tag{71}$$

Recall that eigenvectors are identified only up to a scalar multiple. To uniquely identify an eigenvector, we must stipulate an additional constraint. A common constraint is to require the eigenvector have unit norm. We will employ an alternative. To be precise, we will fix the $j-$th component of v to be 1. To do so, we impose the corrections to e_j, *i.e.* δv_j be perpendicular to e_j. Hence $\langle e_j, \delta v_j\rangle = 0$.

Substituting (70-71) in (69) we obtain

$$\begin{aligned}(D + \epsilon A)(e_j + \epsilon\delta v_j) &= (d_j + \epsilon\delta\lambda_j)(e_j + \epsilon\delta v_j),\\ \Rightarrow (D - d_j I)\delta v_j &= -Ae_j + \delta\lambda_j e_j + \epsilon(\delta\lambda_j\delta v_j - A\delta v_j).\end{aligned} \tag{72}$$

Note the matrix on the left-hand side is a diagonal matrix with one diagonal element equal to zero and hence it is not invertible. Consequently, the Fredholm alternative requires the right-hand side of the above equation be orthogonal to the null-space of $(D - d_j I)^T = D - d_j I$, namely all vectors in the direction of e_j. We can then write

$$0 = -\langle e_j, Ae_j\rangle + \delta\lambda_j\langle e_j, e_j\rangle + \epsilon[\langle e_j, \delta v_j\rangle - \langle e_j, A\delta v_j\rangle]. \tag{73}$$

When we impose the additional constraint of $\delta v_j \perp e_j$, we get the expression of $\delta\lambda_j$

$$\delta\lambda_j = \langle e_j, Ae_j\rangle + \epsilon\langle e_j, A\delta v_j\rangle. \tag{74}$$

Without loss of generality, we can assume $A_{jj} = 0$ as the diagonal elements of A can be absorbed into D and the above expression for $\delta\lambda_j$ simplifies even further. Indeed

$$\delta\lambda_j = \epsilon\langle e_j, A\delta v_j\rangle \tag{75}$$

We can employ this expression in (72) to obtain an equation for δv_j alone

$$(D - d_j I)\delta v_j = -Ae_j - \epsilon A\delta v_j + \epsilon\langle e_j, A\delta v_j\rangle e_j + \epsilon^2\langle e_j, A\delta v_j\rangle\delta v_j \tag{76}$$

Note equations (75-76) are exact relations. We have not employed perturbation theory up to this point! Moreoever, (76) is an equation for the correction to the eigenvector alone. Thus we first solve for the eigenvector and then substitute this expression into the right-hand side of (75) to obtain the eigenvalue.

Remark 6. *We could consider an iterative scheme based on (76) to obtain a numerical method to compute the eigenvector to the required precision. Indeed, that method is simply a Newton-Raphson scheme to solve this equation for* δv_j*.*

Finally, we can expand δv_j in an ϵ power series (perturbatively) to obtain an approximate solution. This leads to the following expressions for λ, v

$$\lambda = d_j - \epsilon^2 \langle e_j, A(D - d_j I)^{-1} A e_j \rangle + \mathcal{O}(\epsilon^3) \tag{77}$$

$$v = e_j - \epsilon (D - d_j I)^{-1} A e_j + \mathcal{O}(\epsilon^2). \tag{78}$$

We can further simplify the expression for λ using the fact that A is a symmetric matrix to get

$$\lambda = d_j - \epsilon^2 \sum_{i \neq j} \frac{(A e_j)_i^2}{(d_i - d_j)} + \mathcal{O}(\epsilon^3). \tag{79}$$