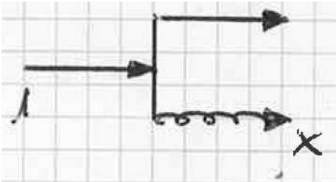
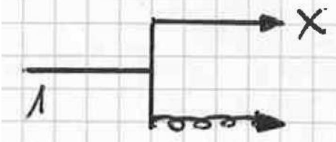
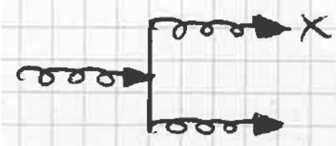
QCD splitting probabilities: $\dfrac{\delta N}{\delta \log \frac{Q^2}{\Lambda^2}} = \dfrac{\alpha_s(Q^2)}{2\pi} P(x) dx$

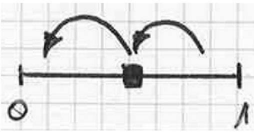
$q \to q + g(x)$  $P_{gq} = \dfrac{4}{3} \dfrac{1 + (1-x)^2}{x}$ bremsstrahl-sing. $x \to 0$

$q \to q(x) + g$  $P_{qq} = \dfrac{4}{3} \dfrac{1 + x^2}{1-x}$ bremsstrahl-sing. $x \to 1$

$g \to q(x) + \bar{q}$  $P_{qg} = P_{\bar{q}g} = \dfrac{1}{2}[x^2 + (1-x)^2]$ finite

$g \to g(x) + g$  $P_{gg} = 6 \dfrac{[1 - x + x^2]^2}{x(1-x)}$ bremsstrahl-sing. $x \to 0, 1$

Altarelli–Parisi master equations for parton densities:



$Q^2 \to Q^2 + \delta Q^2$

$$\frac{\partial q(x, Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_0^1 dy \int_0^1 dz \;\delta_1(x - yz) \left\{ P_{qq}(y) q(z, Q^2) + P_{qg}(y) g(z, Q^2) \right\}$$
$$- \frac{\alpha_s(Q^2)}{2\pi} \int_0^1 dy' \; P_{qq}(y') q(x, Q^2)$$

$$\int_0^1 dy' P_{qq}(y') q(x, Q^2) = \int_0^1 dy \int_0^1 dz \delta_1(x - yz) \delta(y-1) \left[\int_0^1 dy' P_{qq}(y') \right] q(z, Q^2)$$

$$
\begin{aligned}
\frac{\partial q(x,Q^2)}{\partial \log Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\ \delta_1(x-yz)\left\{P^R_{qq}(y)q(z,Q^2)\right.\\
&\qquad\left. +P_{qg}(y)g(z,Q^2)\right\}\\
\frac{\partial g(x,Q^2)}{\partial \log Q^2} &= \frac{\alpha_s(Q^2)}{2\pi}\int_0^1 dy \int_0^1 dz\ \delta_1(x-yz)\left\{P_{gq}(y)\sum_{fl}[q(z,Q^2)+\bar{q}(z,Q^2)]\right.\\
&\qquad\left. +P^R_{gg}(y)g(z,Q^2)\right\}\\
P^R_{qq}(y) &= P_{qq}(y)-\delta(y-1)\int_0^1 dy' P_{qq}(y')\\
P^R_{gg}(y) &= P_{gg}(y)-\delta(y-1)\left[\frac{1}{2}\int_0^1 dy' P_{gg}(y') + N_F\int_0^1 dy' P_{qg}(y')\right]\\
\alpha_s(Q^2) &= \frac{12\pi}{(33-2N_F)\log\frac{Q^2}{\Lambda^2}}
\end{aligned}
$$

partial disentanglement: $\delta = q - q'$ non-singlet

$$
\left.\begin{array}{l}\Sigma = \sum_{fl}(q+\bar{q}) \\ g\end{array}\right\} \text{coupled singlet set}
$$

SOLUTIONS:

transition to moments: $\underline{\underline{q(N,Q^2) = \int_0^1 dx\ x^{N-1} q(x,Q^2)}}$

transforms integro-differential system of equations into system of usual differential equations.

natural variable: $s = \log \frac{\log Q^2}{\log Q_0^2}$ [Q_0 = reference momentum transfer]

[for fixed coupling constant $t = \log Q^2$ would be the natural variable]

1.) Non-singlet density:

$$\frac{\partial}{\partial s}\delta(N,Q^2) = \frac{6}{33-2N_F}\int_0^1 dy\ y^{N-1} P_{qq}^R(y)\delta(N,Q^2)$$

$$\uparrow = \frac{6}{33-2N_F}\frac{4}{3}\left[-\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{j=2}^{N}\frac{1}{j}\right] \equiv -d_{NS}(N)$$

$$\frac{\partial}{\partial s}\delta(N,Q^2) = -d_{NS}(N)\delta(N,Q^2) \Rightarrow \delta = \delta_0 e^{-s d_{NS}}$$

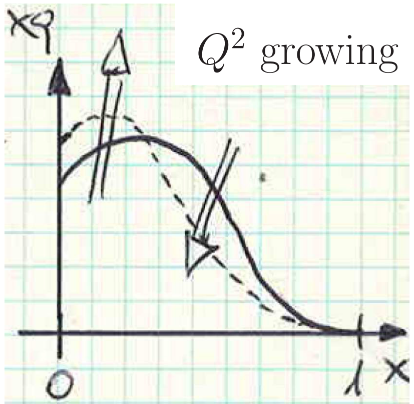
$$\boxed{\begin{aligned}\delta(N,Q^2) &= \delta(N,Q_0^2)\left[\frac{\log Q^2}{\log Q_0^2}\right]^{-d_{NS}} \\ &= \delta(N,Q_0^2)\left[\frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)}\right]^{d_{NS}}\end{aligned}}$$

$\leftarrow$ log. violation of <u>Bjorken scaling</u>

<u>interpretation:</u>

(i) asymptotic freedom $\Rightarrow \left[\frac{\log Q^2}{\log Q_0^2}\right]^{-d}$

fixed coupling $\Rightarrow \left[\frac{Q^2}{Q_0^2}\right]^{-d}$



(ii) $d_{NS}(N=1)=0$: net quark # unchanged

$d_{NS}(N>1)>0$: moments decrease with increasing Q^2

(iii) moment comparison: test of anomalous dimensions

Q^2-dependence of structure functions

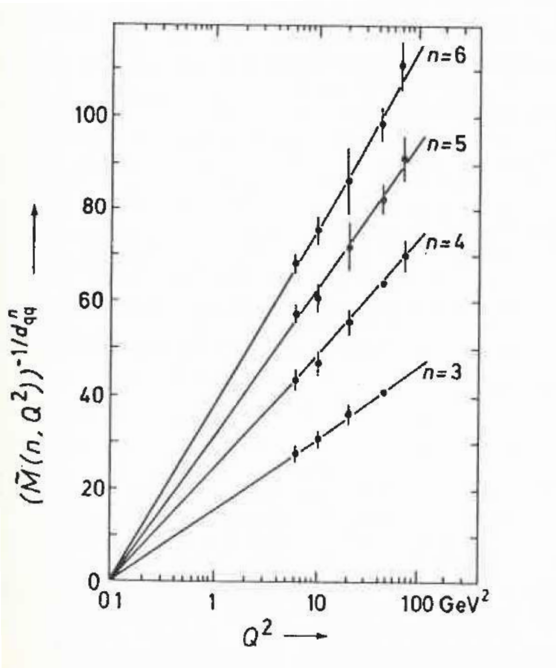


Bild 19-7

Die Momente der Strukturfunktion $F_3^{(\nu N)}$, gemessen in Neutrino-Eisen-Streuung (nach de Groot 1979)

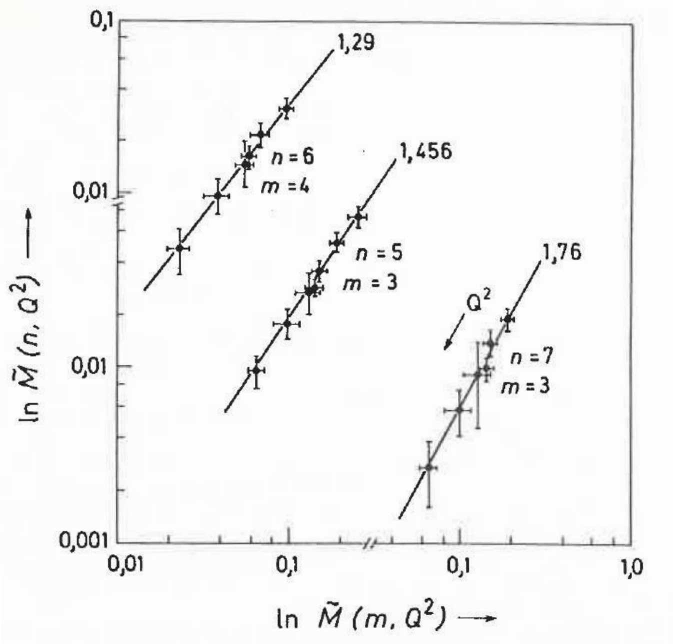


Bild 19-8

Logarithmen von Momenten der Strukturfunktion F_3 gegeneinander aufgetragen. Die QCD-Vorhersagen sind gerade Linien mit berechenbarem Anstieg, wie angegeben (nach Bosetti 1978).

2.) Quark singlet and gluon densities:

$$\frac{\partial}{\partial s}\begin{pmatrix} \Sigma \\ G \end{pmatrix} = -\begin{pmatrix} d_{QQ} & d_{QG} \\ d_{GQ} & d_{GG} \end{pmatrix}\begin{pmatrix} \Sigma \\ G \end{pmatrix} \quad \text{with } \Sigma = \Sigma(N, Q^2) \text{ etc.}$$

$$d_{QQ}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} P_{qq}^R(y) = \frac{4}{33-2N_F}\left[1 - \frac{2}{N(N+1)} + 4\sum_{j=2}^{N}\frac{1}{j}\right] \equiv d_{NS}(N)$$

$$d_{QG}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} 2N_F\; P_{qg}(y) = -\frac{6N_F}{33-2N_F}\frac{N^2+N+2}{N(N+1)(N+2)}$$

$$d_{GQ}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} P_{gq}(y) = -\frac{8}{33-2N_F}\frac{N^2+N+2}{(N-1)N(N+1)}$$

$$d_{GG}(N) = -\frac{6}{33-2N_F}\int_0^1 dy\; y^{N-1} P_{gg}^R(y) = \frac{9}{33-2N_F}\left\{\frac{1}{3} - \frac{4}{N(N-1)} - \frac{4}{(N+1)(N+2)} + 4\sum_{j=2}^{N}\frac{1}{j} + \frac{2N_F}{9}\right\}$$

solution of the systems via exponential ansatz $\Rightarrow$

$$\boxed{\begin{aligned} \Sigma &= \frac{1}{\mu_+ - \mu_-}\left\{[-\mu_-\Sigma_0 + G_0]e^{-d_+ s} + [\mu_+\Sigma_0 - G_0]e^{-d_- s}\right\} \\ G &= \frac{1}{\mu_+ - \mu_-}\left\{\mu_+[-\mu_-\Sigma_0 + G_0]e^{-d_+ s} + \mu_-[\mu_+\Sigma_0 - G_0]e^{-d_- s}\right\} \end{aligned}}$$

eigenvalues: $d_\pm(N) = \frac{1}{2}\left[(d_{GG} + d_{QQ}) \pm \sqrt{(d_{GG} - d_{QQ})^2 + 4 d_{QG} d_{GQ}}\right]$

eigenvectors: $\mu_\pm(N) = \frac{d_\pm - d_{QQ}}{d_{QG}} = \frac{1}{2}\frac{d_{GG} - d_{QQ} \pm \sqrt{(d_{GG} - d_{QQ})^2 + 4 d_{QG} d_{GQ}}}{d_{QG}}$

PHYSICAL CONCLUSIONS:

(a) momentum sum rule:

$$\left.\begin{array}{l} d_-(2) = 0 \\ \mu_+(2) = -1 \end{array}\right\} \underline{\underline{\Sigma(2) + G(2) = 1}} \text{ follows from } \Sigma_0(2) + G_0(2) = 1$$

(b) asymptotic momentum distribution:

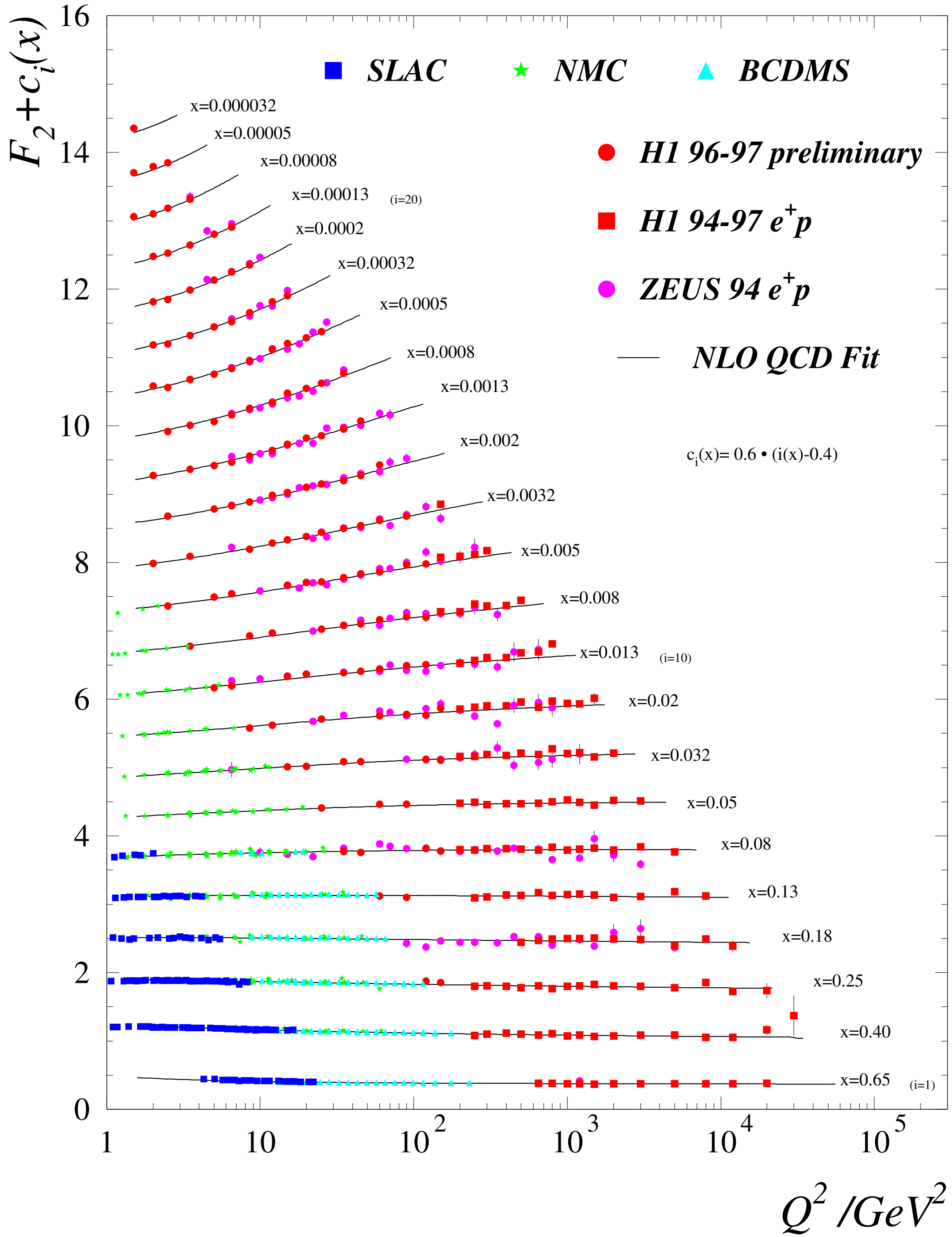
$$\left.\begin{array}{l} d_-(2) = 0 \\ \mu_+(2) = -1 \end{array}\right\} \begin{array}{l} \Sigma(2) \to \dfrac{1}{\mu_-(2) - \mu_+(2)} = \dfrac{3N_F}{16 + 3N_F} = \dfrac{3}{7} \text{ for } N_F = 4 \\[2ex] \underline{\underline{G(2) \to \dfrac{\mu_-(2)}{\mu_-(2) - \mu_+(2)} = \dfrac{16}{16 + 3N_F} = \dfrac{4}{7}}} \text{ for } N_F = 4 \end{array}$$

(c) measurement of all gluon moments:

deep inelastic $\ell\mathcal{N}$ scatt.: no dir. poss. to meas. gluon density.

indirect: • momentum sum rule: $G(2, Q_0^2) = 1 - \Sigma(2, Q_0^2)$

- modifies strength of $\Sigma(N, Q^2)$-var. with Q^2 through buildup of sea density.
- $[G \to Q\bar{Q}$ splitting into heavy quarks$]$



$F_2+c_i(x)$
SLAC
NMC
BCDMS
H1 96-97 preliminary
H1 94-97 e^+p
ZEUS 94 e^+p
NLO QCD Fit
$c_i(x)= 0.6 \bullet (i(x)-0.4)$
x=0.000032
x=0.00005
x=0.00008
x=0.00013 (i=20)
x=0.0002
x=0.00032
x=0.0005
x=0.0008
x=0.0013
x=0.002
x=0.0032
x=0.005
x=0.008
x=0.013 (i=10)
x=0.02
x=0.032
x=0.05
x=0.08
x=0.13
x=0.18
x=0.25
x=0.40
x=0.65 (i=1)
Q^2 /GeV^2

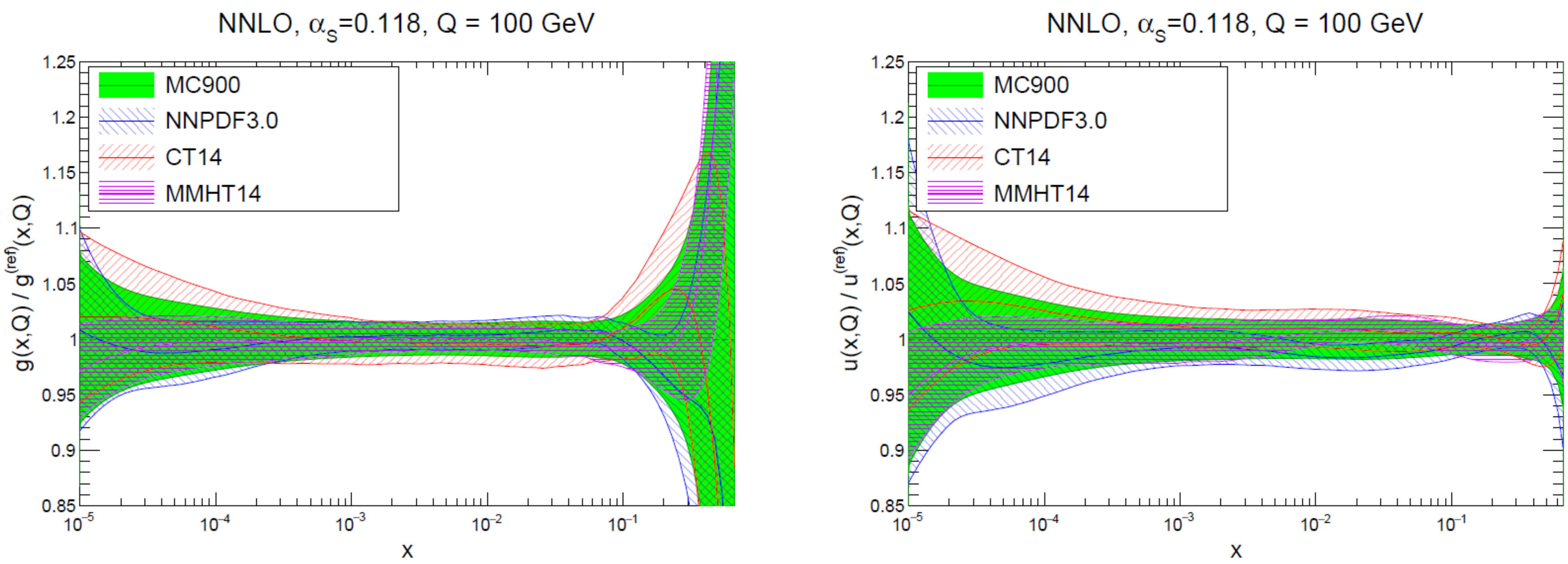


Figure 2: Comparison of the MC900 PDFs with the sets that enter the combination: CT14, MMHT14 and NNPDF3.0 at NNLO. We show the gluon and the up quark at $Q = 100$ GeV. Results are normalized to the central value of the prior set MC900.

PDF4LHC15 PDFs: Comb. of NNPDF3.0, CT14 and MMHT14 PDFs

quark and gluon PDFs known to few-% accuracy for $x \sim 10^{-4} \ldots 0.1$

central PDFs + error PDFs $\to$ PDF+α_s error

global fit to hadronic observables: DIS, DY, jet production,. . .

missing: elw. corr. [$\to \gamma$ PDF], hadronic uncertainties

§7. Factorization Theorems of QCD

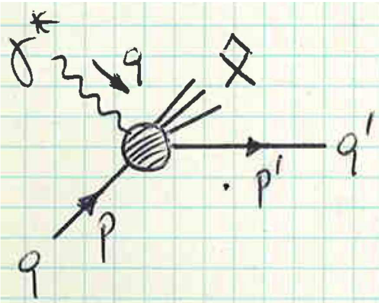
QCD corrections to deep inelastic $\ell\mathcal{N}$ scattering:

$\boxed{\text{DIMENSIONAL REGULARIZATION}}$

idea: analytical continuation 4-dim. $\to$ n-dim. $[n = 4 - 2\epsilon]$; $\displaystyle\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^nk}{(2\pi)^n}$

$\Rightarrow$ UV singularities as poles for $\epsilon \to 0$ [gauge invariance preserved]

deep inelastic $\ell\mathcal{N}$ scattering:



$$z = \frac{-q^2}{2pq} = \frac{Q^2}{2pq} \Rightarrow -\frac{pq}{q^2} = \frac{1}{2z}$$

$$q = p' - p + p_{\hat{X}}$$

parton tensor:

$$\widehat{W}^{\mu\nu} = \widehat{F}_1(z,Q^2)\left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right] + \frac{2z}{Q^2}\widehat{F}_2(z,Q^2)\left(p^\mu + \frac{q^\mu}{2z}\right)\left(p^\nu + \frac{q^\nu}{2z}\right)$$

$$\Rightarrow p^\mu p^\nu \widehat{W}_{\mu\nu} = \frac{Q^2}{4z^2}\left(\frac{\widehat{F}_2}{2z} - \widehat{F}_1\right) \equiv \frac{Q^2}{4z^2}\frac{\widehat{F}_L}{2z}$$

$$-g^{\mu\nu}\widehat{W}_{\mu\nu} = (1-\epsilon)\frac{\widehat{F}_2}{z} - \frac{3-2\epsilon}{2z}\widehat{F}_L$$

$$\boxed{\begin{aligned}\widehat{\mathcal{F}}_1(z,Q^2) &= \widehat{F}_1(z,Q^2)\\ \widehat{\mathcal{F}}_{2,L}(z,Q^2) &= \frac{\widehat{F}_{2,L}(z,Q^2)}{2z}\end{aligned}}$$

structure functions: $\displaystyle\mathcal{F}_i(x,Q^2) = \sum_{q,\bar{q}} e_q^2 \int_0^1 dydz\ \widehat{\mathcal{F}}_i(z,Q^2)\ q(y,Q^2)\ \delta(x - yz)$

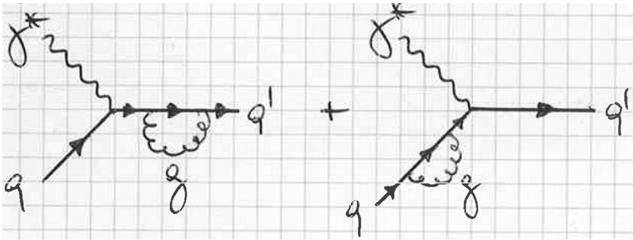
1.) Born term: $\mathcal{M}^{\mu}_{LO} = -iee_q\delta_{ij}\bar{u}(p')\gamma^{\mu}u(p)$

$$\widehat{W}^{\mu\nu}_{LO} = \frac{1}{N_c}\sum \mathcal{M}^{\mu}_{LO}\mathcal{M}^{*\nu}_{LO}\frac{dPS_1(p+q;p')}{8\pi\sigma_0}$$

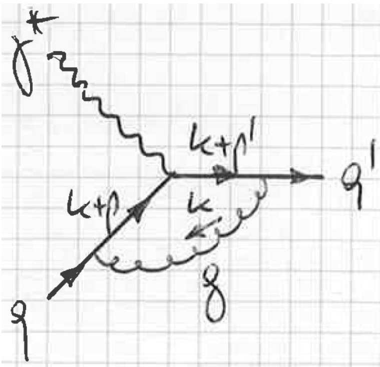
$\Rightarrow$ $\boxed{\widehat{\mathcal{F}}_{L,LO} = 0}$ $\boxed{\widehat{\mathcal{F}}_{1,LO} = \widehat{\mathcal{F}}_{2,LO} = \delta(1-z)}$ $(\sigma_0 = 2\pi\alpha e_q^2)$

2.) QCD corrections:

(i) virtual corrections:



$\equiv 0$ [partons massless]



$$\mathcal{M}_V = i^6(-1)^4 ee_q\bar{g}_s^2(T^aT^a)_{ij}\ \bar{u}(p')\int\frac{d^nk}{(2\pi)^n}\frac{\gamma^{\alpha}(\not{k}+\not{p}')\gamma^{\mu}(\not{k}+\not{p})\gamma_{\alpha}}{k^2(k+p)^2(k+p')^2}\ u(p)$$

dimensionless coupling: $\bar{g}_s^2 = g_s^2\mu^{2\epsilon}$

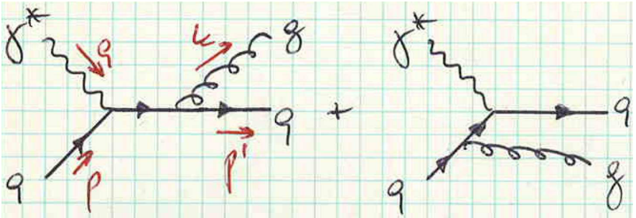
$$\widehat{W}^{\mu\nu}_{V} = \frac{1}{N_c}\sum 2\Re e\mathcal{M}^{\mu}_{V}\mathcal{M}^{*\nu}_{LO}\ \frac{dPS_1(p+q;p')}{8\pi\sigma_0}$$

$$\boxed{\delta\hat{\mathcal{F}}_{2,V} = \delta\hat{\mathcal{F}}_{1,V} = C_F \frac{\alpha_s}{2\pi}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8-2\zeta(2)\right]\delta(1-z)}$$

$$\boxed{\delta\hat{\mathcal{F}}_{L,V} = 0}$$

Infrared divergences will be subtracted by adding the real gluon radiation.

(ii) real corrections:



$$\left.\begin{array}{l} s = (p+q)^2 \\ t = (p'-p)^2 \\ u = (k-p)^2 \end{array}\right\} \Rightarrow s+t+u = q^2$$

$$\mathcal{M}_q^{\mu} = i^3(-1)^2 e e_q \bar{g}_s T^a_{ij}\bar{u}(p')\left\{\frac{\gamma^{\alpha}(\not{p}'+\not{k})\gamma^{\mu}}{(p'+k)^2}+\frac{\gamma^{\mu}(\not{p}-\not{k})\gamma^{\alpha}}{(p-k)^2}\right\}u(p)\epsilon^*_{\alpha}$$

$$\hat{W}^{\mu\nu}_{\gamma q} = \frac{1}{N_c}\int\sum \mathcal{M}_q^{\mu}\mathcal{M}_q^{*\nu}\frac{dPS_2(p+q;p',k)}{8\pi\sigma_0}$$

parametrization: $s = \frac{1-z}{z}Q^2; \quad t = -\frac{Q^2}{z}(1-y); \quad u = -\frac{Q^2}{z}y$

$$\boxed{\delta\hat{\mathcal{F}}_{Lq} = \frac{4z^2}{Q^2}p_{\mu}p_{\nu}\hat{W}^{\mu\nu}_{\gamma q} = \int_0^1 dy\ \frac{4}{3}\frac{\alpha_s}{2\pi}4z^2\frac{-t}{Q^2} = \frac{4}{3}\ \frac{\alpha_s}{2\pi}\ 2z \neq 0} \leftarrow \text{finite}$$

$$\delta\hat{\mathcal{F}}_{2q} = C_F\ \frac{\alpha_s}{2\pi}\ \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\left\{\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{7}{2}\right]\delta(1-z)\right.$$
$$\left.-\left(\frac{1}{\epsilon}+\log z\right)\left(\frac{1+z^2}{1-z}\right)_+ +(1+z^2)\left(\frac{\log(1-z)}{1-z}\right)_+ -\frac{3}{2}\left(\frac{1}{1-z}\right)_+ +3+2z\right\}$$

Altarelli–Parisi splitting functions:

$$P_{qq}(z) = C_F \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dz' \; C_F \; \frac{1+z'^2}{1-z'}$$

$$\Rightarrow \int_0^1 dz \; f(z) \; P_{qq}(z) = \int_0^1 dz \; C_F \frac{1+z^2}{1-z} \left[f(z) - f(1)\right] = \int_0^1 dz \; C_F \left(\frac{1+z^2}{1-z}\right)_+ f(z)$$

$$\Rightarrow \boxed{P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+ = C_F \left\{ \left(\frac{2}{1-z}\right)_+ - 1 - z + \frac{3}{2}\delta(1-z) \right\}}$$

sum virtual + real:

$$\delta \hat{\mathcal{F}}_2^{\gamma q} = \hat{\mathcal{F}}_{2V} + \hat{\mathcal{F}}_{2q} = \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \frac{\alpha_s}{2\pi} \; \left[-\frac{1}{\epsilon} - \log z\right] P_{qq}(z) + C_F \; \frac{\alpha_s}{2\pi} \; \left\{ (1+z^2) \left(\frac{\log(1-z)}{1-z}\right)_+ \right.$$
$$\left. - \frac{3}{2} \left(\frac{1}{1-z}\right)_+ + 3 + 2z - \left(\frac{9}{2} + \frac{\pi^2}{3}\right) \delta(1-z) \right\}$$

The crossed channel $\gamma^* g \to q\bar{q}$ is of the same order in α_s and cannot be distinguished from $\gamma^* \to qg$.