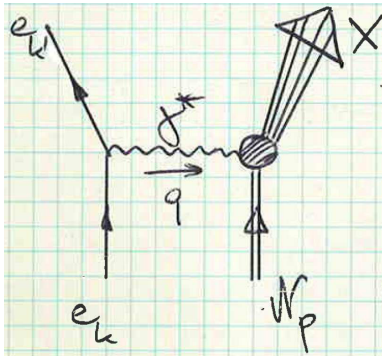


§4. Structure Functions of the Nucleon

Asymptotic freedom:

- (i) α_s small $\Rightarrow 0^{th}$ approx.: approx. free part. at short dist./high energies
 $\Rightarrow$ PARTON MODEL
- (ii) $\log Q^2$ dependence through HOs [w.l.o.g.: elm. structure functions]



$$\mathcal{M}(X) = ie^2 \bar{u}' \gamma^\mu u \frac{1}{q^2} \langle X | j_\mu | \mathcal{N}_p \rangle$$

cross section [$E = e$ lab. energy]

$$d\sigma(e') = \frac{1}{4ME} \frac{d^3k'}{(2\pi)^3 2E'} \frac{1}{4} \sum_X (2\pi)^4 \delta_4(p + q - p_X) |\mathcal{M}_X|^2$$

$$q = k - k' \quad q^2 = -Q^2 < 0$$

$$\frac{1}{4} \sum_X (2\pi)^4 \delta_4(p + q - p_X) |\mathcal{M}_X|^2 = \underbrace{\left(\frac{e^2}{Q^2} \right)^2 \frac{1}{4} \sum_{spins} [\bar{u}' \gamma^\nu u] [\bar{u} \gamma^\mu u']}_{=\mathcal{L}^{\mu\nu} \text{ lepton tensor}} \underbrace{\sum_X \langle \mathcal{N} | j_\mu | X \rangle \langle X | j_\nu | \mathcal{N} \rangle (2\pi)^4 \delta_4(p + q - p_X)}_{=8\pi W_{\mu\nu} \text{ hadron tensor}}$$

lepton tensor: $\mathcal{L}_{\mu\nu} = k_\mu k'_\nu + k_\nu k'_\mu - (kk')g_{\mu\nu} \leftarrow \text{symm. } \mu, \nu; k, k'$

$$\begin{aligned} \text{hadron tensor: } W_{\mu\nu} &= \frac{1}{8\pi} \sum_{spins} \sum_X (2\pi)^4 \delta_4(p + q - p_X) \langle \mathcal{N}_p | j_\mu^{elm} | X \rangle \langle X | j_\nu^{elm} | \mathcal{N}_p \rangle \\ &= \frac{1}{8\pi} \sum_{spins} \int d^4x e^{-iqx} \langle \mathcal{N}_p | \left[j_\mu^{elm}(0), j_\nu^{elm}(x) \right] | \mathcal{N}_p \rangle \end{aligned}$$

properties of $W_{\mu\nu}$:

- (i) symm. tensor in $p_\mu, q_\mu, g_{\mu\nu}$
- (ii) current cons.: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$ [$\partial^\mu j_\mu^{elm} = 0$]
- (iii) tensor real ($\leftarrow$ hermiticity of elm. current)

decomposition in invariants:

$$W_{\mu\nu} = W_1 \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right] + W_2 \left[p_\mu - q_\mu \frac{pq}{q^2} \right] \left[p_\nu - q_\nu \frac{pq}{q^2} \right]$$

$W_i = \text{Lorentz scalar structure functions}$

variables: (i) electron state characterized by energy and scatt. angle

(ii) inv.: $Q^2 = -q^2 = 4EE' \sin^2 \frac{\theta}{2}$ scattering angle

$\nu = pq = M(E - E')$ energy loss in e sector

$$\text{range: } \left. \begin{array}{l} Q^2 \geq 0 \\ \nu \geq 0 \end{array} \right\} \begin{array}{l} (p+q)^2 = W^2 \geq M^2 \text{ (at least } \mathcal{N} \text{ in final state)} \\ M^2 + 2pq + q^2 \geq M^2 \Rightarrow 2\nu \geq Q^2 \\ \qquad \qquad \qquad \underline{\underline{= elastic}} \end{array}$$

(iii) scaling variables:

$$\begin{array}{ll} \text{Bjorken variable} & x = \frac{Q^2}{2\nu} \quad 0 \leq x \leq 1 \\ \text{rel. energy loss} & y = \frac{pq}{pk} \quad 0 \leq y \leq 1 \end{array}$$

$$\begin{array}{l} \text{structure fct.: } F_1(x, Q^2) = W_1(\nu, Q^2) \\ F_2(x, Q^2) = \nu W_2(\nu, Q^2) \end{array}$$

Cross section in high-energy limit:

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2}{Q^4} s_{e\mathcal{N}} \left[(1-y)F_2(x, Q^2) + y^2 x F_1(x, Q^2) \right]$$

interpretation of structure functions:

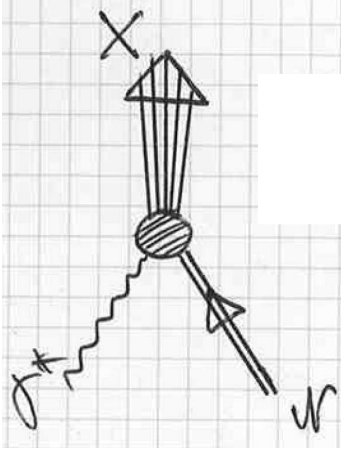
essence of $e\mathcal{N} \rightarrow e' + \text{evth.}$ is $\gamma^* + \mathcal{N} \rightarrow \text{evth.}$ (tot. absorp. cxn of γ^*)

wave fct. of virt. space-like γ 's: $q_\mu = \left(\frac{\nu}{M}; 0, 0, \sqrt{Q^2 + \frac{\nu^2}{M^2}} \right)$ in lab. frame

→ transv. pol.: $\epsilon_\mu(\pm) = \frac{1}{\sqrt{2}}(0; 1, \pm i, 0)$

long. pol.: $\epsilon_\mu(L) = \frac{1}{\sqrt{Q^2}} \left(\sqrt{Q^2 + \frac{\nu^2}{M^2}}; 0, 0, \frac{\nu}{M} \right)$

norm.: $\epsilon_i \epsilon_j^* = \pm \delta_{ij} \quad \epsilon_i q = 0 \quad \epsilon_\pm^* \epsilon_\pm = -1 \quad \epsilon_L^2 = +1$



cxn $\gamma^* + \mathcal{N} \rightarrow \text{everything}$

$$\sigma(\gamma^* \mathcal{N}) \propto \sum_X \epsilon^{*\mu} \langle \mathcal{N} | j_\mu | X \rangle \langle X | j_\nu | \mathcal{N} \rangle \epsilon^\nu (2\pi)^4 \delta_4(p + q - p_X)$$

$$\propto \epsilon^{*\mu} W_{\mu\nu} \epsilon^\nu$$

transv. cxn: $\sigma_\pm = \epsilon_\pm^{*\mu} W_{\mu\nu} \epsilon_\pm^\nu = W_1 = F_1 \geq 0$

$$[\mathcal{P}_{elm} : \sigma_+ = \sigma_- = \frac{1}{2} \sigma_T]$$

long. cxn: $\sigma_L = \epsilon_L^{*\mu} W_{\mu\nu} \epsilon_L^\nu = -W_1 + \left(\frac{\nu^2}{Q^2} + M^2 \right) W_2 \geq 0$

$$\xrightarrow{(Q^2 \gg M^2)} -F_1 + \frac{1}{2x} F_2$$

R ratio:

$$R = \frac{\sigma_L}{\sigma_T} \quad R = \left(\frac{\nu^2}{Q^2} + M^2 \right) \frac{W_2}{W_1} - 1$$

$$\rightarrow \frac{F_2 - 2xF_1}{2xF_1}$$

Experimental results:

1.) Bjorken scaling:

$$\text{Bjorken limit: } \left. \begin{array}{l} Q^2 \text{ large} \\ x \text{ fixed} \end{array} \right\} \left\{ \begin{array}{l} \nu W_2(\nu, Q^2) = F_2(x, Q^2) \rightsquigarrow F_2(x) \\ W_1(\nu, Q^2) = F_1(x, Q^2) \xrightarrow{Bj} F_1(x) \end{array} \right.$$

scaling most pronounced for $x \sim 0.25$

$$\left\{ \begin{array}{l} x \lesssim 0.25 : F_2(x, Q^2) \text{ slightly increasing with } Q^2 \\ x \gtrsim 0.25 : F_2(x, Q^2) \text{ slightly decreasing with } Q^2 \end{array} \right.$$

small log. violation of scaling predicted by QCD

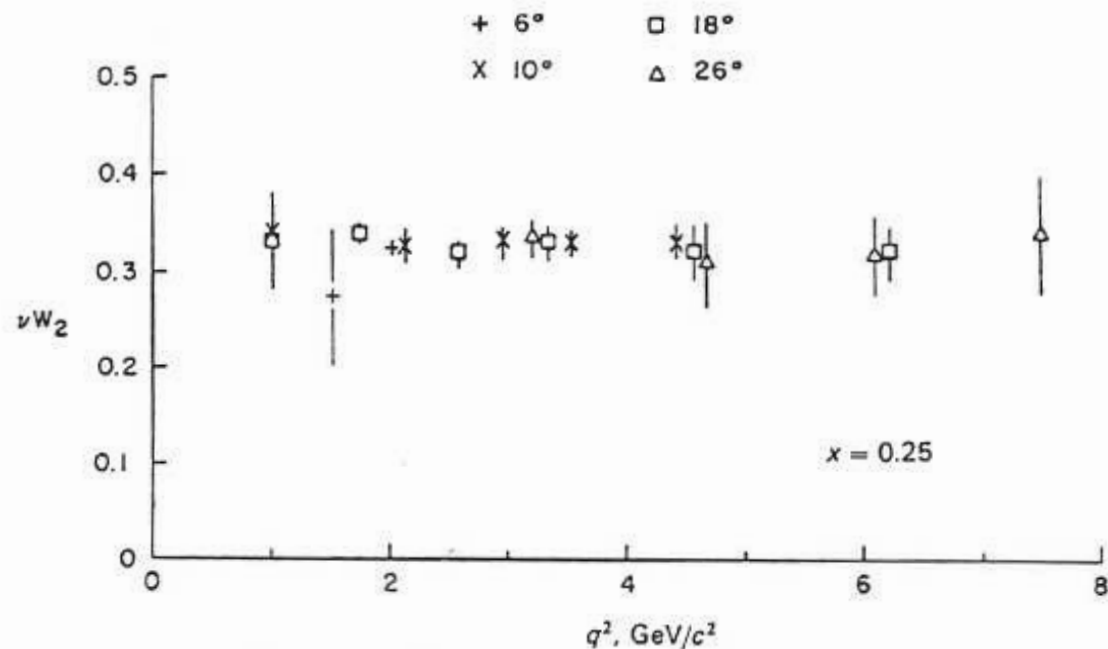
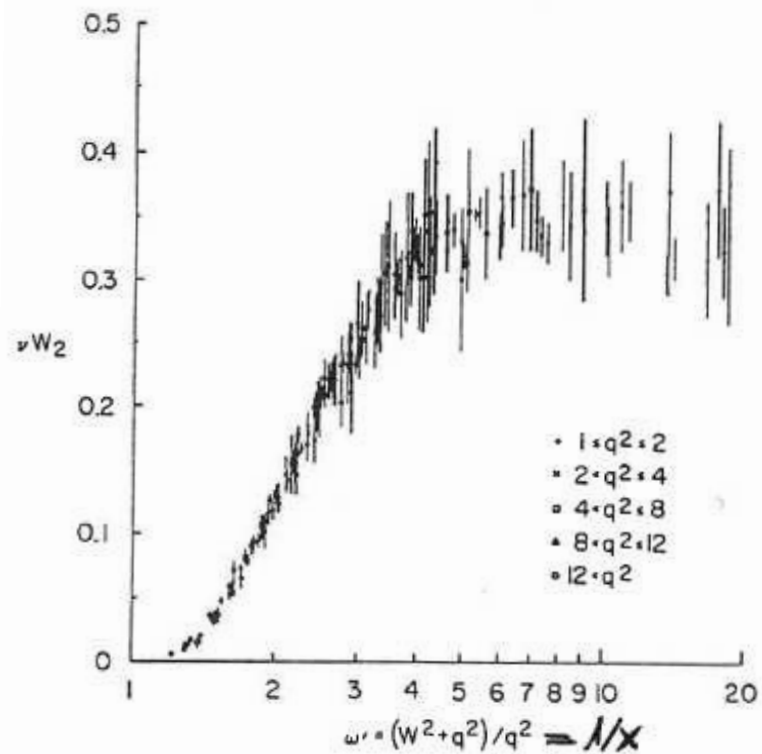
$$2.) \text{ R ratio: } R(x, Q^2) \underset{Bj}{=} \frac{F_2(x) - 2xF_1(x)}{2xF_1(x)}$$

for large Q^2 : $R \rightarrow 0$, i.e. long. abs. cxn vanishes:

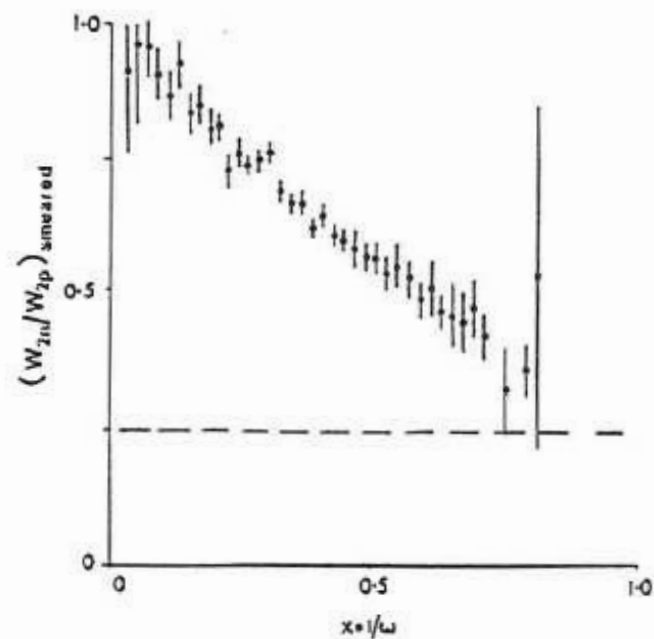
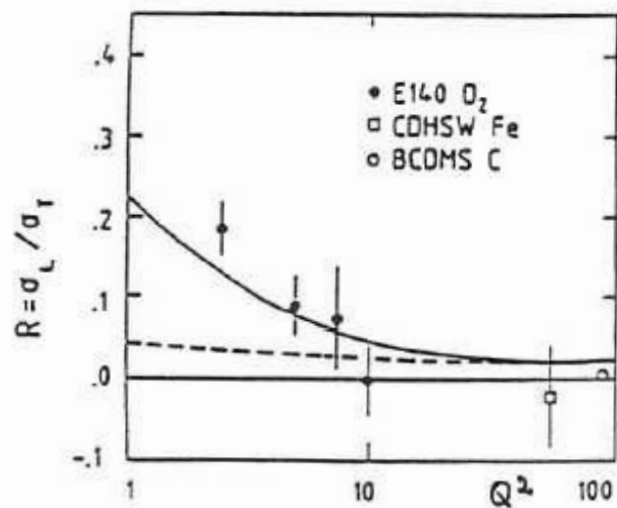
$$\boxed{\text{Callan-Gross relation: } F_2 = 2xF_1}$$

3.) neutron/proton ratio:

$F_2^N(x)/F_2^P(x)$ dec. from value 1 at $x = 0$ to $\gtrsim \frac{1}{4}$ for $x = 1$.



Neutron / Proton Verhältnis:



Classical quark-parton model

basis:

$e + \text{pt-like} \rightarrow e + \text{pt-like}$	$e\mathcal{N} \rightarrow e\mathcal{N}$	$e\mathcal{N} \rightarrow e + \text{evth.}$
$\frac{d\sigma^{pt}}{dQ^2} \sim \frac{1}{Q^4}$	$\frac{d\sigma^{el}}{dQ^2} \sim \frac{1}{Q^4} F(Q^2) ^2$ $\sim \frac{d\sigma^{pt}}{dQ^2} \left(\frac{M^4}{Q^4} \right)^2$	$\frac{d\sigma}{dQ^2} \sim \frac{1}{Q^4} F_2(x)$ $\sim \frac{d\sigma^{pt}}{dQ^2}$

scaling $F_2(x, Q^2) \approx F_2(x) \Rightarrow$ for $Q^2 \rightarrow \infty$ the inclusive cxn behaves analogous to point-like cxn [Q^2 decrease slower by 8 orders than elastic nucleon cxn]

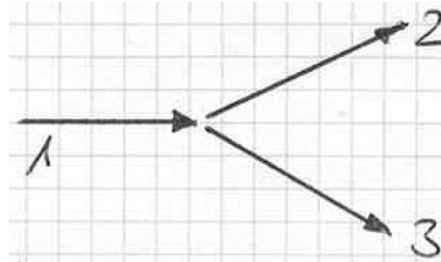
§5. Parton model of deep inelastic lepton-nucleon scattering

In quark picture at high resolution the reaction is built up by superposition of scattering processes off quark constituents: modeling in parton model.

REMARKS

probabilistic picture:

“splitting” of a particle 1 into two constituents 2 and 3 for large P



$$p_1 = \left(P + \frac{m_1^2}{2P}; 0_\perp, P \right)$$

$$p_2 = \left(|x|P + \frac{m_2^2 + k_\perp^2}{2|x|P}; k_\perp, xP \right)$$

$$p_3 = \left(|1-x|P + \frac{m_3^2 + k_\perp^2}{2|1-x|P}; -k_\perp, (1-x)P \right)$$

3-momentum conservation
energy jump

Solution of Born series before introduction of time ordering

$$S_{fi} = \lim_{t \rightarrow +\infty} \langle f | U(t, -\infty) | i \rangle = \lim_{t \rightarrow +\infty} U(t, -\infty)_{fi}$$

$$\begin{aligned}
U(t, -\infty)_{fi} &= \delta_{fi} + (-i) \int_{-\infty}^t dt_1 V_{fi}(t_1) \\
&\quad + (-i)^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \sum_n V_{fn}(t_1) V_{ni}(t_2) + \dots \\
&= \delta_{fi} + (-i) \int_{-\infty}^t dt_1 e^{i(E_f - E_i)t_1} V_{fi} \\
&\quad + (-i)^2 \int_{-\infty}^t dt_1 \int_{-\infty}^{t_1} dt_2 \sum_n e^{i(E_f - E_n)t_1 + i(E_n - E_i)t_2} V_{fn} V_{ni} + \dots \\
&= \delta_{fi} + e^{i(E_f - E_i)t} \left\{ \frac{V_{fi}}{E_i - E_f + i\epsilon} \right. \\
&\quad \left. + \sum_n \frac{V_{fn}}{E_i - E_f + i\epsilon} \frac{V_{ni}}{E_i - E_n + i\epsilon} + \dots \right\}
\end{aligned}$$

$$\lim_{t \rightarrow \infty} \frac{e^{i(E_f - E_i)t}}{E_i - E_f + i\epsilon} = \frac{1}{i} \lim_{t \rightarrow \infty} \int_{-\infty}^t dt_1 e^{i(E_f - E_i)t_1} = -2\pi i \delta(E_f - E_i)$$

$$S_{fi} = \delta_{fi} - 2\pi i \delta(E_f - E_i) \left\{ V_{fi} + \sum_n \frac{V_{fn} V_{ni}}{E_i - E_n + i\epsilon} + \dots \right\}$$

“old-fashioned perturbation theory”

$$\Delta E = E_1 - (E_2 + E_3) = P(1 - |x| - |1 - x|) \text{ for } x < 0 \text{ and } x > 1: \Delta E \sim P$$

$$= \frac{1}{2P} \left[m_1^2 - \frac{m_2^2 + k_\perp^2}{x} - \frac{m_3^2 + k_\perp^2}{1-x} \right] \text{ for } 0 < x < 1: \Delta E \sim P^{-1}$$

leading

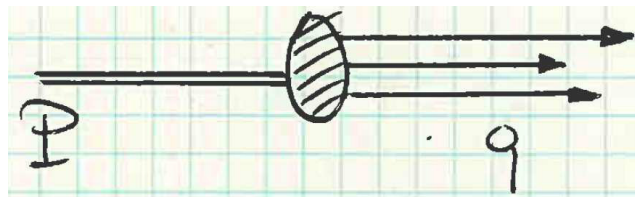
$$\Rightarrow \text{lifetime } \tau_L \sim \frac{1}{\Delta E} \sim \frac{P}{\langle k_\perp^2 \rangle} \text{ very long in } eP\text{-c.m.s.}$$

$[x < 0, x > 1$: one of the daughter particles moves backward]

$\Rightarrow$ For fast moving particles the splitting dominates that makes the daughter particles adopt an energy/momentum fraction x with $0 < x < 1$ parallel to the mother particle.

Quark-Parton Model:

- (i) In fast moving coordinate systems a nucleon can be split into interaction-free parallel partons that scatter leptons incoherently.
- (ii) Partons can be identified with point-like quarks.

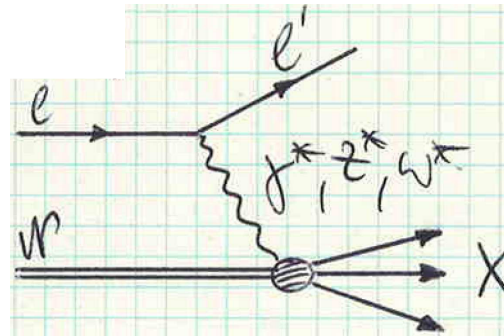


Feynman

Bjorken, Pachos

Deep Inelastic Lepton-Nucleon Scattering

Lorentz invariance
point-like lepton current
spin 1 exchange $\Rightarrow$



$$\frac{d\sigma^{elm}}{dxdy} = \frac{4\pi\alpha^2}{Q^4} s \left\{ (1-y)F_2^{elm}(x, Q^2) + y^2 x F_1^{elm}(x, Q^2) \right\}$$

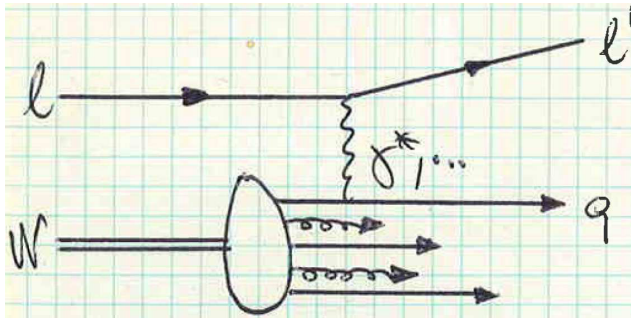
$$\frac{d\sigma_{cc}^{\nu/\bar{\nu}}}{dxdy} = \frac{G_F^2 s}{2\pi} \left\{ (1-y)F_2^{\nu/\bar{\nu}}(x, Q^2) + y^2 x F_1^{\nu/\bar{\nu}}(x, Q^2) \right. \\ \left. \pm \frac{1 - (1-y)^2}{2} x F_3^{\nu/\bar{\nu}}(x, Q^2) \right\}$$

$F_i = F_i(x, Q^2)$ elm. and weak structure functions

transverse: $F_T = F_1$

longitudinal: $F_L = F_2 - 2xF_1$ $R = \frac{F_L}{2xF_T}$

Quark-Parton Picture:



interaction time:

$$\left. \begin{aligned} \nu = pq = P(q^0 + q^3) &= \frac{Q^2}{2x} \\ Q^2 &= -(q^0)^2 + (q^3)^2 \end{aligned} \right\}$$

$$\Delta E^* = -q^0 = xP - \frac{Q^2}{4xP} \sim xP \quad (x > 0)$$

$$\left. \begin{aligned} \tau_{int} &\sim \frac{1}{xP} \\ \tau_L &\sim \frac{P}{\langle k_{\perp}^2 \rangle} \end{aligned} \right\} \quad \boxed{\tau_{int} \ll \tau_L} \rightarrow \underline{\underline{\text{quarks real in short distance processes}}}$$

quark-parton cross sections:

$$\boxed{\nu_{\mu} d \rightarrow \mu^{-} u}$$

$$\frac{d\sigma^q}{dy} = \frac{G_F^2 s_*}{\pi}$$

$$\nu_{\mu} d \rightarrow \mu^{-} u \quad LL$$

$$\bar{\nu}_{\mu} \bar{d} \rightarrow \mu^{+} \bar{u} \quad RR$$

$$y = \frac{pq}{pk} \underset{IMF}{=} \frac{p_q q}{p_q k} \underset{CM}{=} \frac{\epsilon_*^2 (1 - \cos \theta_*)}{2\epsilon_*^2} = \frac{1}{2} (1 - \cos \theta_*)$$

$$\boxed{\bar{\nu}_{\mu} u \rightarrow \mu^{+} d}$$

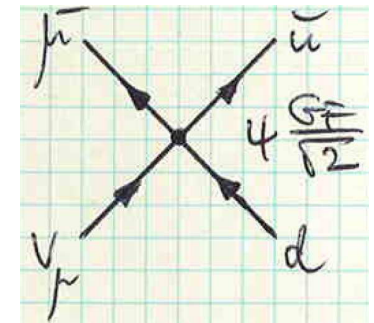
$$\frac{d\sigma^q}{dy} = \frac{G_F^2 s_*}{\pi} (1 - y)^2$$

no backw. scatt.: RL, LR ($\nu \bar{q}$)

$$\boxed{eq \rightarrow eq}$$

both helicities
incoherent

$$\frac{d\sigma^q}{dy} = \frac{2\pi\alpha^2 e_q^2}{Q^4} s_* [1 + (1 - y)^2]$$



Composition:

IMF: $\text{CM}(\ell, \mathcal{N})$

Breit-frame: $q = (0; 0, 0, q)$

etc.

$f_q(\xi)d\xi = \#$ of quarks q in mom. interval $d\xi$ around ξ : $p_q = \xi P$

$$\frac{d\sigma}{dxdy} = \sum_q \int_0^1 d\xi f_q(\xi) \frac{d\sigma^q(s_* = \xi s)}{dy} \delta_1 \left(x - \frac{Q^2}{2\nu} \right)$$

$$\frac{Q^2}{2\nu} = \xi \frac{Q^2}{2\nu_q} = \xi$$

↑ elasticity condition: $(p_q + q)^2 = p_{q'}^2$

$$-Q^2 + 2qp_q = 0 \Rightarrow \frac{Q^2}{2\nu_q} = 1$$

$\frac{d\sigma}{dxdy} = \sum_q f_q(x) \frac{d\sigma^q(s_* = xs)}{dy}$	Bjorken-variable determines the relative momentum of the scattered quark: $\xi = x$
---	---

$$\nu : \frac{d\sigma}{dxdy} = \frac{G_F^2 s}{\pi} \{ x d(x) + (1-y)^2 x \bar{u}(x) \}$$

$$\bar{\nu} : \frac{d\sigma}{dxdy} = \frac{G_F^2 s}{\pi} \{ (1-y)^2 x u(x) + x \bar{d}(x) \}$$

$$e : \frac{d\sigma}{dxdy} = \frac{2\pi\alpha^2 s}{Q^4} \sum_q e_q^2 x f_q(x) [1 + (1-y)^2]$$

Analysis:

$F_i(x, Q^2)$ independent of Q^2 : scaling

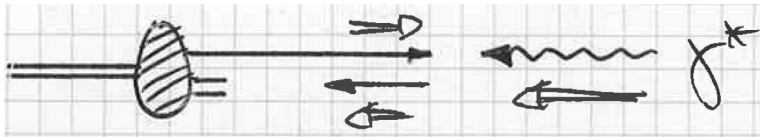
$F_2 = 2xF_1$ Callan–Gross relation

$$F_2^{elm} = \sum_q e_q^2 x f_q(x) \quad F_2^\nu = 2x(d + \bar{u}) \quad xF_3^\nu = +2x(d - \bar{u})$$

$$F_2^{\bar{\nu}} = 2x(u + \bar{d}) \quad xF_3^{\bar{\nu}} = -2x(u - \bar{d})$$

Physical interpretation:

1.) Callan–Gross relation measures quark spin = $\frac{1}{2}$

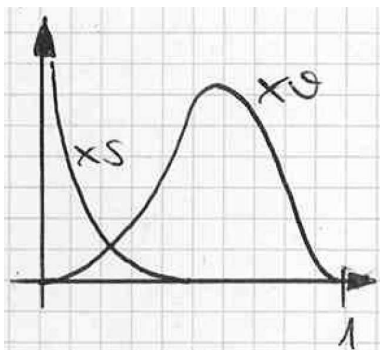


Breit frame $q = (0; 0, 0, q)$: current $\bar{q}\gamma_\mu q$ chirally conserved, massless parallel quarks

$$\Delta S_z = 1 \Rightarrow S_z(\gamma^*) = 1, \neq 0 \Rightarrow \sigma_T \neq 0, \sigma_L = 0 \Rightarrow R = \frac{\sigma_L}{\sigma_T} = 0$$

[spinless partons: $\sigma_T = 0, \sigma_L \neq 0 \Rightarrow R = \infty$]

2.) valence quarks: $f = v + s$ $v(x)$ valence distribution



$$\int_0^1 dx v_d = 1$$

$$\int_0^1 dx v_u = 2$$

$s(x)$ div. sea quarks from quantum fluct.

fractionized electric quark charge:

nuclear target
valence region

$$F_2^{elm} \approx x \left[\frac{4u+d}{9} \frac{1}{2} + \frac{1d+u}{9} \frac{1}{2} \right] = \frac{5}{18} x(u+d)$$

$$\mathcal{N} = \frac{1}{2}(P+N) \quad F_2^\nu \approx 2x \frac{d+u}{2} = x(u+d)$$

$$\boxed{F_2^{elm} \approx \frac{5}{18} F_2^\nu}$$

3.) 3 quarks in nucleon

nuclear target
val. dominance

$$\sigma_\nu \approx \frac{G_F^2 s}{\pi} \int_0^1 dx \, x \frac{u+d}{2} \quad \sigma_{\bar{\nu}} \approx \frac{1}{3} \sigma_\nu$$

$$\sigma_{\bar{\nu}} \approx \frac{1}{3} \frac{G_F^2 s}{\pi} \int_0^1 dx \, x \frac{u+d}{2} \quad \uparrow \text{ spin: quarks} \\ \text{no antiquarks}$$

sum rules: baryon number

$$1 = \int_0^1 dx \, \frac{1}{3} [(u - \bar{u}) + (d - \bar{d}) + (s - \bar{s})]$$

(“exact”)

isospin

$$\pm \frac{1}{2} = \int_0^1 dx \left[\frac{1}{2}(u - \bar{u}) - \frac{1}{2}(d - \bar{d}) \right]$$

strangeness

$$0 = \int_0^1 dx \, (s - \bar{s})$$

solution proton: $\int_0^1 dx (u - \bar{u}) = 2 \int_0^1 dx (d - \bar{d}) = 1 \int_0^1 dx (s - \bar{s}) = 0$

nuclear target: $\int_0^1 dx F_3^\nu = \int_0^1 dx [d + u - \bar{u} - \bar{d}] = \int_0^1 dx [(u - \bar{u}) + (d - \bar{d})]$

Gross-Llewellyn-Smith: $\int_0^1 dx F_3^\nu = 3$

4.) momentum sum rule: $1 = \sum_{q, \bar{q}} \int_0^1 d\xi \xi f_q(\xi) + \int_0^1 d\xi \xi f_g(\xi)$
 flavor-neutral matter: $\uparrow$
 binding energy

measurement: $\int_0^1 d\xi \xi f_g(\xi) \approx \frac{1}{2}$

50% of the nucleon energy in fast moving particles is carried by
flavor-neutral binding energy: GLUONS

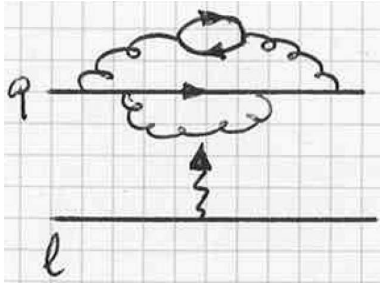
§6. Scaling Violation: Altarelli–Parisi Equations (DGLAP)

idea: parton-quarks are surrounded by a gluon cloud inside nucleon;

at sufficiently large Q^2 more and more quantum fluctuations are resolved

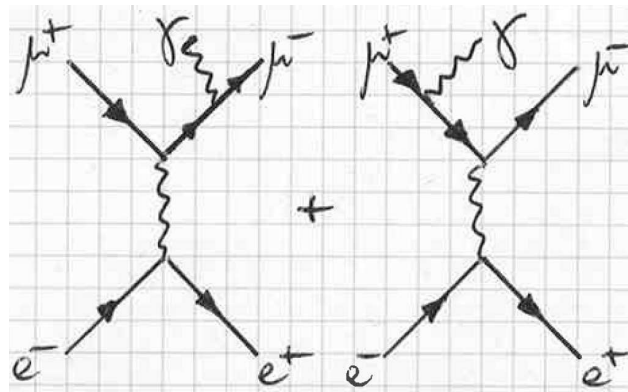
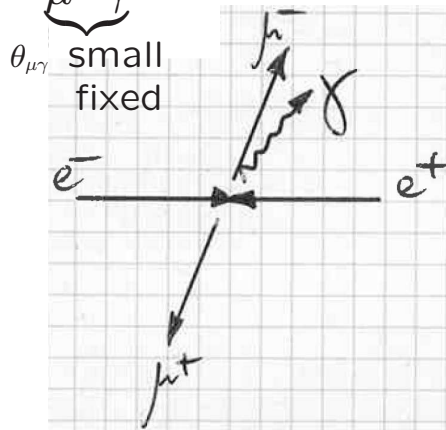
$\Rightarrow$ momentum spectra of quarks and gluons vary with Q^{-1} :

microscopic parton distributions are Q^2 -dependent.



splitting probability:

$$e^+e^- \rightarrow \mu^+ \underbrace{\mu^- \gamma}_{\theta_{\mu\gamma} \text{ small fixed}}$$



$$x_{1,2} = \frac{E_{\pm}}{E} \quad z = \frac{E_{\gamma}}{E} \quad x_{\perp} = \frac{2}{x_1} \sqrt{(1-x_1)(1-x_2)(1-z)} = \frac{p_{\perp}}{E}$$

$$\log x_{\perp}^2 \approx \log(1-x_1)$$

$$d \log p_{\perp}^2 \approx \frac{dx_1}{1-x_1}$$

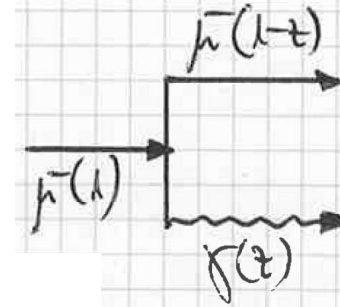
$$x_1 + x_2 + z = 2$$

$$\frac{1}{\sigma_0} \frac{d^2 \sigma}{dx_1 dx_2} = \frac{\alpha}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Fragmentation: $x_2 \approx 1 - z$

$$d\sigma = \sigma_0 \int^{Q^2} \frac{dp_{\perp}^2}{p_{\perp}^2} \frac{\alpha}{2\pi} \frac{1 + (1 - z)^2}{z} dz$$

cxn = μ -pair cxn * particle flux ($\mu \rightarrow \mu\gamma$)



increase of particle flux at $Q^2 \rightarrow Q^2 + \delta Q^2$:

$$\frac{\delta N(\mu \rightarrow \mu\gamma)}{\delta \log Q^2} = \frac{\alpha}{2\pi} \frac{1 + (1 - z)^2}{z} dz$$

quark fragmentation: color average/sum

$$\sum_{k,a} T_{ik}^a T_{kj}^a = \frac{4}{3} \delta_{ij}$$

