pQCD, kinetic theory, hydrodynamics and hadronic collisions

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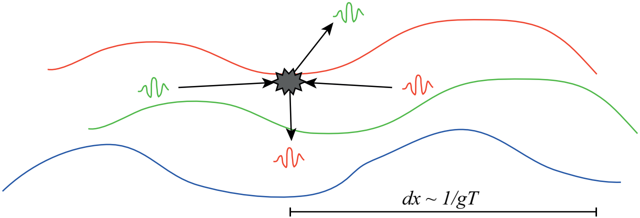
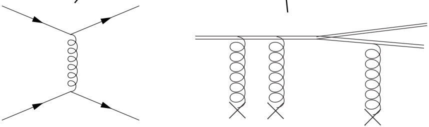
Kinetic theory of hard modes

Arnold, Moore, Yaffe JHEP 0301 (2003) 030



$$\frac{df}{dt} = -C_{2\leftrightarrow 2}[f] - C_{1\leftrightarrow 2}[f]$$

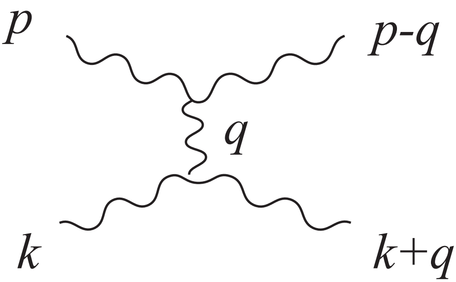


- pQCD kinetic theory for the hard modes

Naive guess:

Naive guess: sum over elementary ($2 \leftrightarrow 2$) tree-level processes

$$C_a^{2\leftrightarrow 2}[f](p) = \frac{1}{2}\frac{1}{2p}\frac{1}{\nu_a}\sum_{bcd}\int_{\mathbf{kp'k'}} |M^{ab}_{cd}|^2 (2\pi)^4 \delta^{(4)}(P+K-P'-K')$$
$$\times \left\{ f_p f_k (1 \pm f'_p)(1 \pm f_{k'}) - f'_p f'_k (1 \pm f_p)(1 \pm f_k) \right\}$$



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$$\times \left\{ f_p f_k (1 \pm f_p')(1 \pm f_{k'}) - f_p' f_k' (1 \pm f_p)(1 \pm f_k) \right\}$$

- Sum over all processes and kinematics
- Matrix elements (in non-relativistic normalization) for the different processes

$$M_{gg}^{gg} = \frac{\alpha^2 \left(3 - \frac{su}{t^2} - \frac{st}{u^2} - \frac{tu}{s^2}\right)}{(2p^0)(2k^0)(2p'^0)(2k'^0)}$$

- Energy and momentum conserving δ-function
- Gain and loss terms

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This is fine :

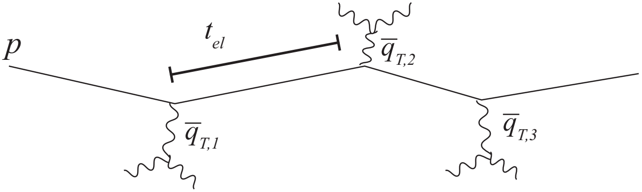
- if all the lines are unaffected by the medium

  Screening of the internal and external lines
- No other processes induced my the medium

  splitting
- All processes happen fast enough that they can be separated

  LPM-suppression

Momentum diffusion coefficient $\hat{q}$



- Random walk in mom. space

$$\mathbf{Q}_\perp = \mathbf{q}_\perp^1 + \mathbf{q}_\perp^2 + \dots$$

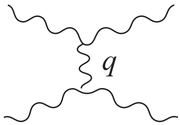
- Incoherent angles add in quadrature:

$$|Q_\perp| \propto \sqrt{\text{number of collisions}} \propto \sqrt{t}$$

- Momentum diffusion coefficient

$$\frac{|Q_\perp|^2}{t} \equiv \hat{q}_\perp \sim \frac{\Delta q_\perp^2}{t_{\mathrm{el}}(q_\perp)}$$

Naive computation of $\hat{q}$



$$\frac{d\Gamma_{el}}{d^2q_\perp} \sim \int dq_z \int d^3k f_k (1 \pm f_{k+q}) \frac{d\sigma}{d^3q} \tag{1}$$

$$\hat{q} \sim \int d^2q_\perp \frac{d\Gamma_{el}}{d^2q_\perp} q_\perp^2 \tag{2}$$

$\Gamma_{\rm el} \equiv \frac{1}{t_{el}}$

For soft scattering $q \ll k, p$

$$\hat{q} \sim \underbrace{\int d^3k f_k (1 + f_k)}_{T_* m^2/\alpha} \times \int d^2q_\perp \frac{d\sigma}{d^2q_\perp} q_\perp^2 \tag{3}$$

The soft scattering is dominated by t-channel coulomb scattering:

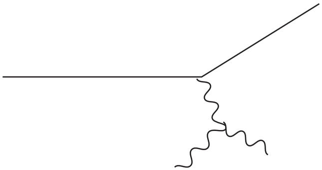
$$\frac{d\sigma}{d^2q_\perp} \sim \frac{\alpha^2}{(q_\perp^2)^2}$$

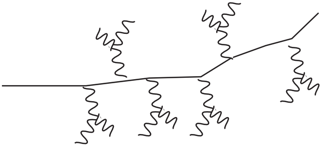
The momentum broadening coefficient has a IR log divergence $\int \frac{dq_\perp}{q_\perp}$!

UV taken care by relaxing the soft assumption

Soft log-digergence

$$\hat{q} \sim \int d^2 q_\perp \frac{\alpha^2}{(q_\perp^2)^2} q_\perp^2$$





$$q_\perp \sim T$$

$$t_{el} \sim \frac{1}{g^4 T}$$

$$\hat{q} \sim g^4 T^3$$

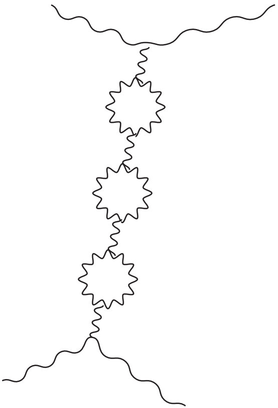
$$q_\perp \sim gT$$

$$t_{el} \sim \frac{1}{g^2 T}$$

$$\hat{q} \sim g^4 T^3$$

- IR divergence is cured by the physics of screening

Effective matrix element



- Interaction of the soft mode non-perturbative with hard modes needs to be resummed

$$\frac{g_{\mu\nu}}{Q^2} \Rightarrow \left[Q^2 + G_J(\omega, k)\right]^{-1}_{\mu\nu}$$

- Because we get a leading order contribution only from the soft region with $q \ll p$, can approximate the full G_J function with hard-loop approximation G_J^{HL}
- Infrared divergence then cancelled by the screening scale

$$\int_m \frac{d^2 q_\perp}{(q_\perp^2 + m^2)^2} q_\perp^2 \sim \log\left(\frac{T}{m}\right)$$

Note, in the following I neglect plasma instabilities

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  splitting

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  LPM-suppression

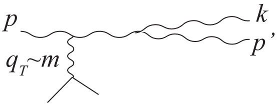
Induced splitting/merging

Screening makes the total scattering rate finite but it is still very fast:

technically speaking, has a log-divergence...

$$\Gamma_{el} \sim \frac{1}{t_{el}} \sim \alpha T_* m^2 \int \frac{d^2 q_\perp}{(q_\perp^2 + m^2)^2} \sim \alpha \frac{T_* m^2}{m^2} \sim \alpha T_*$$

Large angle scattering rate $\alpha^2 T_*$



- Massless onshell particles don't have phase space to decay
- But even a very soft scattering enough to induce a splitting

Induced splitting/merging rate as big as large angle scattering rate:

The rate at which a given particle emits at given k

$$\frac{d\Gamma^{BH}_{\rm split}}{d\log k} \sim \alpha \Gamma_{el} \sim \alpha^2 T$$

BH = Bethe-Heitler

Induced splitting/merging

- For a leading order description, need to include an effective "$1 \leftrightarrow 2$" collision kernel

$$C^{\text{“}1\,\leftrightarrow\,2\text{”}}[f] \sim \int dk \frac{d\Gamma^{\rm BH}_{split}}{dk} \Big\{ f(p)[1{+}f(p{-}k)][1{+}f(k)] \\ - f(p{-}k)f(k)[1{+}f(p)] \Big\},$$

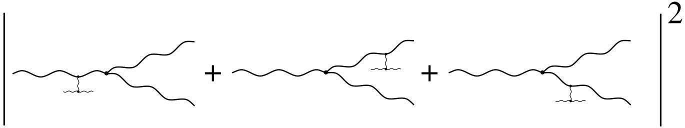
  - The soft scattering that induced the splitting is hidden in Γ^{BH}_{split}

Induced splitting/merging

The effective splitting element: $\gamma = p^2 \frac{d\Gamma^{BH}_{split}}{dp'}$

$$\gamma_{BH}(p; p', p-p') = p^2 \frac{\mathcal{Q}(m_\infty^2/m_D^2)}{4(2\pi)^4} N_c g^2 T_* \frac{p^2}{p'}$$

$$\mathcal{Q}(m_\infty^2/m_D^2) \equiv 8 \int_{p_\perp, q_\perp} \left[\frac{1}{\mathbf{q}_\perp^2} - \frac{1}{\mathbf{q}_\perp^2 + m_D^2}\right] \left(\frac{\mathbf{p}_\perp}{m_\infty^2 + \mathbf{p}_\perp^2} - \frac{\mathbf{p}_\perp - \mathbf{q}_\perp}{m_\infty^2 + (\mathbf{p}_\perp - \mathbf{q}_\perp)^2}\right)^2,$$



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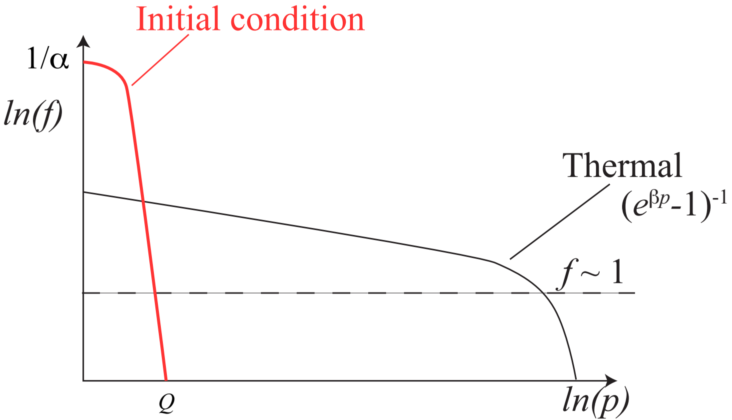
Overoccupied gluons

Overoccupied cascade

What happens if you have too many soft gluons = too strong color fields, $f \sim 1/\alpha$.



Overoccupied cascade

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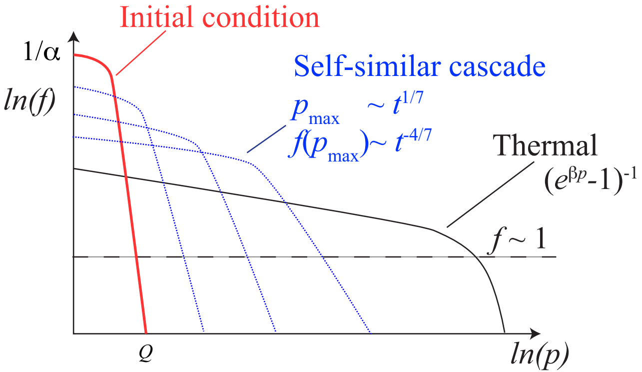
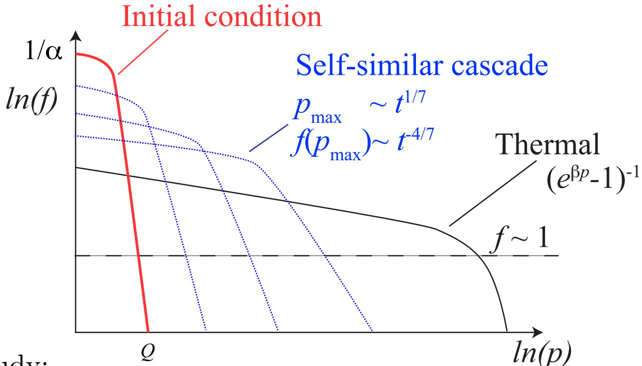


- Cascade because maximal momentum transfer is p_{max}
- Self-similar because initial scattering rate is larger than final

$$\Gamma_{scat}^{init} \gg \Gamma_{scat}^{thermal}$$

Overoccupied cascade



How to study:

- if $f \sim \frac{1}{\alpha}$, no scale separations: ultrasoft∼soft∼hard, but classical fields.
- When $\frac{1}{\alpha} \ll f \ll 1$, scale separations emerge and KT is fine, still classical
- When $f \sim 1$ quantum effects emerge, no more classical field description

Overoccupied cascade:

Elastic and inelastic scattering will transport energy to UV: $p_{max}^2 \sim \hat{q}t$

Interactions conserve energy, $\epsilon \sim f p_{\rm max}^4$

Occupancies are reduced, Bose enchancement reduced: $\hat{q} \sim \alpha^2 f^2 p_{max}^3$

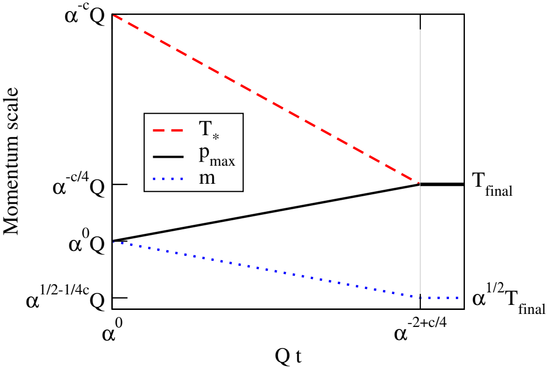
Solution:

$$\begin{aligned} p_{max} &\sim \alpha \epsilon^{2/7} t^{1/7} \\ f &\sim \alpha^{-8/7} \frac{1}{\epsilon^{1/7} t^{4/7}} \end{aligned}$$

The cascade reaches $p_{max} \sim T$ when

$$t \sim \frac{\epsilon^2}{\alpha^2 T^7} \sim \frac{1}{\alpha^2 T}$$

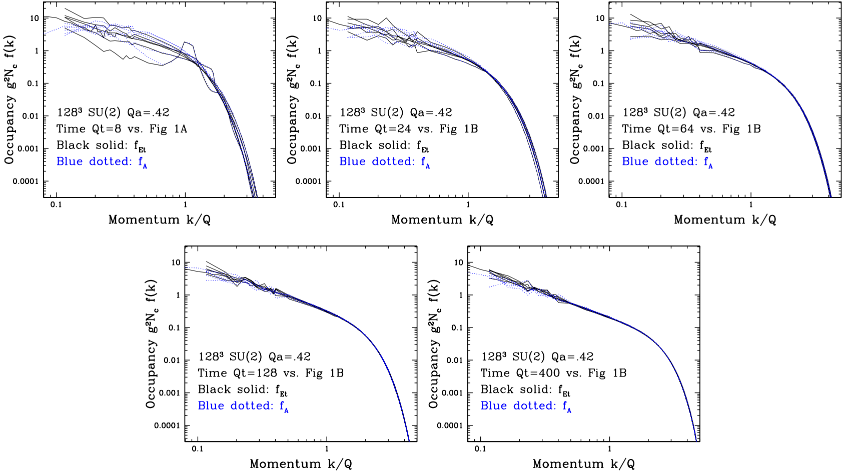
Overoccupied cascade



Overoccupied cascade

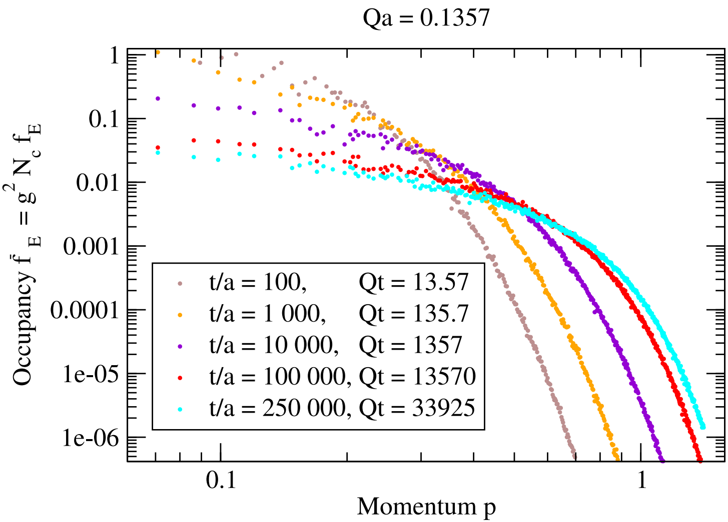
Information on the initial condition lost quickly

AK, Moore 1207.1663



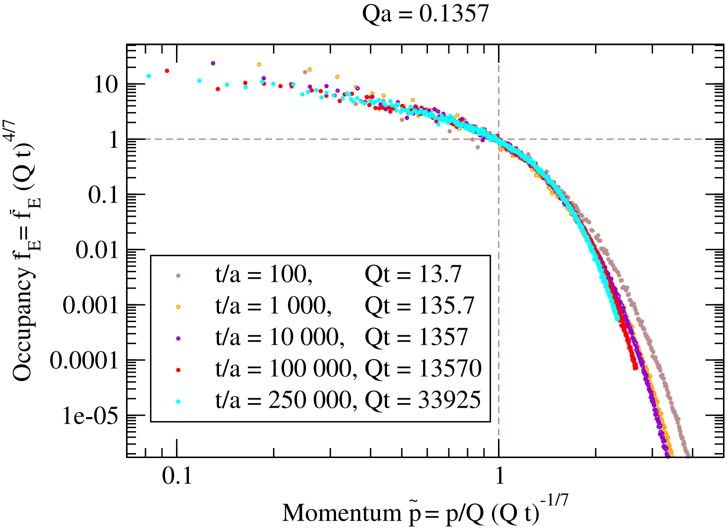
$$\tau_{\text{init}} \sim \left(\frac{Q}{T}\right)^7 \frac{1}{\alpha^2 T} \ll \frac{1}{\alpha^2 T} \sim \tau_{\text{them.}}$$

Overoccupied cascade

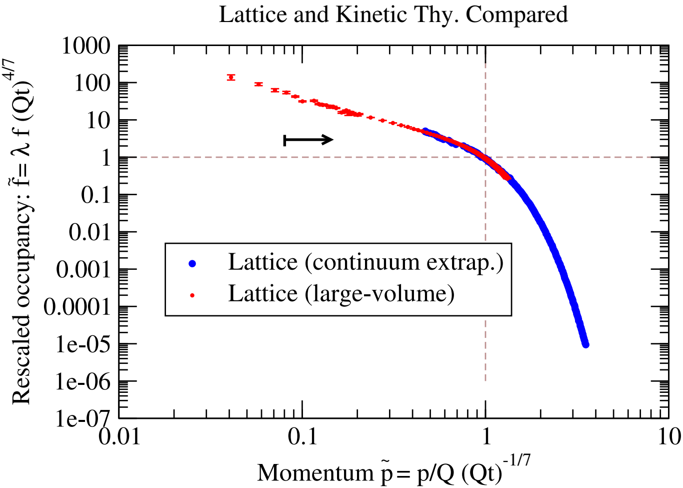


$$t \sim \tau_{\mathrm{pmax}} \sim \left(\frac{p_{\max}}{T}\right)^7 \frac{1}{\alpha^2 T} \Rightarrow p_{\max} \propto (Qt)^{1/7}$$

Overoccupied cascade



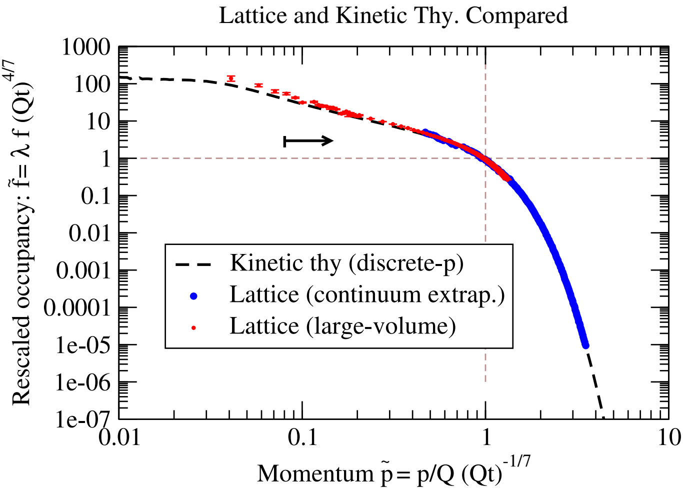
Overoccupied cascade



Form of self-similar cascade from classical lattice simulation,

$$1 \ll f \lesssim 1/\alpha$$

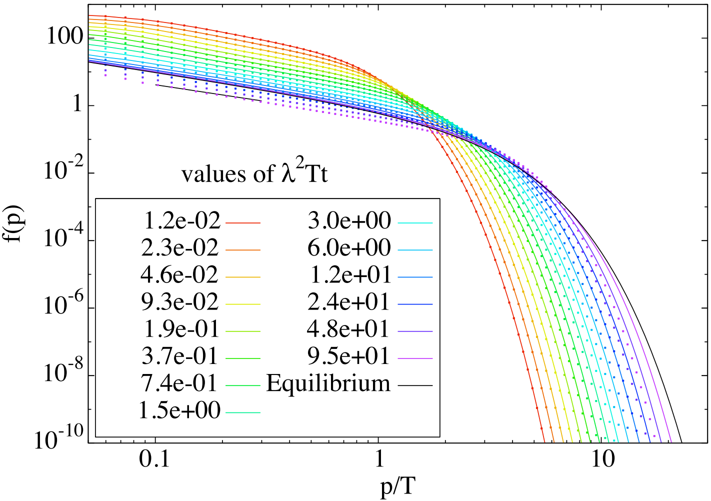
Overoccupied cascade



Same system, very different degrees of freedom

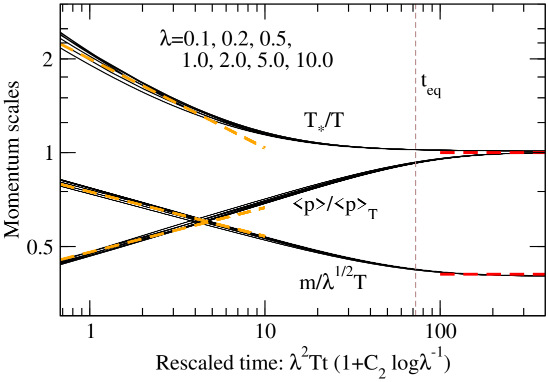
$$1 \lesssim f \ll 1/\alpha$$

Approaching the equilibrium



Thermal equilibrium reached once $f \sim 1$.

Approaching the equilibrium



$$m^2 = \lambda \int_{\mathbf{p}} \frac{f(p)}{p}$$

$$T_* = \frac{\lambda}{2} \int_{\mathbf{p}} f(p)[1 + f(p)]/m^2$$

$$\langle p \rangle = \frac{1}{n} \int_{\mathbf{p}} p\, f(p)$$

Therm. time through the approach of $\langle |p| \rangle - \langle |p| \rangle_T \sim \exp(-t/t_{\mathrm{eq}})$

$$t_{\mathrm{eq}} \approx \frac{72.}{1 + 0.12 \log \lambda^{-1}} \frac{1}{\lambda^2 T}, \qquad \lambda = 4\pi N_c \alpha.$$

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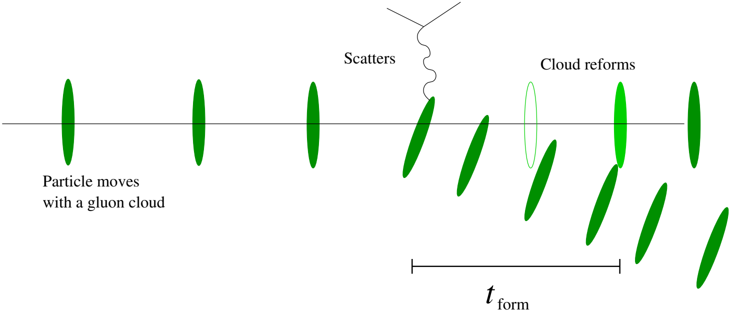
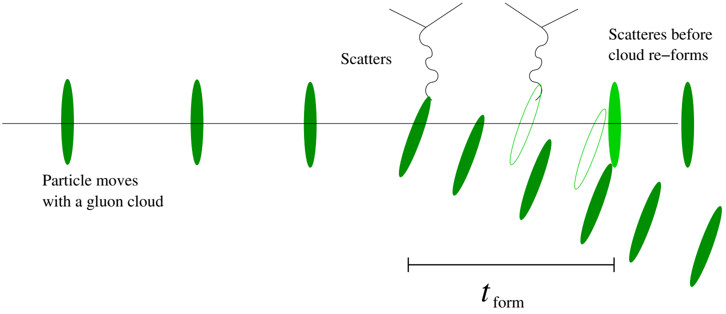
Landau-Pomeranchuck-Migdal suppression



- Scattering kicks a gluon from the virtual cloud
- Cloud reforms when the wave-packets separate

$$t_{\text{form}} \sim \frac{\text{trans. size}}{\text{trans. vel.}} \sim \frac{1/p_\perp}{p_\perp/p} \sim \frac{p}{p_\perp^2}$$

Landau-Pomeranchuck-Migdal suppression



- Frequent or soft scattering: cloud hasn't re-formed Coulomb div.!
- At most one emission per t_{form} $\Rightarrow$Reduced rate:

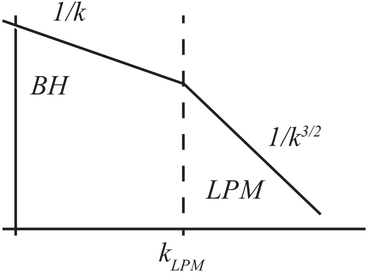
$$p_\perp^2 \equiv \hat{q} t_{\text{form}} \Rightarrow \Gamma_{\text{emit}}(p) \sim g^2 t_{\text{form}}^{-1} \sim g^2 \sqrt{\frac{\hat{q}}{p}}$$

QED: Landau, Pomeranchuk, Migdal 1953. QCD: Baier Dokshitzer Mueller Peigne Schiff hep-ph/9607355

Landau-Pomeranchuck-Migdal suppression

LPM suppression affects those splitting for which formation time is larger than the soft scattering rate: The correct rate is always the smaller one of the two:

$$\frac{d\Gamma_{spit}}{dk} \sim \min\left(\frac{\Gamma_{spit}^{BH}}{dk}, \frac{\Gamma_{spit}^{LPM}}{dk}\right)$$



The transition takes place when

$$k \sim k_{LPM} \sim \frac{m^4}{\hat{q}}$$

Landau-Pomeranchuck-Migdal suppression

Classical rate?

- LPM is quantum mechanical because of the coherence
- Radiation of a charge in classical field

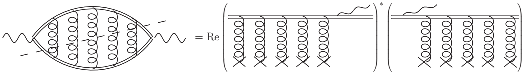
connects to normal classical brehmstrahlung a la Jackson

- BH classical because no coherence between collisions
- From the field point of view, radiation from a choppy quantum field

Effective $C_{1\leftrightarrow 2}$ matrix element revisited

$$\gamma^{p'}_{p,k} \sim \underbrace{\frac{p'^4 + p^4 + k^4}{p'^3 p^3 k^3}}_{\text{DGLAP split-kernel}} \int \frac{d^2 h}{(2\pi)^2} \mathbf{h} \cdot \mathrm{Re}\mathbf{F}(\mathbf{h}; p', p, k)$$

$$2\mathbf{h} = i\delta E(\mathbf{h})\mathbf{F}(\mathbf{h}) + \frac{g^2 N_c}{2} \int \frac{d^2\mathbf{q}}{(2\pi)^2} \left[T_* \left(\frac{1}{\mathbf{q}^2} - \frac{1}{\mathbf{q}^2 + m_D^2} \right) \right]$$
$$\times \left(3\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} - p\mathbf{q}) - \mathbf{F}(\mathbf{h} - k\mathbf{q}) - \mathbf{F}(\mathbf{h} + p\mathbf{q})\right)$$



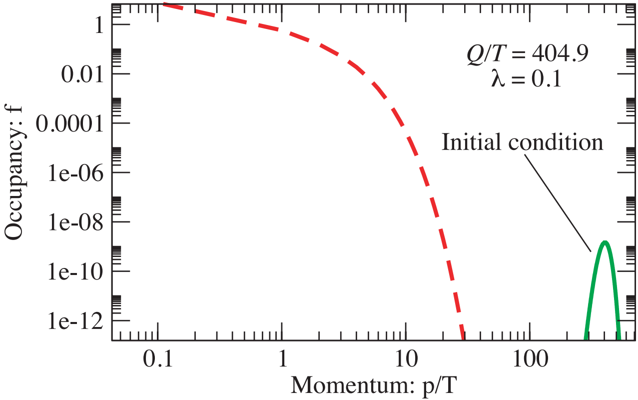
Where sensitivity to the medium comes from

- δE is the difference of energies of one gluon with momentum p' compared the two with k, p': depends on effective masses
- Dependence on $p/T_*, m/T_*$

Under-occupied cascade

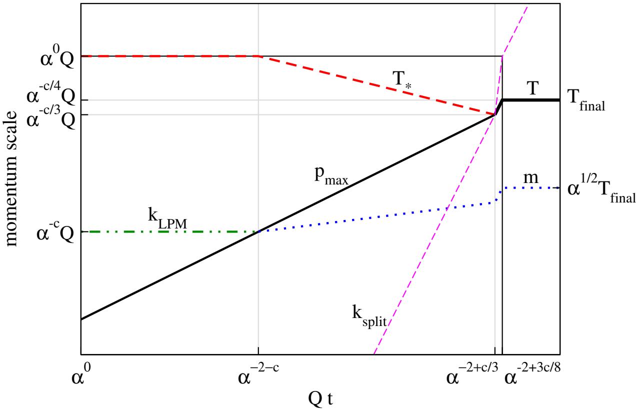
Under-occupied cascade

AK, Lu, PRL 113 (2014) 18, 182301

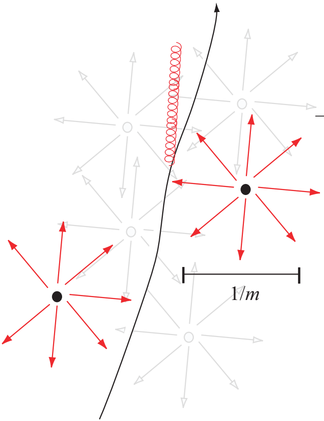


- Start with an underoccupied initial condition $p \sim Q$
- after a very short time, an IR bath is created (1↔ 2–processes)

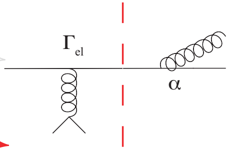
Under-occupied cascade



Underoccupied cascade: Formation of thermal bath



- Soft modes quick to emit



$$\Gamma_{\rm el} \sim \alpha_s^2 \frac{n}{m_D^2} \sim \alpha_s^2 \frac{\int_{\mathbf{p}} f}{\alpha_s \int_{\mathbf{p}} f/p}$$

$$n_{\rm soft} \sim \alpha_s \, \Gamma_{\rm el} \, t$$

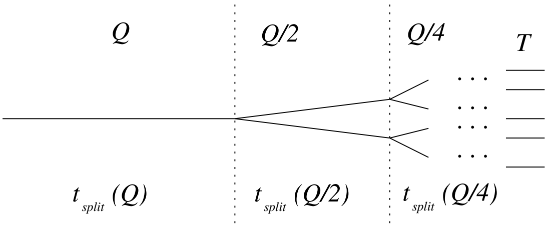
- Low-p: easy to thermalize
- Can dominate the dynamics

  scattering, screening, ...

$\Rightarrow$ Few energetic "jets" propagating in thermal bath

Underoccupied cascade: Radiational breakup

What happens to jet in medium:



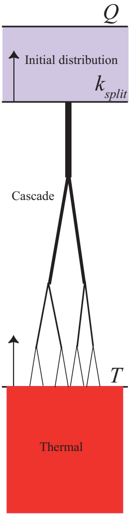
- Successive splittings happen in faster times scales: $t_{split}(p) \sim \frac{1}{\alpha}\sqrt{\frac{p}{\hat{q}}}$

$$t_{\text{split}}(Q) \sim t_{\text{split}}(Q) + t_{\text{split}}(Q/2) + t_{\text{split}}(Q/4) + \ldots \sim t_{\text{split}}(Q)$$

- Once the parton has had time to split once it cascades its energy to IR

Bottom-up thermalization

AK, Moore JHEP 1112 (2011) 044



- Hard particles scatter with soft thermal bath

$$\hat{q} \sim \alpha^2 T^3$$

- Scales below k_{split} have had time to undergo radiational breakup

$$t_{\mathrm{split}}(k_{\mathrm{split}}) \sim t \Rightarrow k_{\mathrm{split}} \sim \alpha^2 \hat{q} t^2$$

- "Falling" particles heat the soft thermal bath

$$T^4 \sim k_{\mathrm{split}} \int d^3p f(p)$$

- Thermalization when hardest scale Q gets eaten $k_{\mathrm{split}}(\tau_0) \sim Q$ at

$$\tau_0 \sim t_{\mathrm{quench}}(Q) \sim \frac{1}{\alpha}\sqrt{\frac{Q}{\hat{q}}} \sim \left(\frac{Q}{T_{\mathrm{final}}}\right)^{1/2} \frac{1}{\alpha^2 T_{\mathrm{final}}}$$

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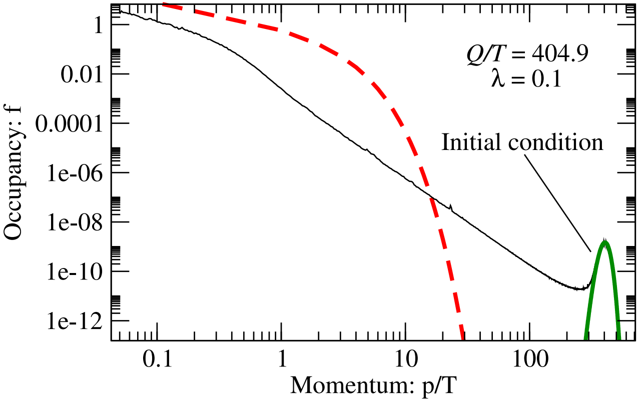
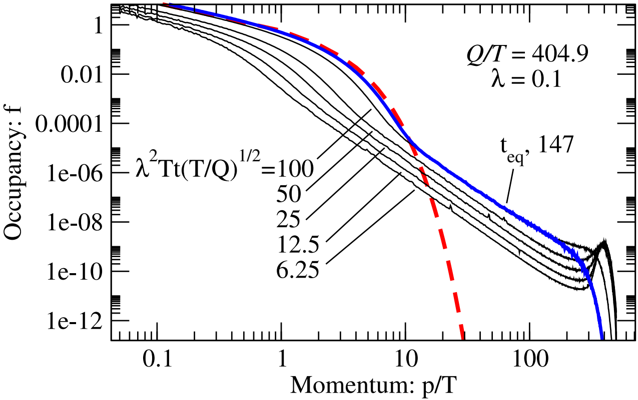
AK, Lu, PRL 113 (2014) 18, 182301



- Start with an underoccupied initial condition $p \sim Q$
- after a very short time, an IR bath is created (1↔ 2–processes)

Bottom-up thermalization

AK, Lu, PRL 113 (2014) 18, 182301



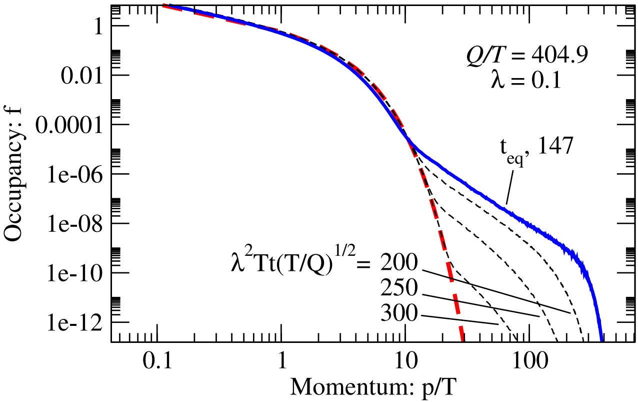
- More energy flows to the IR, temperature increases, “Bottom-up”
- When “bottom” reaches final T, “up” is quenched

AK, Moore JHEP 1112 (2011) 044

$$t_{\rm eq} \sim (Q/T)^{1/2} \frac{1}{\alpha^2 T}$$

Bottom-up thermalization

AK, Lu, PRL 113 (2014) 18, 182301

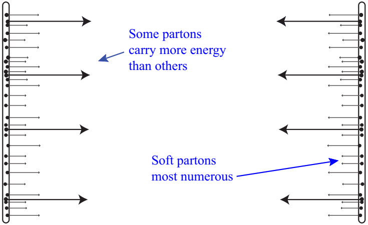


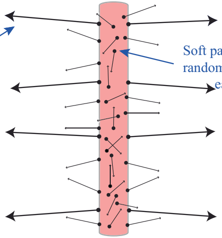
- Hardest scales reach equilibrium last.

Close resemblance to Blaizot, Iancu, Mehtar-tani for jets PRL 111 (2013) 052001

Bottom-up thermalization in nuclear collisions

Initial condition in nuclear collisions



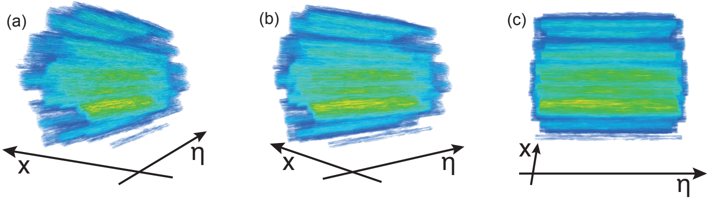


- Parton passing a nucleus gets $\Delta p_T \sim Q_s$
- Time for only 1 interaction $\Rightarrow$ absorption of gluon with soft $p \sim Q_s$
- Scattering α_s-suppressed $\Rightarrow$ need $f \sim 1/\alpha_s$ soft gluons in both nuclei

- Scattering between soft gluons unsuppressed and large angle

$$f(Q_s) \sim 1/\alpha_s, \qquad Q_s \sim 2\text{GeV}$$

Initial condition in nuclear collisions



Schenke, Schlichting, PRC 94 (2016)

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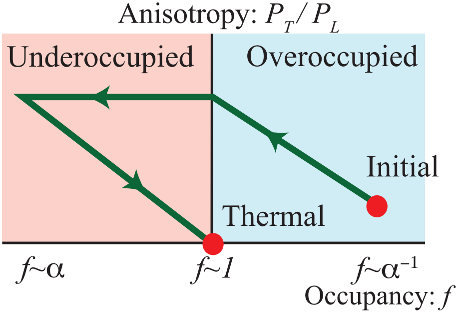
- Systematic EFT framework: "Color-Glass-Condensate"

McLerran, Venugopalan PRD49 (1994), PRD49 (1994)
Gelis et. al Int.J.Mod.Phys. E16 (2007), Ann.Rev.Nucl.Part.Sci. 60 (2010), Lappi PLB703 (2011)

Pre-equilibrium evolution: Secondary interactions



- Expansion makes system underoccupied before thermalizing

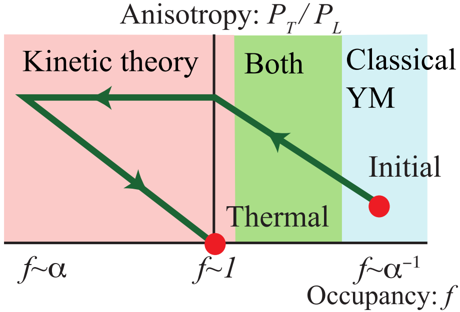
Baier et al Phys.Lett. B502 (2001) 51-58; AK, Moore JHEP 1111 (2011) 120

$$f(Q_s) \ll 1$$

- Underoccupied system thermalizes through *bottom-up thermalization*

$$\tau_{\text{eq}} \sim \frac{1}{\alpha^2 T}\left(\frac{Q}{T}\right)^{1/2}, \quad \epsilon \sim T^4 \sim \frac{Q^4}{\alpha(Qt)} \Rightarrow Qt \sim \alpha^{-13/5}$$

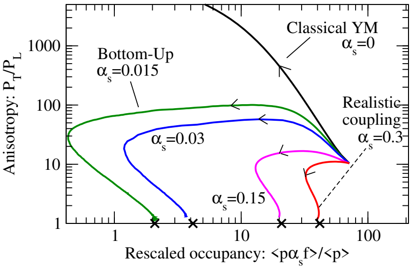
Pre-equilibrium evolution: Secondary interactions



- Degrees of freedom:
  - $f \gg 1$: Classical Yang-Mills theory (CYM)
  - $f \ll 1/\alpha_s$: Eff. Kinetic Theory (EKT)

In practice: Initial condition from classical CGC simulations

Pre-equilibrium evolution: Secondary interactions



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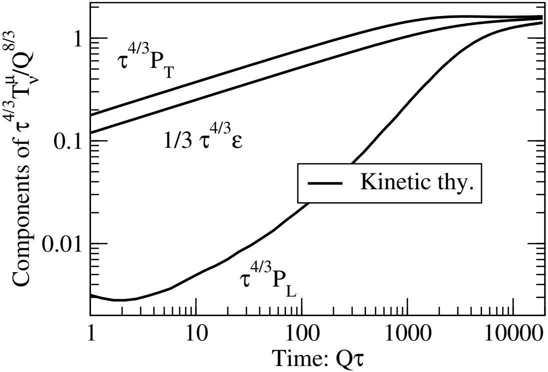
In practice: Initial condition from classical CGC simulations

Smooth approach to hydrodynamics

AK, Zhu, PRL 115 (2015)

$$\alpha_s = 0.03$$

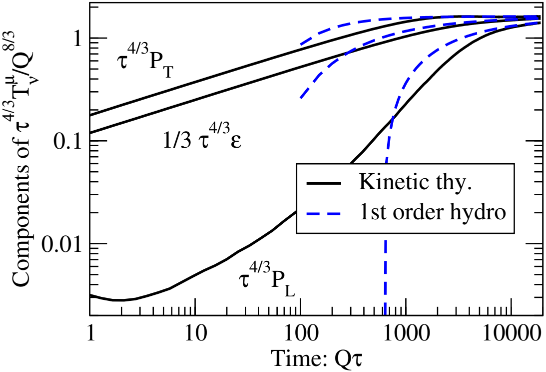


- Kinetic theory converges to hydro smoothly and automatically

Smooth approach to hydrodynamics

AK, Zhu, PRL 115 (2015)

$$\alpha_s = 0.03$$



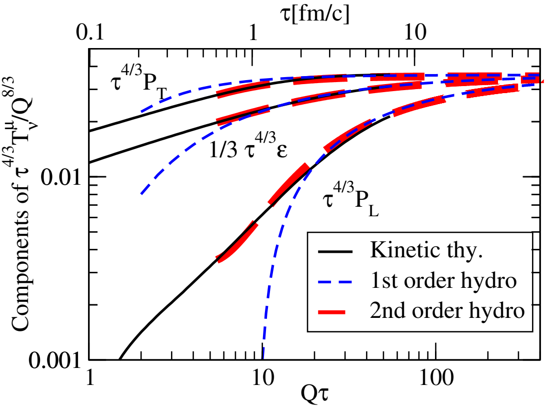
- Kinetic theory converges to hydro smoothly and automatically
- Approach to hydro fixed by perturbative η/s

$$\partial_\tau \epsilon = -\frac{4}{3}\frac{\epsilon}{\tau} + \frac{4\eta}{3\tau^2}, \qquad P_L = \frac{\epsilon}{3} - \frac{4\eta}{3\tau}$$

Smooth approach to hydrodynamics

AK, Zhu, PRL 115 (2015)

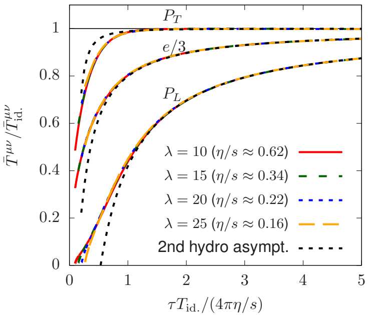
$$\alpha_s = 0.3$$



- Hydro far from thermal equilibrium $P_L/P_T \ll 1$

Similar findings in strongly coupled models Chesler, Yaffe, PRL 102 (2009)

Smooth approach to hydrodynamics AK et al. 1805.01604, PRL in print

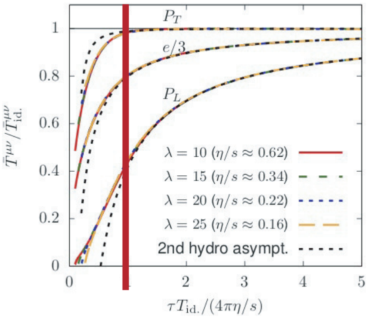


- Hydrodynamization governed by the mean free path

$$l_{mfp} \sim \frac{1}{\alpha_s^2 T} \sim \frac{4\pi\eta}{sT}$$

$$\tau_i \approx 1.1\mathrm{fm/c}\left(\frac{4\pi(\eta/s)}{2}\right)^{3/2}\left(\frac{\langle\tau s\rangle}{4.1\mathrm{GeV}^2}\right)^{-1/2}$$

Smooth approach to hydrodynamics AK et al. 1805.01604, PRL in print



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Connection of fluid dynamics and far-from-equilibrium dynamics