Relativistic dissipative hydrodynamics from kinetic theory

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Fluid dynamics

- Dynamics of a system exhibiting collective behaviour.
- An effective theory describing the long-wavelength, low-frequency limit of the microscopic dynamics of a system.
- Relativistic hydrodynamics finds applications in cosmology, astrophysics, and the physics of high-energy heavy-ion collisions.
- It has been used to study ultra-relativistic heavy-ion collisions with considerable success.
- The theory is formulated as an order-by-order expansion in powers of gradients with ideal hydro being zeroth-order.
- First-order relativistic Navier-Stokes theory has acausal behavior which is rectified in second-order theory.
- The second-order Israel-Stewart theory is generally applied in heavy-ion collisions.

Relativistic fluid dynamics

- For relativistic systems, the mass density $\rho(t, \vec{x})$ is not a good degree of freedom.
- For large kinetic energy, replace $\rho(t, \vec{x})$ by energy density $\epsilon(t, \vec{x})$.
- Similarly, $\vec{v}(t, \vec{x})$ should be replaced by the Lorentz 4-vector for the velocity.

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \frac{1}{\sqrt{1-\vec{v}^2}}\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix} = \gamma(\vec{v})\begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}$$

- The four velocity u^μ is timelike: $u^2 \equiv u^\mu g_{\mu\nu} u^\nu = 1$.
- Hydrodynamic equations are essentially conservation equations:
  - Energy-momentum conservation: $\partial_\mu T^{\mu\nu} = 0$.
  - Current conservation: $\partial_\mu N^\mu = 0$.
- $T^{\mu\nu}$: Energy-momentum tensor, N^μ: Charge current.

Relativistic ideal fluids

- The energy-momentum tensor of an ideal fluid can be written in terms of the available tensor degrees of freedom:

$$T^{\mu\nu}_{(0)} = c_1 u^\mu u^\nu + c_2 g^{\mu\nu}$$

- In local rest frame, i.e., $u^\mu = (1,\ 0,\ 0,\ 0)$,

$$T^{\mu\nu}_{(0)} = \mathrm{diag}(\epsilon,\ P,\ P,\ P) \Rightarrow c_1 = \epsilon + P,\ c_2 = -P.$$

- Energy-momentum tensor for the ideal fluid, $T^{\mu\nu}_{(0)}$ is

$$T^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}\ ; \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

- $\Delta^{\mu\nu} u_\mu = \Delta^{\mu\nu} u_\nu = 0$ and $\Delta^{\mu\nu}\Delta^{\alpha}_{\nu} = \Delta^{\mu\alpha}$, hence serves as a projection operator on the space orthogonal to the fluid velocity u^μ.
- Similarly, $N^{\mu}_{(0)} = nu^\mu$.
- Fluids are in general dissipative; dissipation needs to be included.

Ideal and dissipative hydrodynamics

- Dissipation can be included in the energy momentum tensor and conserved current as

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \; ; \quad N^{\mu} = N^{\mu}_{(0)} + n^{\mu}$$

Ideal	**Dissipative**
$T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$	$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$
$N^\mu = nu^\mu$	$N^\mu = nu^\mu + n^\mu$
Unknowns: $\underbrace{\epsilon,\ P,\ n,\ u^\mu}_{1+1+\ 1+\ 3} = 6$	$\underbrace{\epsilon,\ P,\ n,\ u^\mu,\ \Pi,\ \pi^{\mu\nu},\ n^\mu}_{1+1+\ 1\ +\ 3\ +\ 1\ +\ 5\ +\ 3} = 15$
Equations: $\underbrace{\partial_\mu T^{\mu\nu} = 0,\quad \partial_\mu N^\mu = 0,\quad EOS}_{4\quad+\quad1\quad+\quad1} = 6$	
Closed set of equations	9 more equations required

- Landau frame chosen: $T^{\mu\nu} u_\nu = \epsilon u^\mu$.

Some thermodynamic relations

- An elegant way of obtaining Π, $\pi^{\mu\nu}$ and n^{μ} builds upon the second law of thermodynamics: entropy must always increase locally.
- First law of thermodynamics: $\delta E = \delta Q - P\delta V + \mu\delta N$.
- For reversible processes, $\delta Q = T\delta S$.
- First law in differential form: $dE = TdS - PdV + \mu dN$.
- Energy is an extensive function of the variables $(V,\ S,\ N)$:

$$E(\lambda V,\ \lambda S,\ \lambda N) = \lambda E(V,\ S,\ N) \ \Rightarrow\ E = -PV + TS + \mu N$$

- Hence for $s \equiv S/V$, $\epsilon \equiv E/V$ and $n \equiv N/V$,

$$\epsilon + P = Ts + \mu n \ \Rightarrow\ s = \frac{\epsilon + P - \mu n}{T}$$

- Other useful identities: $dP = sdT + nd\mu, \quad d\epsilon = Tds + \mu dn$

Dissipative equations [L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 1987]

- Second law in covariant form: $\partial_\mu S^\mu \geq 0$, where

$$S^\mu = s\, u^\mu \quad ; \quad s = \frac{\epsilon + P - \mu n}{T}.$$

- Demanding second-law from this entropy current,

$$\Pi = -\zeta\theta, \quad n^\alpha = \lambda T \nabla^\alpha(\mu/T), \quad \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle},$$

  where,

$$\theta \equiv \partial_\mu u^\mu, \quad \nabla^\alpha \equiv \Delta^{\alpha\beta}\partial_\beta, \quad \nabla^{\langle\mu}u^{\nu\rangle} \equiv (\nabla^\mu u^\nu + \nabla^\nu u^\mu)/2 - \Delta^{\mu\nu}\theta/3.$$

- The transport coefficients $\eta, \zeta, \lambda \geq 0$.
- In the non-relativistic limit, above equations reduces to the Navier-Stokes equations.
- Beautiful and simple but flawed! Exhibits acausal behavior.

Acausality problem [P. Romatschke, Int. J. Mod. Phys. E 19, 1 (2010)]

- Consider small perturbations of the energy density and fluid velocity,

$$\epsilon = \epsilon_0 + \delta\epsilon(t, x), \quad u^\mu = (1, \mathbf{0}) + \delta u^\mu(t, x).$$

- For a particular direction y, we get a diffusion-type equation

$$\partial_t \delta u^y - \frac{\eta_0}{\epsilon_0 + P_0} \partial_x^2 \delta u^y = \mathcal{O}(\delta^2).$$

- Use mixed Laplace-Fourier wave ansatz to study the individual modes

$$\delta u^y(t, x) = \exp(-\omega t + ikx) f_{\omega,k}.$$

- We obtain the "dispersion-relation" for the diffusion equation

$$\omega = \frac{\eta_0}{\epsilon_0 + P_0} k^2.$$

- The speed of diffusion of a mode with wavenumber k

$$v_T(k) = \frac{d\omega}{dk} = 2\frac{\eta_0}{\epsilon_0 + P_0} k.$$

- Increases $\propto k$ without bound: acausal behavior.

Maxwell-Cattaneo law

- One possible way out is the “Maxwell-Cattaneo” law,

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu} u^{\nu\rangle}.$$

- The diffusion equation becomes a relaxation-type equation.
- A new transport coefficient enters the theory: the relaxation time τ_π.
- The effect of this modification on the dispersion relation for the perturbation δu^y becomes,

$$\omega = \frac{\eta_0}{\epsilon_0 + P_0} \frac{k^2}{1 - \omega\tau_\pi}.$$

- The above equation describes propagating waves with a maximum propagation speed

$$v_T^{\max} \equiv \lim_{k\to\infty} \frac{d|\omega|}{dk} = \sqrt{\frac{\eta_0}{(\epsilon_0 + P_0)\tau_\pi}}.$$

- Interestingly, for all known fluids the limiting value of $v_T^{\max} < 1$.

Muller-Israel-Stewart theory

- While Maxwell-Cattaneo law is successful in solving the acausality problem, it does not follow from a first-principles framework.
- Desirable to derive some variant of Maxwell-Cattaneo law which preserves causality: Muller-Israel-Stewart theory.
- Assuming entropy current to be algebraic in the hydrodynamic degrees of freedom,

$$S^\mu = s u^\mu - \frac{\beta_2}{2T}\pi_{\alpha\beta}\pi^{\alpha\beta}u^\mu + \mathcal{O}(\pi^3).$$

- Demanding second law of thermodynamics, $\partial_\mu S^\mu \geq 0$,

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} - \tau_\pi\theta\pi^{\mu\nu} - \eta\pi^{\mu\nu}Tu^\mu\partial_\mu(\beta_2/T).$$

- The relaxation time can be related as: $\tau_\pi = 2\eta\beta_2$.

Structure of a general second-order equation

- A general structure, allowed by symmetry, can be written for second-order evolution equation.
- Write down all possible second-order terms formed by the gradients of ideal hydrodynamic quantities [S. Bhattacharyya, JHEP 1207 (2012) 104].
- All possible terms in evolution equation for shear stress tensor allowed by conformal symmetry can be written as

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\nabla^{\langle\mu}u^{\nu\rangle} + \delta_{\pi\pi}\theta\pi^{\mu\nu} + \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} + \lambda_{\pi u}\dot{u}^\mu\dot{u}^\nu + \lambda_{\pi\pi}\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} + \lambda_{\pi\omega}\omega_\gamma{}^{\langle\mu}\omega^{\nu\rangle\gamma}.$$

- The coefficients of each term are the transport coefficients which can be obtained from the underlying microscopic theory.
- The transport coefficients are obtained from microscopic theory using Kubo relations.

Transport coefficients and Kubo relations

- Kubo formulas, in terms of n-point correlation functions, can be written to calculate the transport coefficients [G. D. Moore & K. A. Sohrabi, PRL 106 (2011) 122302].
- For eg., the coefficient of shear viscosity can be obtained as:

$$\eta = i \lim_{\omega\to 0} \lim_{k_z\to 0} \bar{G}^{xy,xy}(\omega, k_z)$$

where $\bar{G}^{xy,xy}(\omega, k_z)$ is the Fourier transform of

$$G^{\mu\nu,\alpha\beta}(x, x') = -i\theta(x_0 - x'_0)\langle[T^{\mu\nu}(x), T^{\alpha\beta}(x')]\rangle$$

- n-point correlators can be evaluated for weakly coupled theories [G. D. Moore & K. A. Sohrabi, JHEP 1211 (2012) 148].
- Some progress for first-order transport coefficients in case of strongly coupled theories on lattice [Harvey B. Meyer, PRD 76 (2007) 101701; PRL 100 (2008) 162001].

Muller-Israel-Stewart theory from Boltzmann H-function

- Apart from η, a new transport coefficient enters the theory: τ_π.
- Can be obtained from kinetic theory using Boltzmann equation, $\tau_\pi = 3\eta/4P$ [W. Israel and J. M. Stewart, Annals Phys. 118, 341 (1979)].
- Demand $\partial_\mu S^\mu \geq 0$ from entropy four-current given by Boltzmann H-function [**AJ**, R. S. Bhalerao and S. Pal, PRC 87, 021901(R) (2013)]:

$$S^\mu = -\int dp\, p^\mu\, f(\ln f - 1).$$

- The shear relaxation time remains the same, we obtain 'finite' bulk relaxation time as a bonus

$$\tau_\pi = \frac{3\eta}{4P}, \quad \tau_\Pi = \frac{\zeta}{P}.$$

- Several phenomenological implications of τ_Π: finite in ultra-relativistic $(m/T \to 0)$ limit, avoids cavitation [R. S. Bhalerao, **AJ**, S. Pal, and V. Sreekanth, PRC 88, 044911 (2013)].

Relativistic kinetic theory

- Kinetic theory: calculation of macroscopic quantities by means of statistical description in terms of distribution function.
- Let us consider a system of relativistic particles of rest mass m with momenta $\mathbf{p}$ and energy p^0

$$p^0 = \sqrt{\mathbf{p}^2 + m^2}$$

- For large no. of particles, introduce a function $f(x, p)$ which gives a distribution of the four-momenta $p = p^\mu = (p^0, \mathbf{p})$ at each space-time point.
- $f(x, p)\Delta^3 x \Delta^3 p$ gives average no. of particles at any given time in the volume element $\Delta^3 x$ at point x with momenta in the range $(\mathbf{p}, \mathbf{p} + \Delta\mathbf{p})$.
- Statistical assumptions:
  - No. of particles contained in $\Delta^3 x$ is large ($N \gg 1$).
  - $\Delta^3 x$ is small compared to macroscopic volume ($\Delta^3 x / V \ll 1$).

Relativistic kinetic theory: Particle four-flow

- To describe a non-uniform system, $n(x)$ is introduced: $n(x)\Delta^3 x$ is avg. no. of particles in volume $\Delta^3 x$ at x.
- Similarly particle flow $\mathbf{j}(x)$ is defined as the particle current along (x,y,z) directions.
- These two local quantities, particle density and particle flow constitute a four-vector field: $N^\mu = (n, \mathbf{j})$
- With the help of distribution function, the particle density and particle flow is given by:

$$n(x) = \frac{g}{(2\pi)^3}\int d^3p \; f(x,p); \quad \mathbf{j}(x) = \frac{g}{(2\pi)^3}\int d^3p \; \mathbf{v} \, f(x,p)$$

where $\mathbf{v} = \mathbf{p}/p^0$ is the velocity.

- Particle four-flow can be written in a unified way

$$N^\mu(x) = \frac{g}{(2\pi)^3}\int \frac{d^3p}{p^0} p^\mu \; f(x,p)$$

Relativistic kinetic theory: Energy-momentum tensor

- Energy per particle is p^0, the average can be written as

$$T^{00}(x) = \frac{g}{(2\pi)^3}\int d^3p \; p^0 \; f(x,p)$$

- Similarly energy flow and momentum density are defined as

$$T^{0i}(x) = \frac{g}{(2\pi)^3}\int d^3p \; p^0 \; v^i \; f(x,p); \quad T^{i0}(x) = \frac{g}{(2\pi)^3}\int d^3p \; p^i \; f(x,p)$$

- For momentum flow (flow in direction j of momentum in direction i), we have

$$T^{ij}(x) = \frac{g}{(2\pi)^3}\int d^3p \; p^i \; v^j \; f(x,p); \qquad \left[v^j = \frac{p^j}{p^0}\right]$$

- Combining all this in compact covariant form:

$$T^{\mu\nu}(x) = \frac{g}{(2\pi)^3}\int \frac{d^3p}{p^0} p^\mu \; p^\nu \; f(x,p)$$

Dissipative quantities and Grad's 14-moment method

- With $f = f_0 + \delta f$, the dissipative quantities can be written as

$$n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f, \quad \Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f.$$

- The equilibrium distribution functions are

$$f_0 = [\exp\{y_0(x,p)\} + r]^{-1}, \quad y_0 = -\beta(u\cdot p) + \alpha, \quad r = 0, \pm 1$$

- Away from equilibrium, $f = [\exp\{y(x,p)\} + r]^{-1}$, where

$$\phi(x,p) \equiv y(x,p) - y_0(x,p) = \varepsilon(x) - \varepsilon_\mu(x) p^\mu + \varepsilon_{\mu\nu}(x) p^\mu p^\nu + \cdots$$

- Taylor expanding around equilibrium upto linear in ϕ

$$f = f_0 + \delta f, \quad \delta f = f_0 \tilde{f}_0 \phi, \quad \text{where, } \tilde{f}_0 = 1 - r f_0$$

- ε, ε_μ and $\varepsilon_{\mu\nu}$ can be expressed in terms of Π, n_μ and $\pi_{\mu\nu}$ as

$$\varepsilon = A_0\Pi, \quad \varepsilon_\mu = A_1 \Pi u_\mu + B_0 n_\mu, \quad \varepsilon_{\mu\nu} = A_2(3u_\mu u_\nu - \Delta_{\mu\nu})\Pi - B_1 u_{(\mu} n_{\nu)} + C_0 \pi_{\mu\nu}$$

- A_0, A_1, A_2, B_0, B_1 and C_0 can be determined using constraints.

Low "density" fluids of massless particles

- Close to equilibrium ($\delta f/f_0 \ll 1$), $f = f_0 + \delta f$ and for $\mu_b = m = 0$,

$$\cancel{n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f}, \quad \cancel{\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f}, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f.$$

- Boltzmann equation in the relxn. time approx. is solved iteratively:

$$p^\mu \partial_\mu f = -\frac{u\cdot p}{\tau_R}(f - f_0) \;\Rightarrow\; f = f_0 - (\tau_R/u\cdot p)\, p^\mu \partial_\mu f$$

- Expand f about its equilibrium value: $f = f_0 + \delta f^{(1)} + \delta f^{(2)} + \cdots$,

$$\delta f^{(1)} = -\frac{\tau_R}{u\cdot p}\, p^\mu \partial_\mu f_0 \ , \quad \delta f^{(2)} = \frac{\tau_R}{u\cdot p} p^\mu p^\nu \partial_\mu \left(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\right) .$$

- Substituting $\delta f = \delta f^{(1)} + \delta f^{(2)}$ [**AJ**, PRC 87, 051901(R) (2013)],

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \frac{4}{3}\pi^{\mu\nu}\theta + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}, \qquad \beta_\pi = \frac{4P}{5}.$$

[G. S. Denicol, T. Koide and D. H. Rischke, PRL 105, 162501 (2010)]

Second-order viscous corrections

- For a system of massless particles at vanishing chemical potential, $\delta f = \delta f^{(1)} + \delta f^{(2)}$ can be written as

$$\delta f = \frac{f_0 \beta}{2\beta_\pi (u\cdot p)} p^\alpha p^\beta \pi_{\alpha\beta} - \frac{f_0\beta}{\beta_\pi}\Bigg[\frac{\tau_\pi}{u\cdot p} p^\alpha p^\beta \pi^\gamma_\alpha \omega_{\beta\gamma} - \frac{5}{14\beta_\pi (u\cdot p)} p^\alpha p^\beta \pi^\gamma_\alpha \pi_{\beta\gamma}$$

$$+\frac{\tau_\pi}{3(u\cdot p)} p^\alpha p^\beta \pi_{\alpha\beta}\theta - \frac{6\tau_\pi}{5} p^\alpha \dot{u}^\beta \pi_{\alpha\beta} + \frac{(u\cdot p)}{70\beta_\pi}\pi^{\alpha\beta}\pi_{\alpha\beta} + \frac{\tau_\pi}{5} p^\alpha \left(\nabla^\beta \pi_{\alpha\beta}\right)$$

$$-\frac{3\tau_\pi}{(u\cdot p)^2} p^\alpha p^\beta p^\gamma \pi_{\alpha\beta}\dot{u}_\gamma + \frac{\tau_\pi}{2(u\cdot p)^2} p^\alpha p^\beta p^\gamma \left(\nabla_\gamma \pi_{\alpha\beta}\right)$$

$$-\frac{\beta + (u\cdot p)^{-1}}{4(u\cdot p)^2\beta_\pi} p^\alpha p^\beta p^\gamma p^\delta \pi_{\alpha\beta}\pi_{\gamma\delta}\Bigg] + \mathcal{O}(\delta^3).$$

[R. S. Bhalerao, **AJ**, S. Pal, and V. Sreekanth, PRC 89, 054903 (2014)]

- The first-order correction can be compared to that due to Grad's 14-moment approximation

$$\delta f^{(1)}_{CE} = \frac{5 f_0}{2(u\cdot p)(\epsilon + P)T} p^\alpha p^\beta \pi_{\alpha\beta}, \qquad \delta f^{(1)}_{G} = \frac{f_0}{2(\epsilon + P)T^2} p^\alpha p^\beta \pi_{\alpha\beta}$$

Checks on the viscous corrections

1. The viscous correction δf should satisfy the definition of shear stress tensor
$$\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp\, p^\alpha p^\beta\, \delta f.$$
2. Moreover, at each order in gradients, the form of δf should satisfy
   - The matching condition $\epsilon = \epsilon_0$
$$\int dp\, (u \cdot p)^2\, \delta f_i = 0$$
   - The Landau frame definition $u_\nu T^{\mu\nu} = \epsilon u^\mu$
$$\int dp\, \Delta_{\mu\alpha} u_\beta\, p^\alpha p^\beta\, \delta f_i = 0.$$
3. Indeed we show that these conditions are valid for $i = 1$ and 2.
[R. S. Bhalerao, **AJ**, S. Pal, and V. Sreekanth, PRC 89, 054903 (2014)]

Effect of viscous corrections on observables

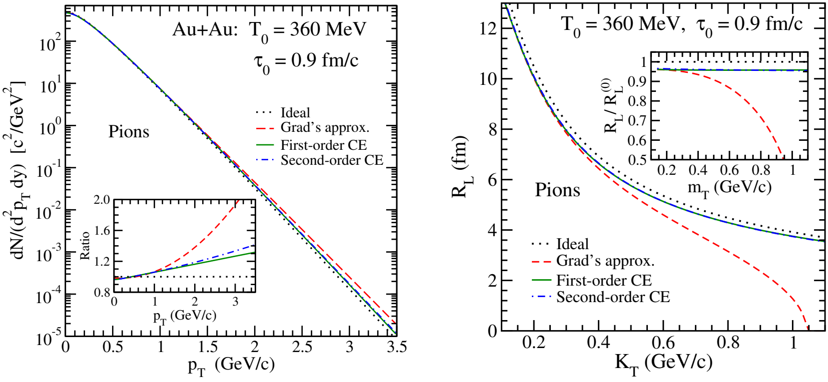


Figure: Effect of viscous corrections on pion spectra and longitudinal HBT radii.

[R. S. Bhalerao, AJ, S. Pal, and V. Sreekanth, PRC 89, 054903 (2014)]

- Grad's approximation for δf violate $1/\sqrt{m_T}$ scaling of the longitudinal HBT radii.

Higher-order hydrodynamics

- Third-order equation for shear stress tensor [**AJ**, PRC 88, 021903(R) (2013)]:

$$
\begin{aligned}
\dot{\pi}^{\langle\mu\nu\rangle} = &-\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{63}\pi^{\mu\nu}\theta^2 \\
&+\tau_\pi\Bigg[\frac{50}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\sigma_{\rho\gamma} - \frac{76}{245}\pi^{\mu\nu}\sigma^{\rho\gamma}\sigma_{\rho\gamma} - \frac{44}{49}\pi^{\rho\langle\mu}\sigma^{\nu\rangle\gamma}\sigma_{\rho\gamma} \\
&-\frac{2}{7}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{2}{7}\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} + \frac{26}{21}\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta - \frac{2}{3}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}\theta\Bigg] \\
&-\frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) \\
&-\frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right).
\end{aligned}
$$

- Improved accuracy compared to second-order equations.

- Neccessary for incorporation of colored noise in fluctuating hydro evolution [J. Kapusta and C. Young, Phys. Rev. C 90, 044902 (2014)].

Checks on the third-order equation

1. Number of new transport coefficients at third-order:
   - 14 new transport coefficients obtained; 15 predicted from conformal analysis [S. Grozdanov and N. Kaplis, PRD 93, 066012 (2016)].
   - Misses $\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma}$ similar to $\omega^{\rho\langle\mu}\omega^{\nu\rangle}{}_{\rho}$ at second-order.
   - Complete third-order viscous hydrodynamic eqation from kinetic theory.

2. The values of the transport coefficients [W. Florkowski, R. Ryblewski and M. Spalinski, PRD 94, 114025 (2016)]:

n	RTA BE	2nd-order	3rd-order
0	$2/3$	$2/3$	$2/3$
1	$4/45$	$4/45$	$4/45$
2	$16/945$	$16/945$	$16/945$
3	$-208/4725$	-304/33075	$-208/4725$

   - Correct transport coefficients.

Bjorken Flow

- In Milne coordinates: proper time $\tau = \sqrt{t^2 - z^2}$, spacetime rapidity $\eta = tanh^{-1}(z/t)$, $t = \tau \ cosh\ \eta$, $z = \tau \ sinh\ \eta$ and the metric is given by $g_{\mu\nu}$ =diag$(1, -1, -1, -\tau^2)$.
- Boost invariance ($v^z = z/t$) for hydro translates into

$$u^z = \frac{z}{\tau}, \quad u^\eta = -u^t \frac{sinh\ \eta}{\tau} + u^z \frac{cosh\ \eta}{\tau} = 0 \ \Rightarrow \ u^\mu = (1, 0, 0, 0)$$

- In center of the fireball, stress energy tensor in local comoving frame has the form: $T^{\mu\nu} = diag(\epsilon,\ p_T,\ p_T,\ p_L)$.

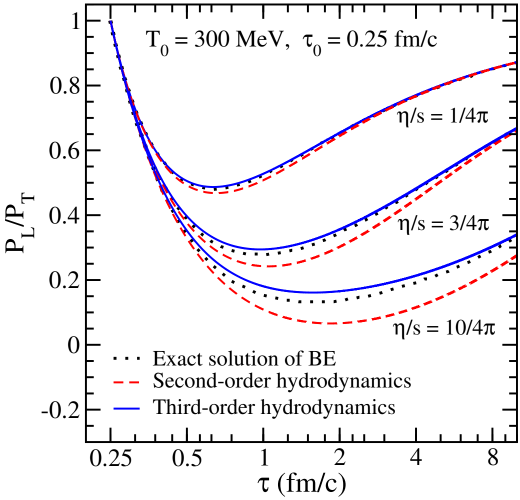
$$P_T = P + \pi/2 \quad ; \quad P_L = P - \pi \quad ; \quad \frac{P_L}{P_T} = \frac{P - \pi}{P + \pi/2} \ .$$

- The evolution equations for ϵ, $\pi \equiv -\tau^2 \pi^{\eta\eta}$ and Π becomes

$$u_\nu \partial_\mu T^{\mu\nu} = 0 \ \Rightarrow \ \frac{d\epsilon}{d\tau} = -\frac{1}{\tau} \left(\epsilon + P - \pi\right),$$

$$\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + \beta_\pi \frac{4}{3\tau} - \lambda \frac{\pi}{\tau} - \chi \frac{\pi^2}{\beta_\pi \tau}.$$

One dimensional evolution of pressure anisotropy



Exact solution of the BE:
[W. Florkowski, R. Ryblewski and M. Strickland, PRC 88, 024903 (2013); NPA 916, 249 (2013); W. Florkowski, E. Maksymiuk, R. Ryblewski and M. Strickland, PRC 89,054908 (2014); W. Florkowski and E. Maksymiuk, JPG 42, 045106 (2015)]

Figure: [**AJ**, PRC 88, 021903(R) (2013)]

Low density fluids of massive particles

- Massive particles $m \neq 0$ and low net Baryon number density $\mu_b = 0$

$$\cancel{n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f}, \quad \Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f, \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f.$$

- Second-order evolution equations are obtained as,

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu},$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \tau_{\pi\pi}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}.$$

- In relaxation-time approximation, $\tau_\Pi = \tau_\pi = \tau_R \;\Rightarrow\; \zeta/\eta = \beta_\Pi/\beta_\pi$.
- For $m/T \ll 1$,

$$\frac{\zeta}{\eta} = \Lambda\left(\frac{1}{3} - c_s^2\right)^2, \qquad \Lambda = \begin{cases} 75 & \text{for MB} \\ 48 & \text{for FD} \\ \infty & \text{for BE} \\ 15 & \text{by Weinberg} \end{cases}$$

[**AJ**, R. Ryblewski, M. Strickland, PRC 90, 044908 (2014); W. Florkowski, **AJ**, E. Maksymiuk, R. Ryblewski, M. Strickland, PRC 91, 054907 (2015)]

One dimensional evolution

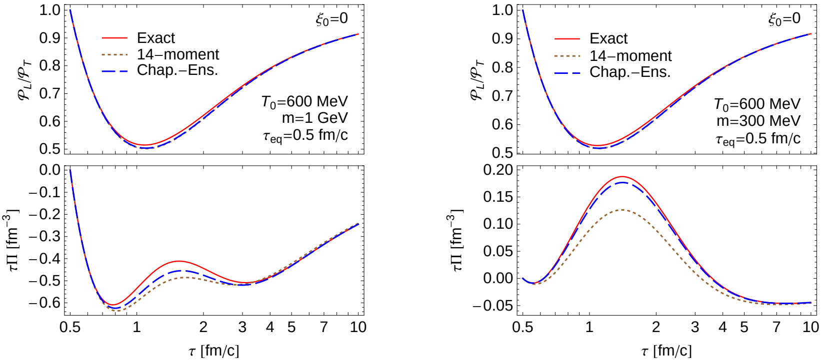


Figure: [**AJ**, R. Ryblewski, M. Strickland, PRC 90, 044908 (2014); W. Florkowski, **AJ**, E. Maksymiuk, R. Ryblewski, M. Strickland, PRC 91, 054907 (2015)].

- Chapman-Enskog method performs better than moment method.
- Result valid for all distributions.

High density fluids of massless particles

- Massless particles $m = 0$ and net Baryon number density $\mu_b \neq 0$,

$$n^\mu = \Delta^\mu_\alpha \int dp\, p^\alpha\, \delta f, \quad \cancel{\Pi = -\frac{1}{3}\Delta_{\alpha\beta}\int dP\, p^\alpha p^\beta\, \delta f,} \quad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\, \delta f.$$

- Second-order evolution equations are obtained as,

$$\dot{n}^{\langle\mu\rangle} + \frac{n^\mu}{\tau_n} = \beta_n \nabla^\mu \alpha - n_\nu \omega^{\nu\mu} - n^\mu\theta - \frac{9}{5} n_\nu \sigma^{\nu\mu},$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}.$$

- Charge: $\kappa_n/\eta = \beta_n/\beta_\pi$; heat: $\kappa_q/\eta = (\beta_n/\beta_\pi)[(\epsilon + P)/nT]^2$.

- Wiedemann-Franz law [**AJ**, B. Friman, K. Redlich, PLB 751, 548 (2015)]:

$$\frac{\kappa_q}{\eta} = C\frac{\pi^2 T}{\mu^2}, \quad C = \begin{cases} 37/27 & \text{for 2 flavor QGP, } \mu/T \ll 1 \\ 95/81 & \text{for 3 flavor QGP, } \mu/T \ll 1 \\ 5/3 & \text{for } \mu/T \gg 1 \\ 8/9 & \text{AdS/CFT [Son \& Starinets, JHEP 0603, 052 (2006)]} \end{cases}$$

(μ: quark chemical potential)

Charge and heat conductivity

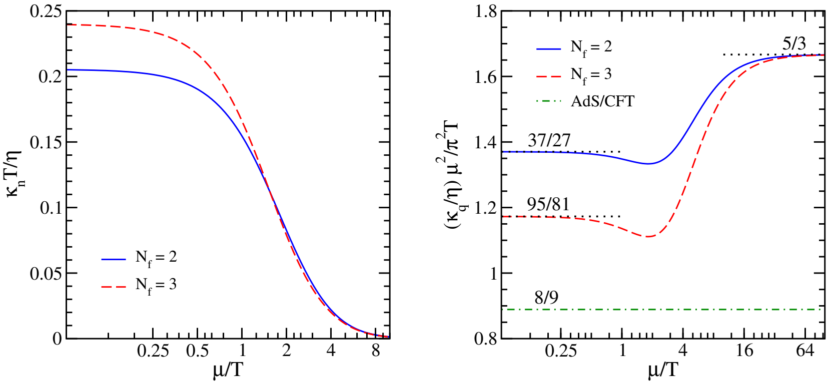


Figure: [**AJ**, B. Friman, K. Redlich, PLB 751, 548 (2015)].

- At high densities, charge conductivity of QGP is small compared to shear viscosity.
- Intriguing similarity with AdS/CFT results for heat conductivity.

Relativistic hydrodynamics with spin [W. Florkowski, B. Friman, **AJ**, R. Ryblewski and E. Speranza, PRC 97(R), 041901 (2018); PRD 97, 116017 (2018).]

- For a system of particles with intrinsic spin, one must consider angular momentum conservation along with energy and momentum,

$$\partial_\lambda J^{\lambda,\mu\nu} = 0, \qquad J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}.$$

- For spin-1/2 prticles, we have

$$N^\mu = \int dP\, p^\mu \left[\mathrm{tr}(X^+) - \mathrm{tr}(X^-)\right], \quad T^{\mu\nu} = \int dP\, p^\mu p^\nu \left[\mathrm{tr}(X^+) + \mathrm{tr}(X^-)\right],$$

$$\text{and} \quad S^{\lambda,\mu\nu} = \int dP\, p^\lambda\, \mathrm{tr}\left[(X^+ - X^-)\hat{\Sigma}^{\mu\nu}\right],$$

where

$$X^\pm = \exp\left[\pm\alpha(x) - \beta_\mu(x) p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right].$$

- We found that, from a thermodynamic point of view, such a system can be seen as a two component mixture of scalar particles.

Anisotropic metric and aHydro [A. Dash and **AJ**, PRD 97, 104005 (2018)]

- We consider the Bianchi type-I metric:

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2.$$

- The energy-momentum tensor and conserved charge current can be calculated for a locally equilibriated fluid as (axis-symmetric case):

$$\varepsilon = \varepsilon_0 \left(\frac{n}{n_0}\right)^{4/3} R(\xi), \quad \mathcal{P}_T = 3\mathcal{P}_0 \left(\frac{n}{n_0}\right)^{4/3} \left[\frac{R(\xi)}{3} + \xi\, R'(\xi)\right],$$

$$\mathcal{P}_L = 3\mathcal{P}_0 \left(\frac{n}{n_0}\right)^{4/3} \left[\frac{R(\xi)}{3} - 2\xi\, R'(\xi)\right], \quad R(\xi) = \frac{1}{2\xi^{\frac{1}{3}}}\left[1 + \frac{\xi \arctan\sqrt{\xi - 1}}{\sqrt{\xi - 1}}\right],$$

where $A(t) = B(t)$ and $\xi \equiv C^2(t)/A^2(t)$.

- Identical to anisotropic hydrodynamics.
- For Kasner metric, we get two cases: longitudinal Bjorken expansion and a new transverse expansion with longitudinal contraction.
- Relevant for Horava-Lifshitz hydrodynamics, future work.

Summary

- QGP is a phase of QCD which can be created in relativistic heavy-ion collisions.
- The goal is to extract the transport properties of QGP.
- Relativistic hydrodynamics can be applied to study the evolution of QGP.
- First-order (Navier-Stokes) theory leads to violation of causality.
- Second-order (Israel-Stewart) theory restores causality.
- Derivation of Israel-Stewart theory in several ways.
- A third-order evolution equation for shear stress tensor.
- Scaling relations for the ratio of bulk viscosity and shear viscosity.
- Scaling relations for the ratio of conductivity and shear viscosity.

Thank you!

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Backup slide: Exact solution of Boltzmann equation

- Exact solution of BE in RTA in one-dimensional scaling expansion:

$$f(\tau) = D(\tau, \tau_0) f_{\rm in} + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_R(\tau')} D(\tau, \tau') f_0(\tau'),$$

where, $f_{\rm in}$ and τ_0 is the initial distribution function and time, and

$$D(\tau_2, \tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_R(\tau'')}\right]$$

- The damping function, $D(\tau_2, \tau_1)$, has the properties $D(\tau, \tau) = 1$, $D(\tau_3, \tau_2) D(\tau_2, \tau_1) = D(\tau_3, \tau_1)$, and

$$\frac{\partial D(\tau_2, \tau_1)}{\partial \tau_2} = -\frac{D(\tau_2, \tau_1)}{\tau_R(\tau_2)}.$$

- To obtain the exact solution, the Boltzmann relaxation time is taken to be the same as the shear relaxation time ($\tau_R = \tau_\pi$).

Backup slide: Non-local Collision term

- Collision term generalised to include non-local effects by including gradients of $f(x,p)$

$$C[f]_{\rm gen} = C[f] + \partial_\mu \left(A^\mu f\right) + \partial_\mu \partial_\nu \left(B^{\mu\nu} f\right)$$

Where A^μ and $B^{\mu\nu}$ are tensor coefficients of the non-local terms.

- This form of collision term explicitly derived for $2 \leftrightarrow 2$ elastic collision:

$$C[f] = \frac{1}{2}\int dp' dk\, dk'\ W_{pp' \to kk'} \left(f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} - f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} \right)$$

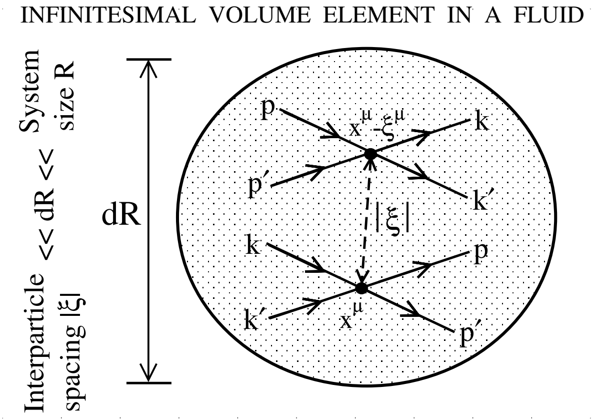
Where, $W_{pp' \to kk'}$ is the transition matrix element and $f_k = f(k,x)$.

- Probability of the process $(kk' \to pp') \propto f_k f_{k'} \tilde{f}_p \tilde{f}_{p'} \leftarrow$ occurs at x

  Probability of the process $(pp' \to kk') \propto f_p f_{p'} \tilde{f}_k \tilde{f}_{k'} \leftarrow$ also occurs at x

[**AJ**, R. S. Bhalerao and S. Pal, PLB 720, 347 (2013)]

Backup slide: Non-local effects



Assumption that the two proccesses ($kk' \to pp'$) and ($pp' \to kk'$) occur at the same space-time point relaxed to include a seperation ξ.

[**AJ**, R. S. Bhalerao and S. Pal, PLB 720, 347 (2013)]

Backup slide: Dissipative equations with non-locality

- Final evolution equations for the dissipative fluxes:

$$\Pi = \tilde{a}\Pi_{\rm NS} - \beta_{\dot{\Pi}}\tau_{\Pi}\dot{\Pi} + \tau_{\Pi n} n\cdot\dot{u} - l_{\Pi n}\partial\cdot n - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi n} n\cdot\nabla\alpha$$
$$+ \lambda_{\Pi\pi}\pi_{\mu\nu}\sigma^{\mu\nu} + \Lambda_{\Pi\dot{u}}\dot{u}\cdot\dot{u} + \Lambda_{\Pi\omega}\omega_{\mu\nu}\omega^{\nu\mu} + (8 \text{ terms}),$$

$$n^{\mu} = \tilde{a}n^{\mu}_{\rm NS} - \beta_{\dot{n}}\tau_{n}\dot{n}^{\langle\mu\rangle} + \lambda_{nn}n_{\nu}\omega^{\nu\mu} - \delta_{nn}n^{\mu}\theta + l_{n\Pi}\nabla^{\mu}\Pi - l_{n\pi}\Delta^{\mu\nu}\partial_{\gamma}\pi^{\gamma}_{\nu}$$
$$- \tau_{n\Pi}\Pi\dot{u}^{\mu} - \tau_{n\pi}\pi^{\mu\nu}\dot{u}_{\nu} + \lambda_{n\pi}n_{\nu}\pi^{\mu\nu} + \lambda_{n\Pi}\Pi n^{\mu} + \Lambda_{n\dot{u}}\omega^{\mu\nu}\dot{u}_{\nu}$$
$$+ \Lambda_{n\omega}\Delta^{\mu}_{\nu}\partial_{\gamma}\omega^{\gamma\nu} + (9 \text{ terms}),$$

$$\pi^{\mu\nu} = \tilde{a}\pi^{\mu\nu}_{\rm NS} - \beta_{\dot{\pi}}\tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle} + \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} + l_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} + \lambda_{\pi\pi}\pi^{\langle\mu}_{\rho}\omega^{\nu\rangle\rho}$$
$$- \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha - \tau_{\pi\pi}\pi^{\langle\mu}_{\rho}\sigma^{\nu\rangle\rho} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \Lambda_{\pi\dot{u}}\dot{u}^{\langle\mu}\dot{u}^{\nu\rangle}$$
$$+ \Lambda_{\pi\omega}\omega^{\langle\mu}_{\rho}\omega^{\nu\rangle\rho} + \chi_1\dot{b}_2\pi^{\mu\nu} + \chi_2\dot{u}^{\langle\mu}\nabla^{\nu\rangle}b_2 + \chi_3\nabla^{\langle\mu}\nabla^{\nu\rangle}b_2.$$

- Where $\tilde{a} = (1-a)$, $\dot{X} = u^{\mu}\partial_{\mu}X$ and "8 terms" ("9 terms") involve second-order, scalar (vector) combinations of derivatives of b_1, b_2.