pQCD, kinetic theory, hydrodynamics and hadronic collisions

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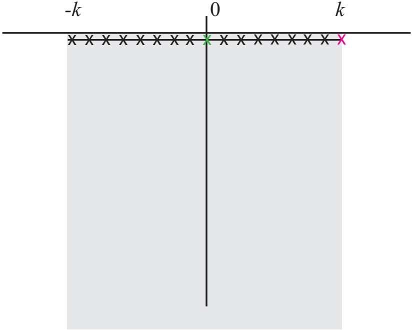
Bengaluru, April 2019

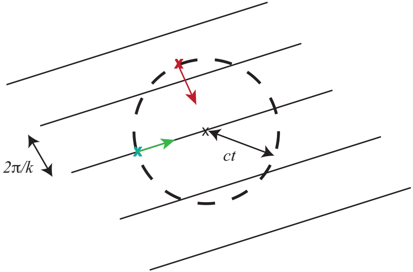
- System in thermal equilibrium T
- Push it out of equilibrium by an external field: δA_μ, $\delta h_{\mu\nu}$

$$\delta A_0(x) = \delta A_0 e^{i\mathbf{k}\cdot\mathbf{x}}$$

$E_z = \partial_z A_0(x)$

- Compute the retarded correlation function $G_J^{\mu\nu} = \langle J^\mu J^\nu \rangle_R = \frac{\delta J^\mu}{\delta A_\nu}$





$$G_J^{0,0}(\omega,k) = -m^2\left[1 + \frac{\omega}{2k}\log\left(\frac{\omega-k}{\omega+k}\right)\right], \quad m^2 = g^2\frac{8\pi}{(2\pi)^3}\int dp p f(p)$$

$\langle J^\mu J^\nu \rangle = \Pi^{\mu\nu}_{HTL}$

What about in interacting kinetic theory?
Simple toy model of interactions, Relaxation Time Approximation

$$\underbrace{p^\mu \partial_\mu f(\mathbf{x},\mathbf{p},t)}_{\text{free streaming}} + \underbrace{F^i \nabla_i^{(p)} f(\mathbf{x},\mathbf{p},t)}_{\text{external force}} = \frac{p^0}{\tau_R}(f - f_{eq})$$

- Interactions try to make distribution fall on equilibrium f_{eq} in a time scale of τ_R

Linearize and Fourier transfrom:

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v})\delta f + \frac{1}{p}F^i\nabla_i^{(p)} f = -\frac{1}{\tau_R}(\delta f - \delta f_{eq})$$

- δf_{eq}: As the distribution has changed, so have the conserved quantities *locally*. The equilibrium to which the system wants to relax differs at different points in space.

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In diffusion channel $G_J^{0,0} = \frac{\delta J^0}{\delta A_0}$:

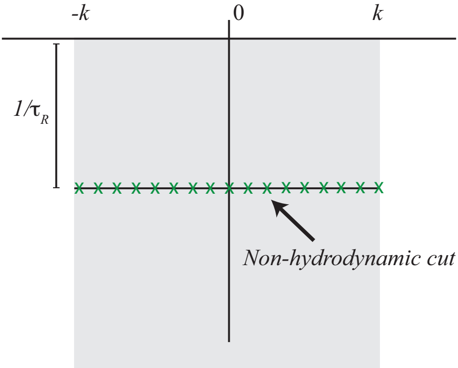
$$f_{eq} = f_{eq}^g + \delta f_{eq} = e^{\frac{u\cdot p - \delta\mu}{T}} = e^{-\beta p} + \frac{\delta\mu}{T_0}e^{-\beta p}\dots$$

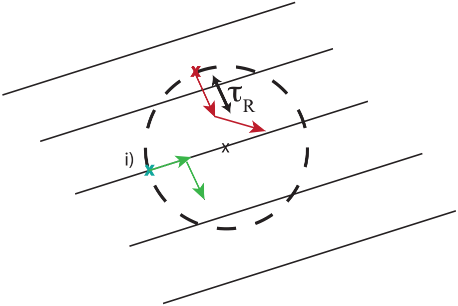
$$n + \delta n = \int_p (f_{eq}^g + \delta f_{eq}) = n_0 + \frac{\delta\mu}{T_0}n_0$$

- $\delta\mu$ depends on δn depends on δf depends on $\delta\mu$...

similarly for δT, δu

Kinetic theory, relaxation time model (large $\tau_R k \gg 1$)





The particles get deviated in time τ, the cut gets an imaginary part

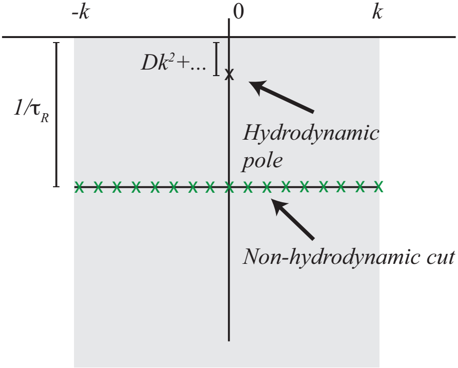
$$\delta f = -i \frac{F^i \nabla_i^{(p)} f_{eq}^g + \frac{1}{\tau_R} f_{eq} \frac{\delta n}{n}}{-\omega + \mathbf{v} \cdot \mathbf{k} - i/\tau_R}$$

Romatshke EPJ. C76 (2016)

Because of the particle number conservation δf contains δn which is a function of δf. Solve self consistently:

$$\delta n = -i \int \frac{d^3p}{(2\pi)^3} \frac{F^i \nabla_i^{(p)} f_{eq}^g}{-\omega + \mathbf{v}\cdot\mathbf{k} - i/\tau_R} - i\delta n \int \frac{d^3p}{(2\pi)^3} \frac{\frac{1}{\tau_R} f_{eq} \frac{1}{n}}{-\omega + \mathbf{v}\cdot\mathbf{k} - i/\tau_R}$$

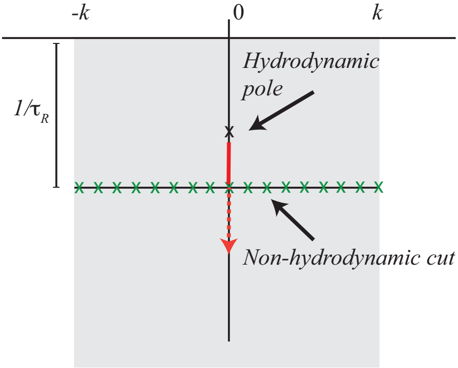
Kinetic theory, relaxation time model (small $\tau_R k \ll 1$)



A new pole appears

$$\delta n = \frac{-i \int \frac{d^3p}{(2\pi)^3} \frac{F^i \nabla_i^{(p)} f_{eq}^g}{-\omega + \mathbf{v}\cdot\mathbf{k} - i/\tau_R}}{1 + i \int \frac{d^3p}{(2\pi)^3} \frac{\frac{1}{\tau_R} f_{eq} \frac{1}{n}}{-\omega + \mathbf{v}\cdot\mathbf{k} - i/\tau_R}}$$

Kinetic theory, relaxation time model ($t_R k \sim 1$)



Pole moves to the second Riemann sheet when crosses the cut.

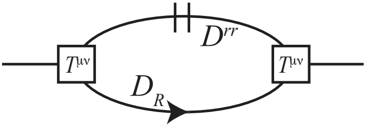
For those k, where pole exists, late time behaviour hydrodynamical

"hydrodynamical transition" for $k \sim 1/\tau_R$

Romatshke EPJ. C76 (2016)

What about in field theory?

Free QFT, calculation:



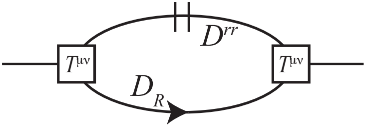
$$G_R(t,\mathbf{k}) \sim \int_p V(p,k) D_R(t,\mathbf{p}-\mathbf{k}) D^{rr}(t,\mathbf{p}),$$

$$\sim \int_p \frac{-iV(p,k)}{2E_{p-k}E_p}\theta(t)\left(e^{iE_{p-k}t} - e^{-iE_{p-k}t}\right)$$

$$[\frac{1}{2} + n(E_p)]\left(e^{iE_pt} + e^{-iE_pt}\right) \ ,$$

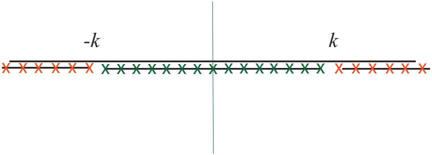
Arnold, Yaffe PRD **57** (1998) 1178

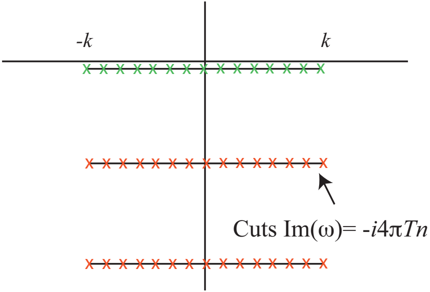
Free QFT, calculation:



$$
\begin{aligned}
G_R(t,\mathbf{k}) &\sim \int_p V(p,k) D_R(t,\mathbf{p}-\mathbf{k}) D^{rr}(t,\mathbf{p}), \\
&\sim \int_p \frac{-iV(p,k)}{2E_{p-k}E_p}\theta(t)[\frac{1}{2}+n(E_p)] \\
&\qquad \times \Big(e^{i(E_{p-k}+E_p)t} - e^{-i(E_{p-k}+E_p)t} \\
&\qquad\quad e^{i(E_{p-k}-E_p)t} + e^{-i(E_{p-k}-E_p)t} \Big) \\
&\equiv C(t,\mathbf{k}) + D(t,\mathbf{k})
\end{aligned}
$$

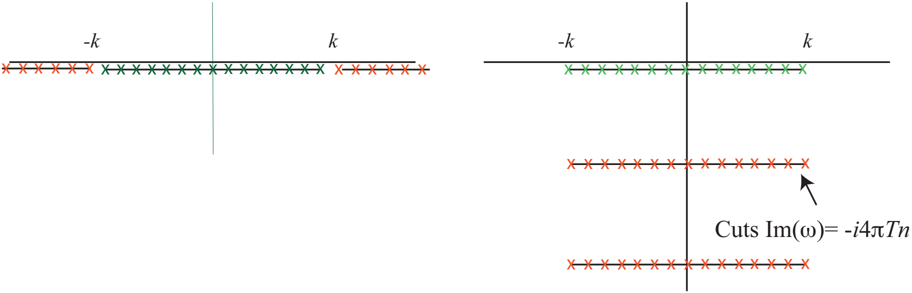
Arnold, Yaffe PRD **57** (1998) 1178





Rapidly oscillating:

$$D(\omega, k) \sim \int_p \frac{-iV(p,k)}{2E_{p-k}E_p} \left(\frac{1}{2} + n(E_p)\right) \times \left[\frac{E_p + E_{p-k}}{(E_p + E_{p-k})^2 - \omega^2}\right]$$



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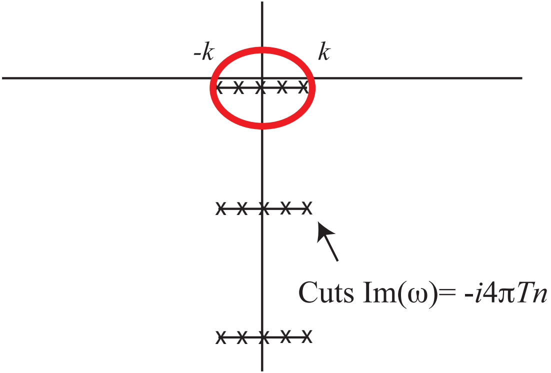
$$D(\omega,k) \sim \int_p \frac{-iV(p,k)}{2E_{p-k}E_p}\left(\frac{1}{2}+n(E_p)\right) \times \left[\frac{E_p+E_{p-k}}{(E_p+E_{p-k})^2-\omega^2}\right]$$

Can be analytically continued:

$k = 0$

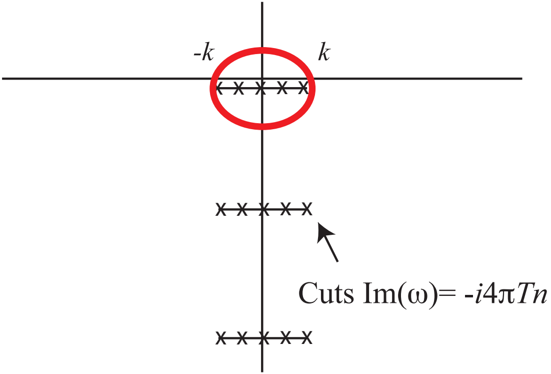
$$n(p)+\frac{1}{2} = \sum_{n=-\infty}^{\infty} \frac{\beta p}{(2\pi n)^2+(\beta p)^2}$$

$$\int dp \frac{\beta p}{(2\pi n)^2+(\beta p)^2}\frac{2p}{(2p)^2-\omega^2} = \frac{\pi}{2}\frac{1}{4\pi n - i\beta\omega}$$



For $\omega, k \ll T$, the correlation functions are described by kinetic theory

$$C(\omega, k) \sim \int_p \frac{-iV(p,k)}{2E_{p-k}E_p} n(E_p) \left[\frac{E_p - E_{p-k}}{(E_p - E_{p-k})^2 - \omega^2} \right]$$
$$\sim \int_p \frac{V(p,0)}{E_p^2} n(E_p) \left[\frac{1}{i\omega - i\mathbf{v}\cdot\mathbf{k}} \right]$$



For $\omega, k \ll T$, the correlation functions are described by kinetic theory

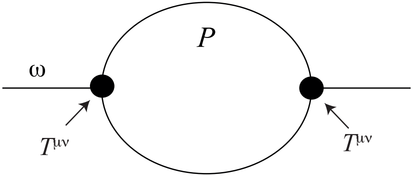
To see details at the $\omega \sim 1/t_{\rm scat}$, interactions in the kinetic theory must be taken into account within kinetic theory.

Why kinetic theory:

$$G(\omega = 0, k = 0) \sim \int_P f[1+f]\rho(P)\rho(P)$$

- Free spectral function: $\rho_{\rm free} \sim 2\pi\delta(P^2)$

- Divergent: $G_R \sim \int_p f(1+f)\delta(P^2)\delta(P^2)$



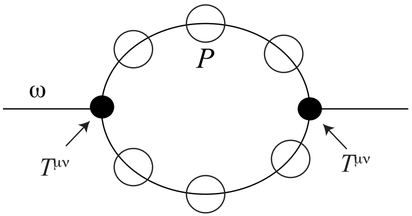
Jeon PRD47 (1993)

Why kinetic theory:

$$G(\omega = 0, k = 0) \sim \int_P f[1 + f]\rho(P)\rho(P)$$

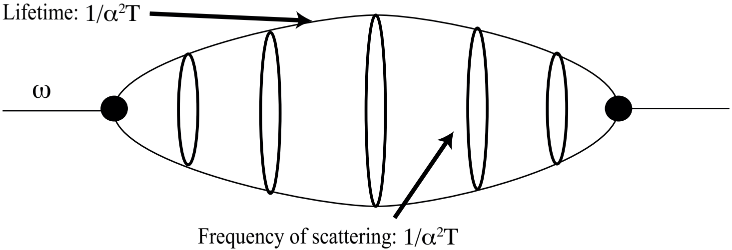
- Resummed spectral function: $\rho_{\mathrm{resum}} \sim \frac{p^0\Gamma}{(P^2)^2+\Gamma^2(p^0)^2}$, $\Gamma \sim \frac{1}{\alpha^2 T}$

- Finite but large: $G \sim \int_P f(1+f)\left[\frac{p^0\Gamma}{(P^2)^2+\Gamma^2(p^0)^2}\right]^2 \sim \frac{T^5}{\Gamma}$



Jeon PRD47 (1993)

More resummations needed!



Both lines long lived $(\alpha^2 T)^{-1}$, of the order of scattering time

Complicated resummation can be dressed in form of an effective kinetic theory:

- Diagrammatic resummation (in $\lambda\phi^4$)

  Jeon PRD52 (1995)

- Interpretation of the diagrammatic resummation in terms of effective kinetic theory

  Jeon, Yaffe PRD53 (1996)

- Generalization to gauge theories through power counting, pQCD effective kinetic theory

  Arnold et al. JHEP 0301 (2003) 030

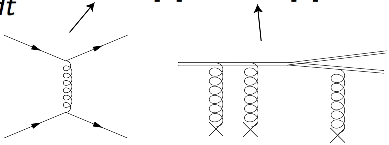
pQCD kinetic theory

QCD Effective kinetic theory

Arnold et al. JHEP 0301 (2003)

- QFT → Transport theory → fluid dynamics

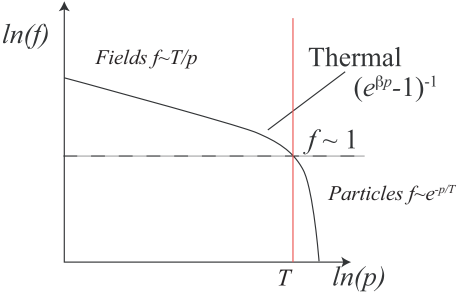
$$\frac{df}{dt} = -C_{2\leftrightarrow 2}[f] - C_{1\leftrightarrow 2}[f]$$



- Soft and collinear divergences lead to nontrivial matrix elements
  soft: screening, Hard-loop; collinear: LPM, ladder resum = Schrödinger equiation
- Only free parameter α_s; LO accurate in the $\alpha_s \to 0$, $\alpha_s f \to 0$ limit.

Scales in thermal QCD

Free theory in thermal equilibrium T:



- For bosons the distribution function $n_B(p) = \frac{1}{e^{\beta p}-1}$
  - for $p \gg T$, classical particles

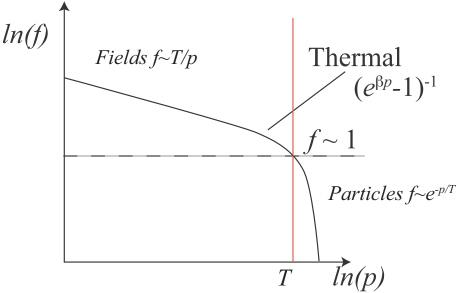
$$n_B(p) \approx e^{-\beta p}$$

  - for $p \ll T$, classical fields

$$n_B(p) \sim T/p$$

Scales in thermal QCD

Free theory in thermal equilibrium T:



- Most quantities dominated by particles at the scale T:
  - Energy density

$$\epsilon \sim \int \frac{d^3p}{(2\pi)^3 f} \sim T^4$$

  - Number density

$$n \sim \int \frac{d^3p}{(2\pi)^3} f$$

  - Interparticle distance

$$\Delta x \sim n^{1/3} \sim T$$

- mean free path $\lambda_{mfp} \sim \frac{1}{\alpha^2 T}$

  Wave packets overlap, but collisions can be treated separately

Interaction scales:

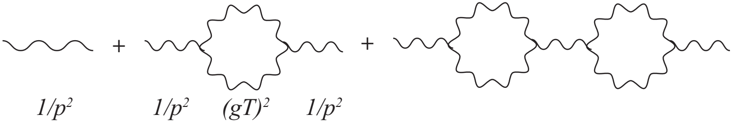
Interactions rise from $D_\mu = \partial_\mu + igA_\mu$

In medium, there are always statistical field fluctuations:

$$\underbrace{\langle A(\mathbf{x},t)A(\mathbf{x},t)\rangle}_{\int_p a^\dagger a} \sim \int_p \frac{1}{p}(\frac{1}{2} + f(p)) \sim T^2$$

For typical modes: $p \sim T, \quad p + gA \sim T + gT$

- Interactions with medium lead to small modifications of dispersion



- modes with $p \sim T$, get a small thermal mass $p^2 + m^2$ from interaction with the medium $m^2 \sim g^2 \int_p \frac{f(p)}{p}$

Interaction scales:

For *soft modes:* $p \sim gT$, $p + gA \sim gT + gT$

- Non-perturbative interaction with typical modes at scale T
- Interactions among soft modes still perturbative

$$\langle A(\mathbf{x},t)A(\mathbf{x},t)\rangle_{gT} \sim \int_p^{gT} \frac{f_p}{p} \sim g^3T^3\frac{T}{g^2T^2} \sim gT^2 \tag{1}$$

so that $\underbrace{p}_{gT} + ig\underbrace{A_{soft}}_{g^{3/2}T}$

- but the expansion parameters if now only g

- Soft modes are classical fields $f \gg 1$, can use classical methods.
- The wavelength of the typical modes is $\frac{1}{p} \gg \frac{1}{T}$, the hard modes appear as classical particles to soft modes
  - At time scales of interest $1/gT$, the hard particles don't interact $\lambda_{mfp} \sim \frac{1}{g^4 T}$.
  - In linear level, QCD is like QED and the interaction between the soft and the hard modes is given by the linear response

$$J^{\mu}_{a}(\omega, k) = G^{\mu\nu}_{J,ab}(\omega, k) A^{b}_{\nu}(\omega, k)$$

+ a delta function in color

Interactions scales

ultrasoft modes: $p \sim g^2 T$

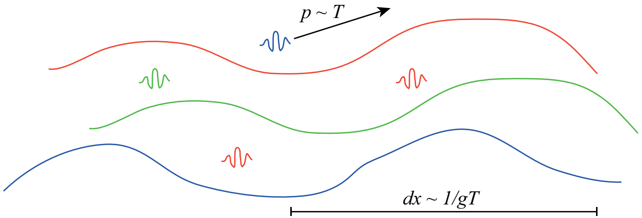
Interaction even among ultrasoft modes is nonperturbative

$$\langle A(\mathbf{x},t)A(\mathbf{x},t)\rangle_{g^2T} \sim \int_p^{g^2T} \frac{f_p}{p} \sim g^6 T^3 \frac{T}{g^4 T^2} \sim g^2 T^2$$

so that $\underbrace{p}_{g^2T} + ig\underbrace{A_{soft}}_{g^2T}$

No expansion parameter, but classical fields

Scales in thermal equilibrium at weak coupling

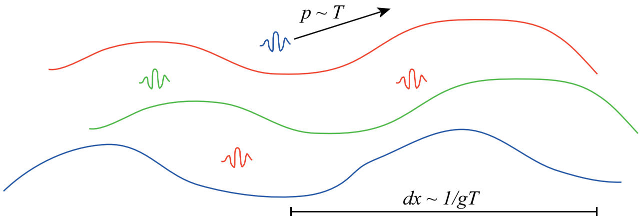


Degrees of freedom at weak coupling $g \ll 1$:

- Hard *particle* modes $p \sim T$: kinetic theory
- Soft (bosonic) *field* modes $p \sim m \sim gT$: classical field theory

$$n_B(p) = \frac{1}{e^{\omega/T} - 1} \sim \frac{T}{\omega} \sim \frac{1}{g}$$

Scales in thermal equilibrium at weak coupling



- Soft fields evolve according to classical nonabelian field equations

$$D_\mu F^{\mu\nu}(t,\mathbf{x}) = J^\mu(t,\mathbf{x})$$

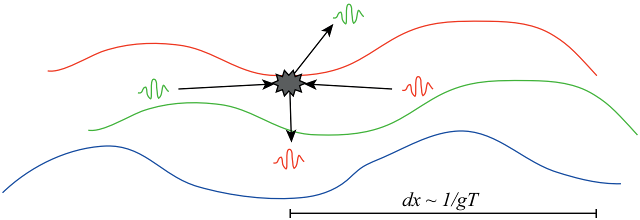
- Hard modes see soft fields as classical fields: Hard loop theory

$$J^{\mu,a}_{ind.}(t) = \delta^{ab}\int dt' G^{HL}_{\mu\nu}(t,t')\,\delta A^\nu_b(t'), \quad G^{HL}_{\mu\nu} \sim g^2 T^2$$

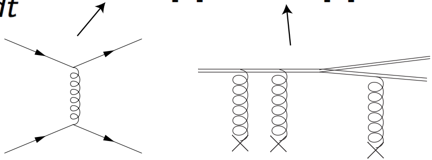
Blaizot, Iancu, Phys.Rept. 359 (2002) 355-528

Kinetic theory of hard modes

Arnold, Moore, Yaffe JHEP 0301 (2003) 030



$$\frac{df}{dt} = -C_{2\leftrightarrow 2}[f] - C_{1\leftrightarrow 2}[f]$$



- pQCD kinetic theory for the hard modes

Contribution from different scales:

- In principle, all the aspects are present in any perturbative calculation
- In practice, to low orders in g for some quantities, one may get away just from looking the hard (or soft, ultrasoft) sectors