Hydrodynamic attractors

Michal P. Heller

aei.mpg.de/GQFI

1610.02023: lecture notes emphasizing the holographic component

1707.02282: review emphasizing the hydrodynamic component

1609.04803v2 and **1802.08225:** more recent results

1906.xxxxx upcoming work with Jefferson, Spalinski & Svensson

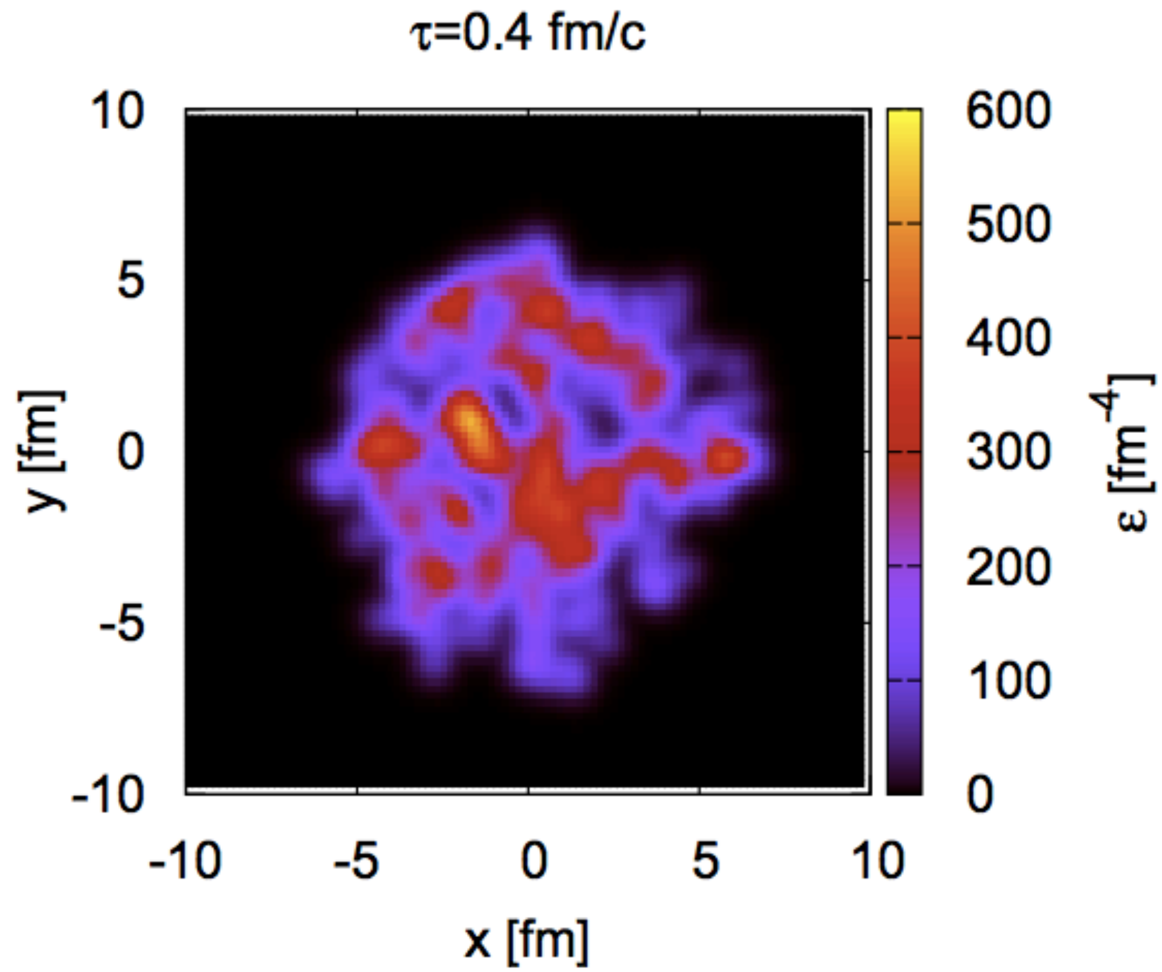
Introduction & motivation

// in this lecture vast majority of results concern conformally-invariant theories //

Motivation: hydrodynamic simulations for HIC

Generic energy density at the moment hydrodynamics simulation starts:

1009.3244 by Schenke, Jeon & Gale

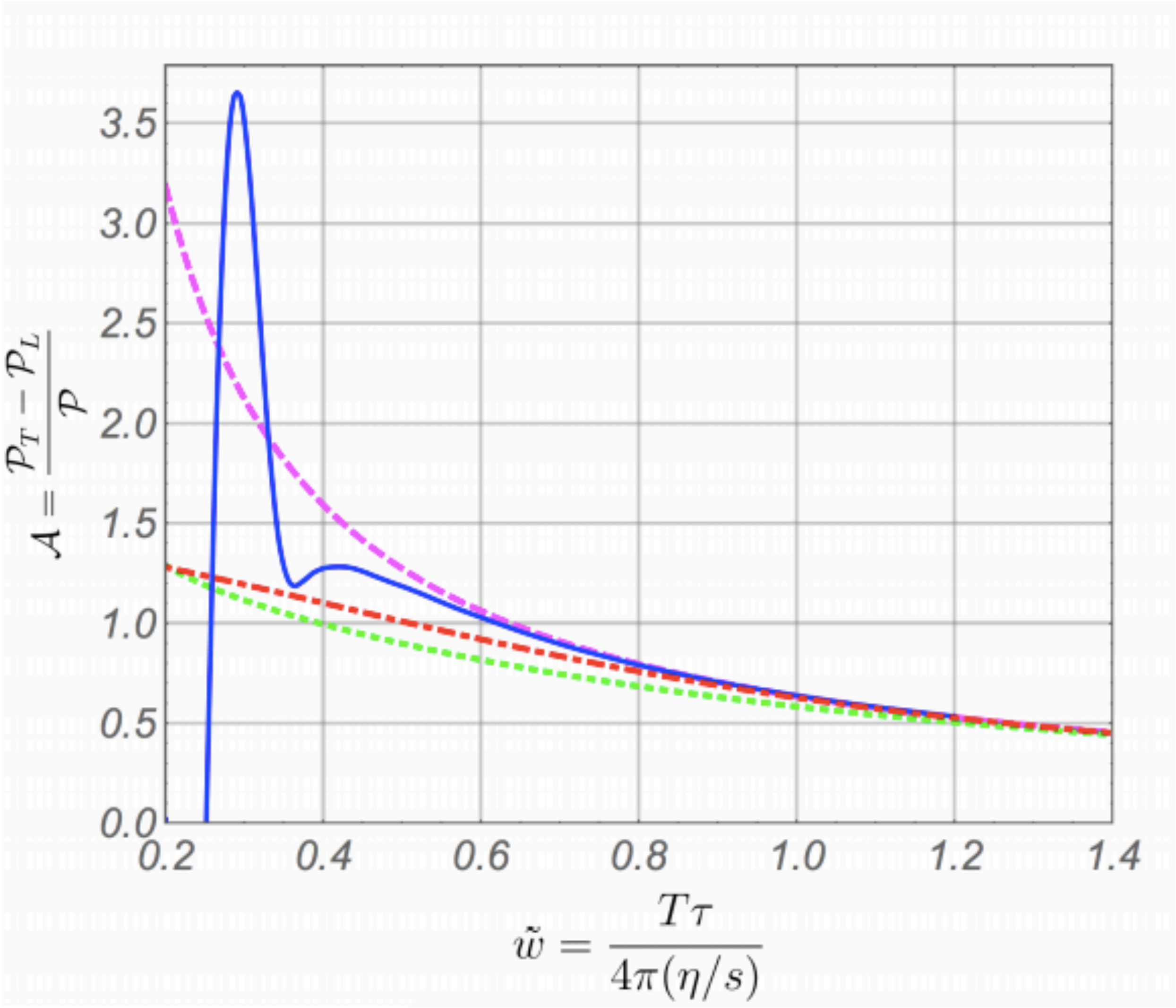


Simulations like this match the data, however, they apply hydrodynamics in the regime of large gradients: $\frac{\Delta\mathcal{E}}{\Delta d} \times \left(\frac{\mathcal{E}_{\rm av}}{(\frac{1}{\mathcal{E}_{\rm av}/10})^{1/4}} \right)^{-1} \approx 1$. Does it even make sense?

Hydrodynamization

Ab initio studies in holography and later studies in other models show that viscous hydro can work even when deviations from local equilibrium are large:

0906.4426, 1011.3562 by Chesler & Yaffe; **1103.3452** with Janik & Witaszczyk



sample n-eq states in:

N=4 SYM

EKT with $\eta/s = 0.624$

RTA with $\eta/s = 0.624$

viscous hydro prediction:

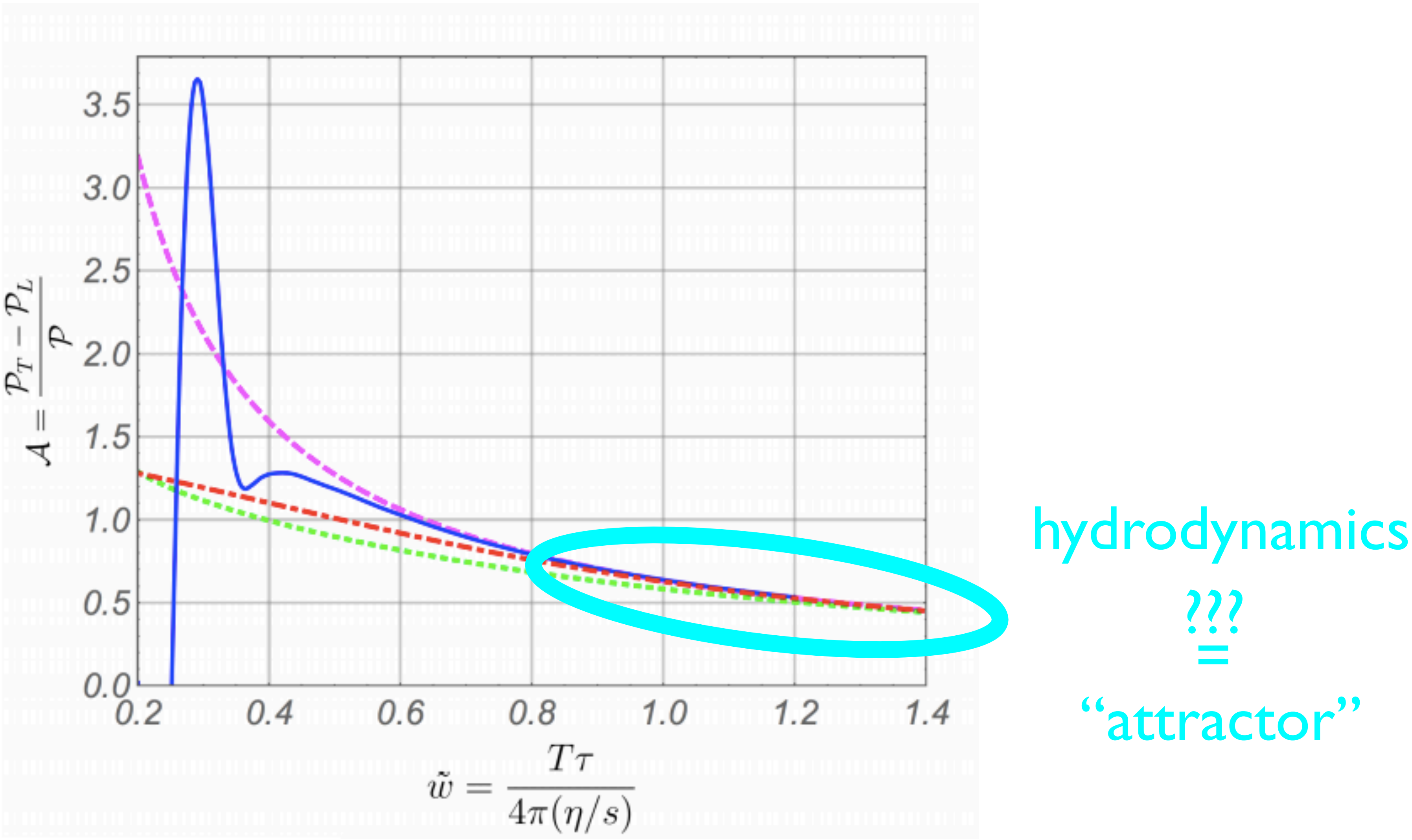
$$\frac{\Delta\mathcal{P}}{\mathcal{E}/3} = \frac{2}{\pi}\tilde{w}^{-1}$$

plot from **1609.04803v2** with Kurkela, Spalinski & Svensson

Preview: hydrodynamic attractors

Viscous hydrodynamics works despite huge gradients in the system:

0906.4426, 1011.3562 by Chesler & Yaffe; **1103.3452** with Janik & Witaszczyk



plot from **1609.04803v2** with Kurkela, Spalinski & Svensson

Relativistic hydrodynamics - textbook definition

hydrodynamics is an EFT of the slow (?) evolution of conserved currents in collective media close to equilibrium (?)

DOFs: always local energy density ϵ and local flow velocity u^μ $(u_\nu u^\nu = -1)$

EOMs: conservation eqns $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ for $\langle T^{\mu\nu} \rangle$ expanded in gradients

$\pi^{\mu\nu}$

$$\langle T^{\mu\nu} \rangle = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\, \sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla \cdot u) + \ldots$$

microscopic input:

EoS ($P(\epsilon) = \frac{1}{3}\epsilon$ for CFTs)

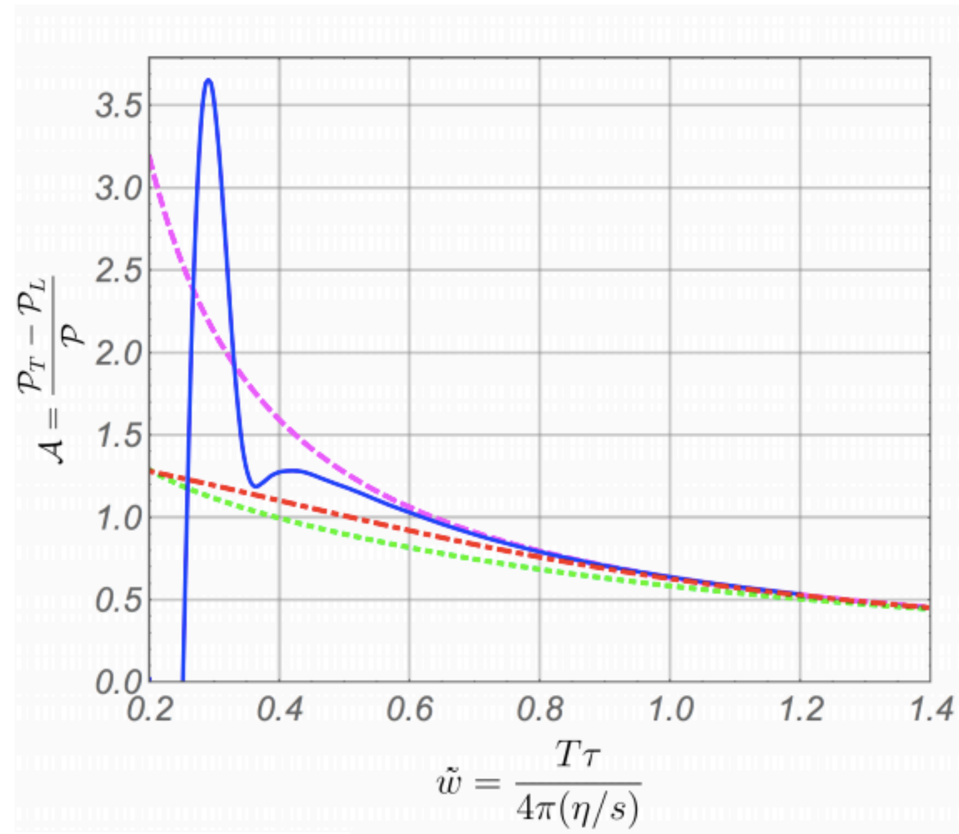
shear viscosity contribution

$$\frac{\Delta \mathcal{P}}{\mathcal{E}/3} = \frac{2}{\pi} \tilde{w}^{-1}$$

bulk viscosity (vanishes for CFTs)

This talk will be a success if:

You understand better this and similar plots:



You get an idea about recent developments on what hides here

$$\langle T^{\mu\nu}\rangle = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} - \eta(\epsilon)\,\sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla\cdot u) + \ldots$$

Hydrodynamic & transient modes

Theories of viscous hydrodynamics

The crucial subtlety: $\nabla_\mu\Big(\,\epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu\,\} - \eta(\epsilon)\,\sigma^{\mu\nu} + ..\Big) = 0$ does not have a well-posed initial value problem ⟶ hydrodynamic theories

Overall idea: make $\pi^{\mu\nu}$ obey an independent PDE ensuring its ↘ to $-\eta\,\sigma^{\mu\nu}$

$$(\tau_\pi\, u^\alpha \mathcal{D}_\alpha + 1)\;[\pi^{\mu\nu} - (-\eta\,\sigma^{\mu\nu})] = 0 \longrightarrow \pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha\, \pi^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha\,(\eta\,\sigma^{\mu\nu})$$

decay timescale

Müller 1967, Israel 1976, Israel & Stewart 1976

New incarnation: **0712.2451** by Baier, Romatschke, Son, Starinets & Stephanov

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha \pi^{\mu\nu} + \lambda_1 \pi^{\langle\mu}{}_\alpha \pi^{\nu\rangle\alpha} + \lambda_2 \pi^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha}$$

Modes in BRSSS theory

Mode = solution of linearized equations of finite-T state without any sources

Technical issue: tensor perturbs.⟶ channels (**here and later sound channel**):

Assuming momentum along x³ direction $e^{-i\,\omega\,x^0 + i\,k\,x^3}$: δT, δu^3 & $\delta\pi^{33}$

conservation

+

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha \pi^{\mu\nu} + \lambda_1 \pi^{\langle\mu}{}_\alpha{}^{\nu\rangle\alpha} + \lambda_2 \pi^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha}$$

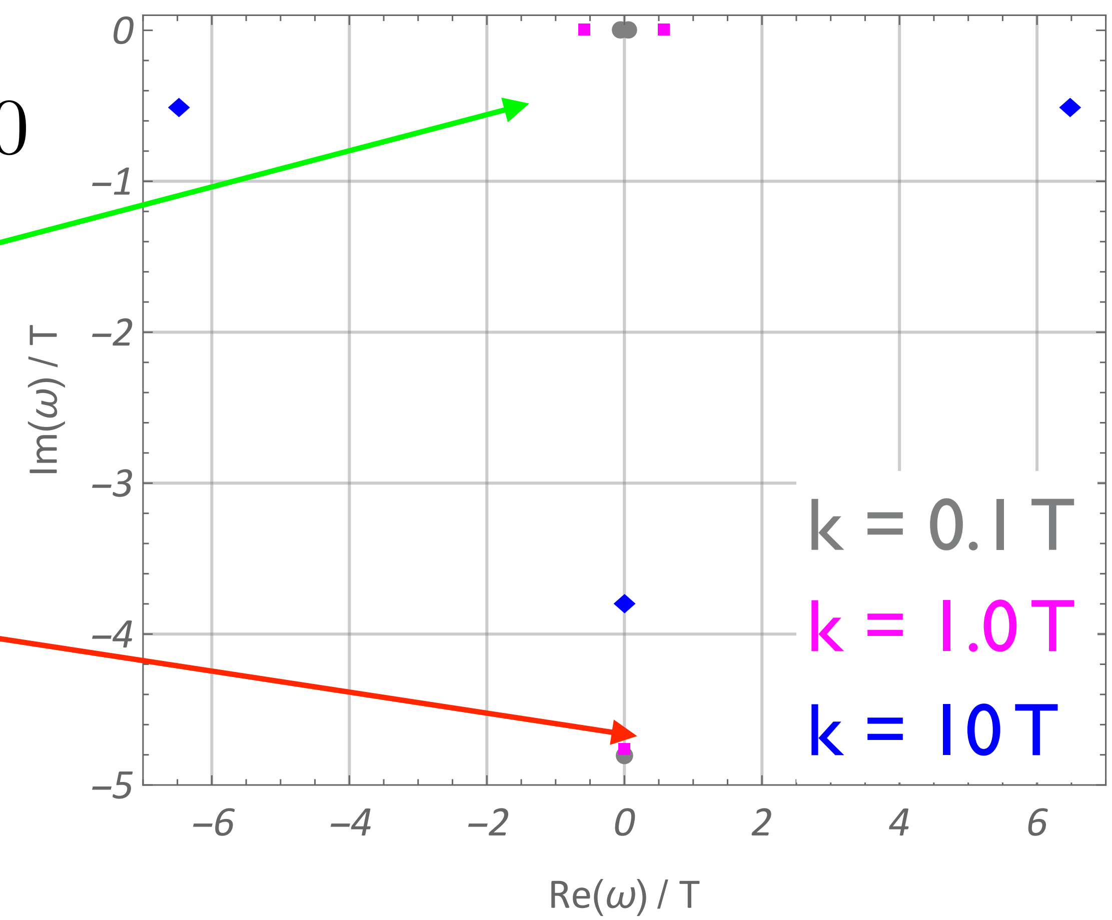
$$\omega^3 + (\ldots)\,\omega^2 + (\ldots)\omega + (\ldots) = 0$$

two modes:

hydro (sound wave)

transient (pure decay):

$$\omega\Big|_{k=0} = -i\,\frac{1}{\tau_\pi}$$



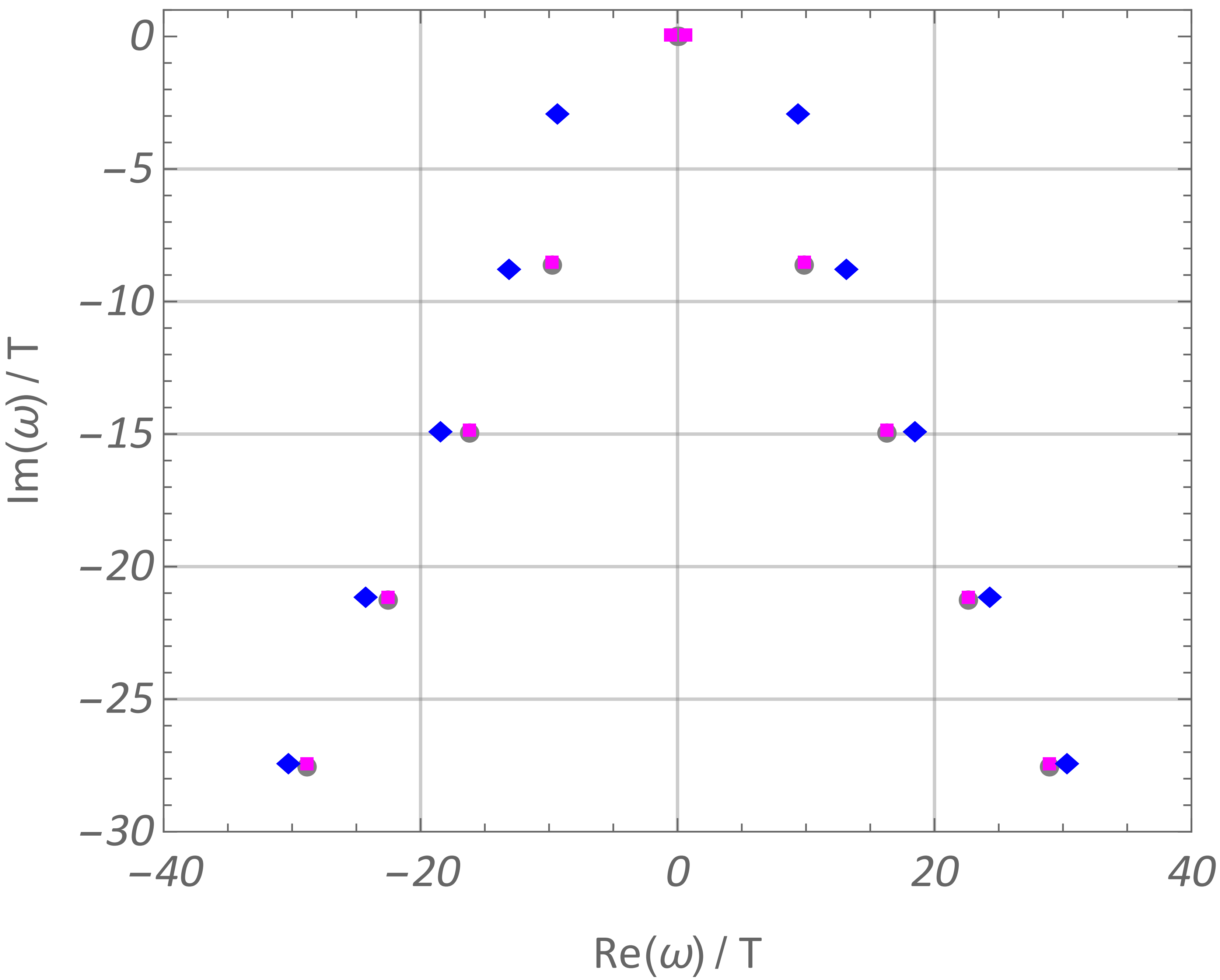
Modes in Einstein-Hilbert holography = QNMs

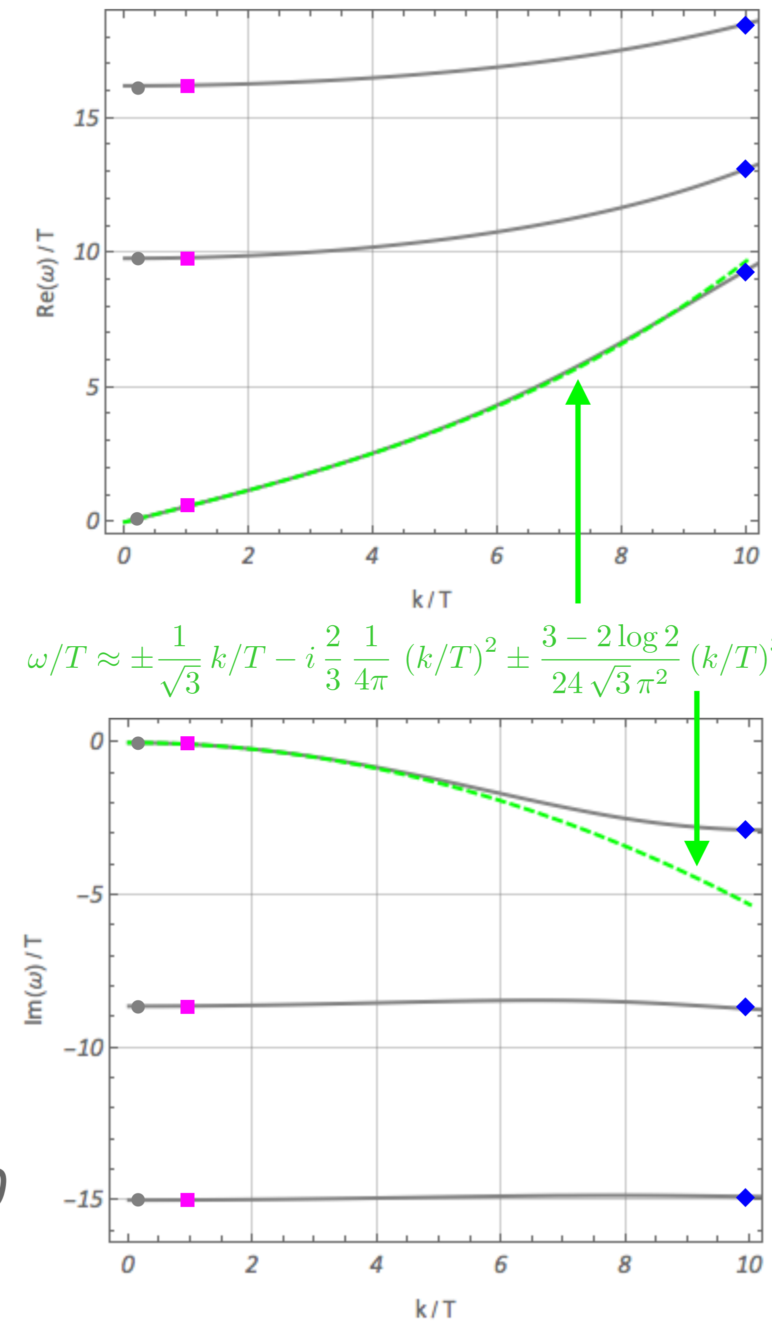
hep-th/0506184 by Kovtun & Starinets

$$ds^2 = \frac{L^2}{u^2}\left\{-2dx^0du - (1-\pi^4T^4u^4)\left(x^0\right)^2 + d\vec{x}^2\right\} + \delta g_{ab}(u)\, e^{-i\,\omega\, x^0 + i\,k\,x^3}$$

vanishing at the boundary

ingoing (regular) at the horizon



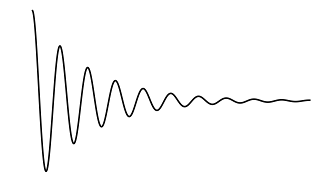


$$\omega/T \approx \pm\frac{1}{\sqrt{3}}\,k/T - i\,\frac{2}{3}\,\frac{1}{4\pi}\,(k/T)^2 \pm \frac{3-2\log 2}{24\sqrt{3}\,\pi^2}\,(k/T)^3$$

HJSW theory and its modes

1409.5087 with Janik, Spalinski & Witaszczyk (see also **1104.2415** by Noronha & Denicol)

MIS/BRSSS idea: $\pi^{\mu\nu}$ decays exponentially to $-\eta\,\sigma^{\mu\nu}$. In holography:



HJSW: relaxation-type eqn. ⟶ damped harmonic oscillator-type eqn. for $\pi^{\mu\nu}$:

$$\left\{\left(\frac{1}{T}\mathcal{D}\right)^2 + 2\,\Omega_I\,\frac{1}{T}\mathcal{D} + |\Omega|^2\right\}\pi^{\mu\nu} = \eta\,|\Omega|^2\,\sigma^{\mu\nu} - c_\sigma\,\frac{1}{T}\mathcal{D}\left(\eta\,\sigma^{\mu\nu}\right) + \ldots \quad \text{with} \quad \frac{1}{T}\,\omega^1_{QNM}\big|_{k=0} = \pm\Omega_R + i\,\Omega_I$$

↓ linearization

$$\omega^4 + (\ldots)\,\omega^3 + (\ldots)\,\omega^2 + (\ldots)\,\omega + (\ldots) = 0$$

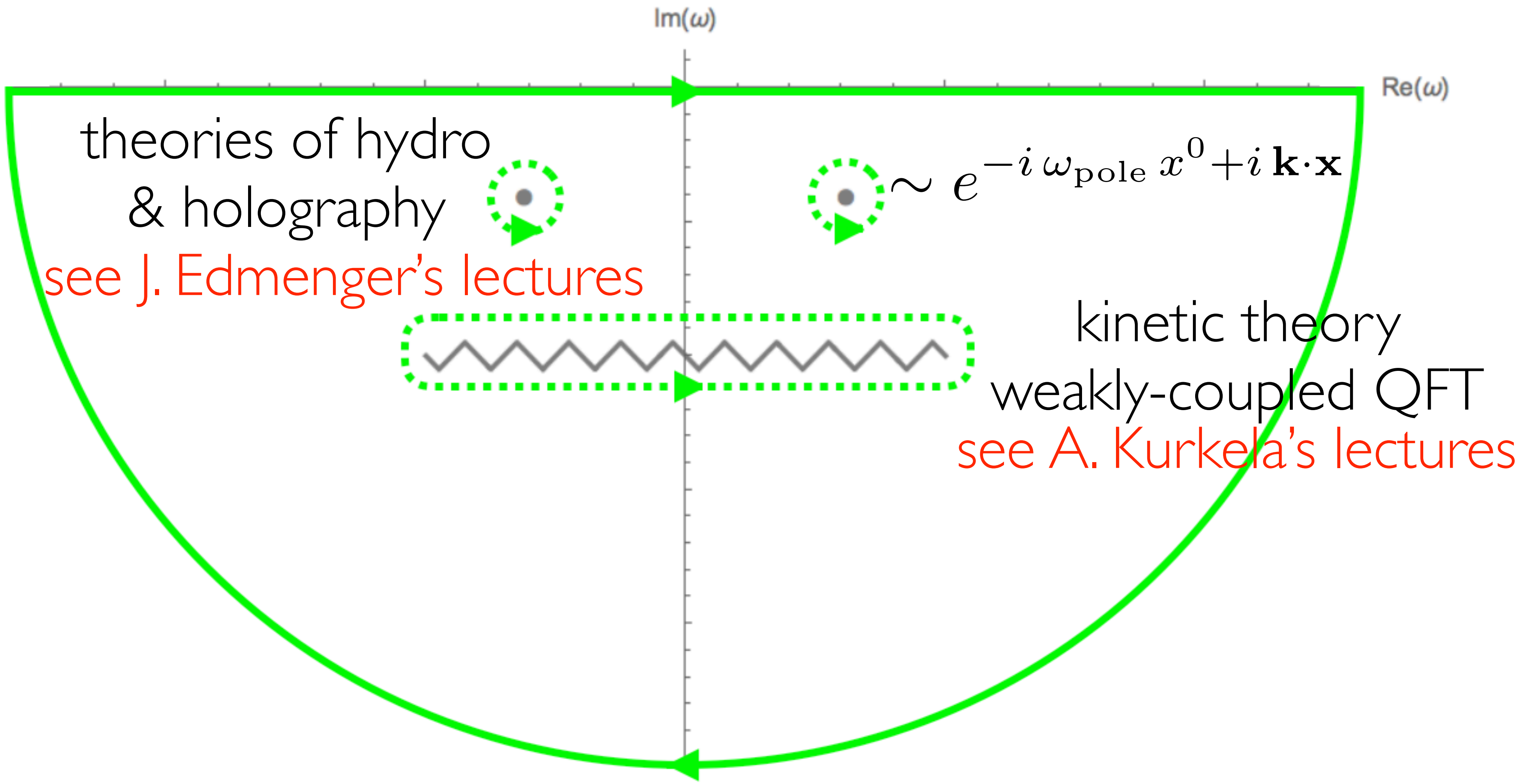
→ **hydrodynamics (sound wave)**

→ **transient (decay + oscillation)**

Tested using holography ✓ (one need to specify not only $\pi^{\mu\nu}$ but also $\partial_0\,\pi^{\mu\nu}$)

Modes in linear response theory

$$\delta\langle\hat{T}^{\mu\nu}\rangle(x) = -\frac{1}{2\times(2\pi)^4}\int d^3k \int d\omega\, e^{-i\,\omega\, x^0 + i\,\mathbf{k}\cdot\mathbf{x}}\, G_R^{\mu\nu,\,\alpha\beta}(\omega,\mathbf{k})\,\delta g_{\alpha\beta}(\omega,\mathbf{k})$$

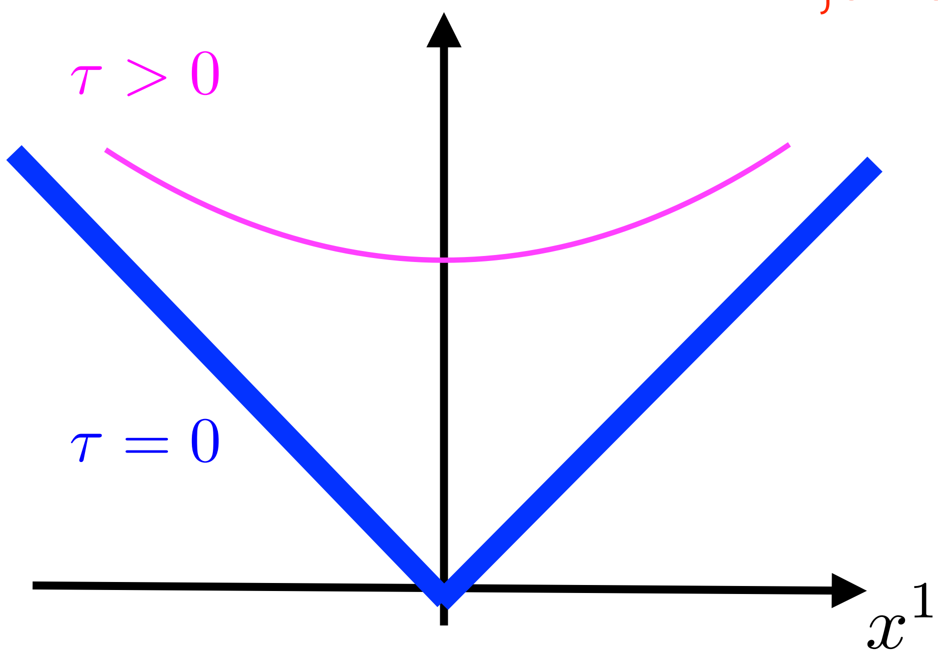


Hydrodynamics at large orders

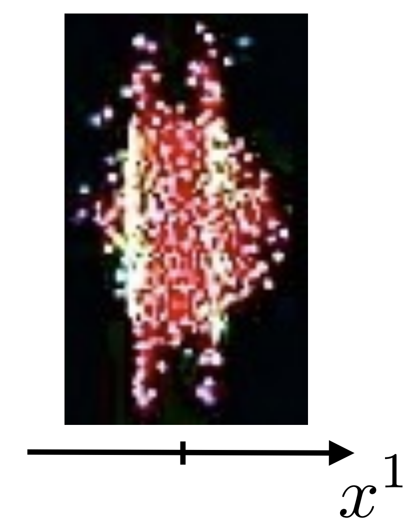
1503.07514 with Spalinski

1302.0697 with Janik & Witaszczyk

Boost-invariant flow Bjorken 1982



const x^0 slice:



Boost-invariance: in $\left(\tau \equiv \sqrt{x_0^2 - x_1^2}, \quad y \equiv \operatorname{arctanh}\frac{x_1}{x_0}, \, x_2, \, x_3\right)$ coords no y-dep

In a CFT: $\langle T^\mu_{\ \nu}\rangle = \operatorname{diag}\left\{-\mathcal{E}(\tau), -\mathcal{E} - \tau\,\dot{\mathcal{E}}, \, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}}, \, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}}\right\}$

$$\langle T^2_{\ 2}\rangle - \langle T^y_{\ y}\rangle \equiv \Delta\mathcal{P} \qquad\qquad \left(\frac{\mathcal{E}(\tau)}{\frac{3}{8}\pi^2 N_c^2}\right)^{1/4} \equiv T$$

and via scale-invariance $\dfrac{\Delta\mathcal{P}}{\mathcal{E}/3} \equiv \boxed{\mathcal{A}}$ is a function of $\boxed{w} \equiv \tau\, T$

Gradient expansion: series in $\dfrac{1}{w}$ **1103.3452** with Janik & Witaszczyk

Large order gradient expansion: BRSSS

1503.07514 with Spalinski

conservation (always the same) ⟶ $\frac{\tau}{w}\frac{dw}{d\tau} = \frac{2}{3} + \frac{1}{18}\mathcal{A}$

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha \pi^{\mu\nu} +$$
$$+\lambda_1 \pi^{\langle\mu}{}_\alpha \pi^{\nu\rangle\alpha} + \lambda_2 \pi^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha}$$

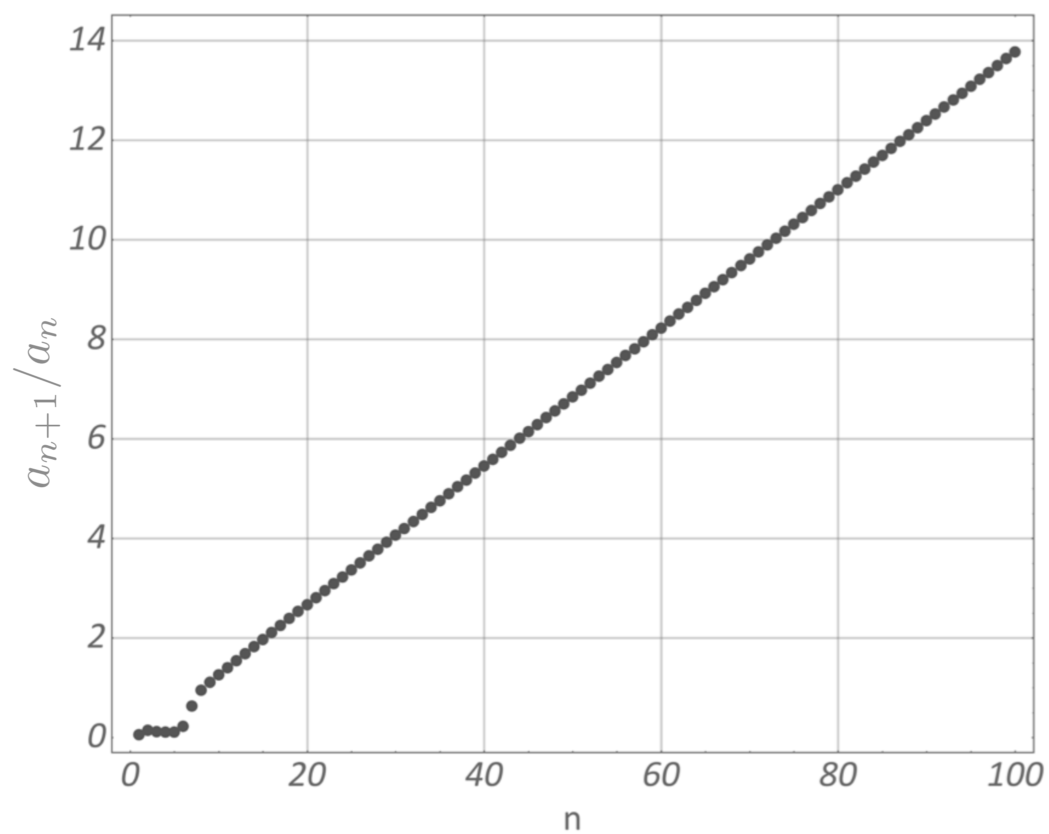
$$\rightarrow \quad C_{\tau_\pi} w\,(1+\frac{1}{12}\mathcal{A})\,\mathcal{A}' + \left(\frac{1}{3}C_{\tau_\pi} + \frac{1}{8}\frac{C_{\lambda_1}}{C_\eta}w\right)\mathcal{A}^2 + \frac{3}{2}w\,\mathcal{A} - 12\,C_\eta = 0$$

$$\left(\eta = C_\eta\,\mathcal{S}, \qquad \tau_\pi = \frac{C_{\tau_\pi}}{T}, \qquad \lambda_1 = C_{\lambda_1}\frac{\eta}{T}\right)$$

(with $C_\eta = \frac{1}{4\pi}$, $C_{\tau_\pi} = \frac{2-\log 2}{2\pi}$, $C_{\lambda_1} = \frac{1}{2\pi}$)

$$\mathcal{A}(w) \approx \sum_{n=1}^{\infty}\frac{a_n}{w^n} = 8\,C_\eta\frac{1}{w} + \frac{16}{3}C_\eta\,(C_{\tau_\pi} - C_{\lambda_1})\frac{1}{w^2} + \ldots$$

(note that a_n's do not depend on ini. cond.)



Divergent series: $a_n \sim n!$

Hydrodynamics and transient modes: BRSSS

1503.07514 with Spalinski

Key observations:

- $\sum_{n=1}^{\infty} \frac{a_n}{w^n}$ does not make sense without a resummation
- there must be sth else that cares about ini. cond.

resurgence ↕

When we linearize our eom on top of $\sum_{n=1}^{\infty} \frac{a_n}{w^n}$ we get:

integration const. (ini. cond.) → σ

further hydro dressing (another div. series) → $\{\dots\}$

$$\delta \mathcal{A} = \sigma \, e^{-\frac{3}{2} \, \frac{1}{C_{\tau\pi}} \, w} \, w^{\frac{C_\eta - 2\, C_{\lambda_1}}{C_{\tau\pi}}} \left\{ 1 + \sum_{j=1}^{\infty} \frac{a_j^{(1)}}{w^j} \right\}$$

In equilibrium one has $e^{-\frac{1}{C_{\tau\pi}} \, T\, t}$

It is still true here, but only at a given instance: $e^{-\frac{1}{C_{\tau\pi}} \, \int_{\tau_i}^{\tau} T(\tau')\, d\tau'}$

Using $T = \frac{\Lambda}{(\Lambda\tau)^{1/3}} \left(1 - C_\eta \frac{1}{(\Lambda\,\tau)^{2/3}} + \dots\right)$ one gets $e^{-\frac{3}{2} \, \frac{1}{C_{\tau\pi}} \, w} \, w^{\frac{C_\eta}{C_{\tau\pi}}} \dots$

see also **hep-th/0606149** by Janik & Peschanski

Transseries and resurgence 1503.07514 with Spalinski

$$\mathcal{A}(w) \approx \sum_{n=1}^{\infty} \frac{a_n}{w^n} \xrightarrow{\text{Borel trafo.}} B\mathcal{A}(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n!}\, \xi^n \approx \frac{b_0 + \ldots + b_{100}\, \xi^{100}}{c_0 + \ldots + c_{100}\, \xi^{100}}$$

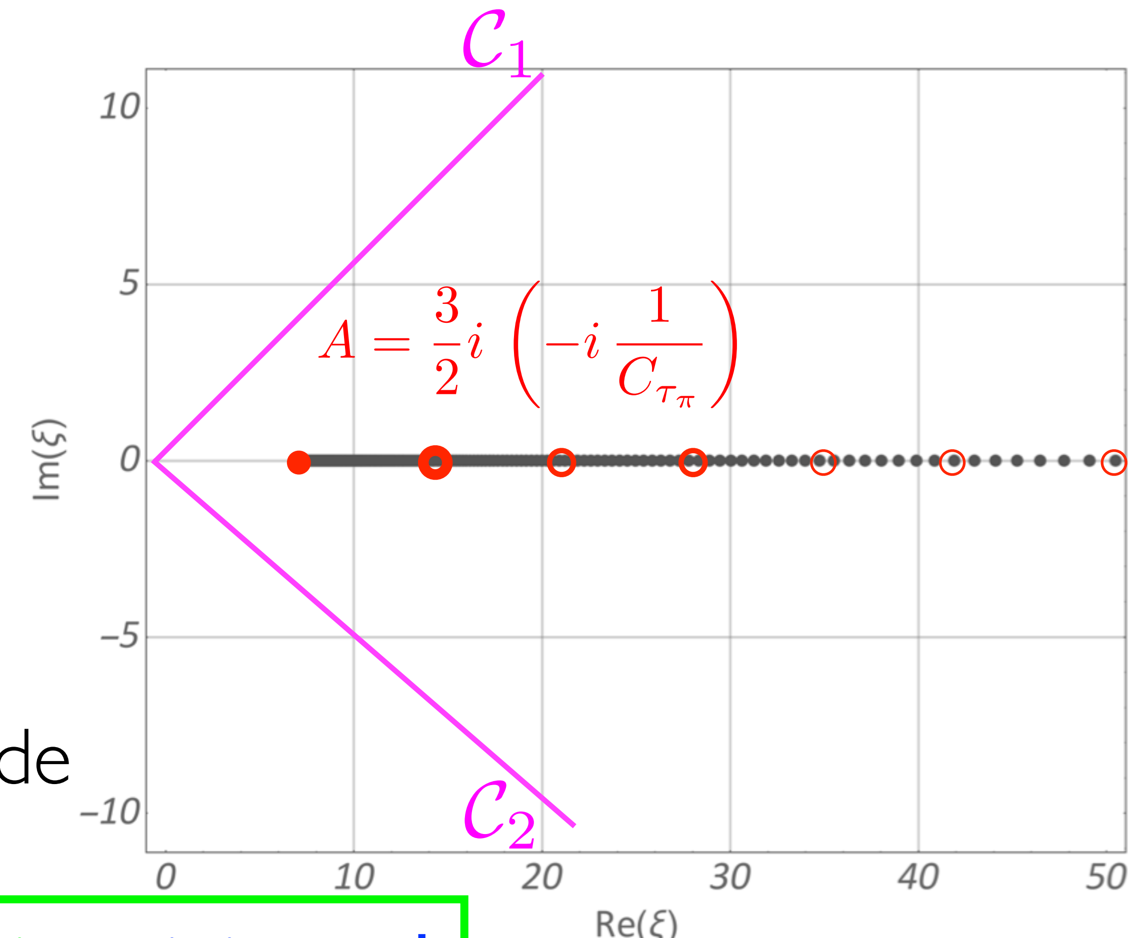
Borel (re)summation:

$$\left(\int_{\mathcal{C}_1} d\xi - \int_{\mathcal{C}_2} d\xi \right) \left[w\, e^{-w\,\xi} B\mathcal{A}(\xi) \right]$$

$$\sim e^{-\frac{3}{2}\frac{1}{C_{\tau_\pi}} w}\, w^{\frac{C_\eta - 2 C_{\lambda_1}}{C_{\tau_\pi}}} \ldots$$

$\equiv A$, $\equiv \beta$

Ambiguity in resummation ~ transient mode



nonlinear effects

~ resum. ambig. + ini. cond.

$$\text{Transseries: } \mathcal{A}(w) = \sum_{j=0}^{\infty} \sigma^j\, e^{-j\, A\, w}\, w^{j\,\beta}\, \Phi_{(j)}(w)$$

~ 1/w expansions

Resurgence: transseries yields an unambiguous answer up to 1 real int. const.

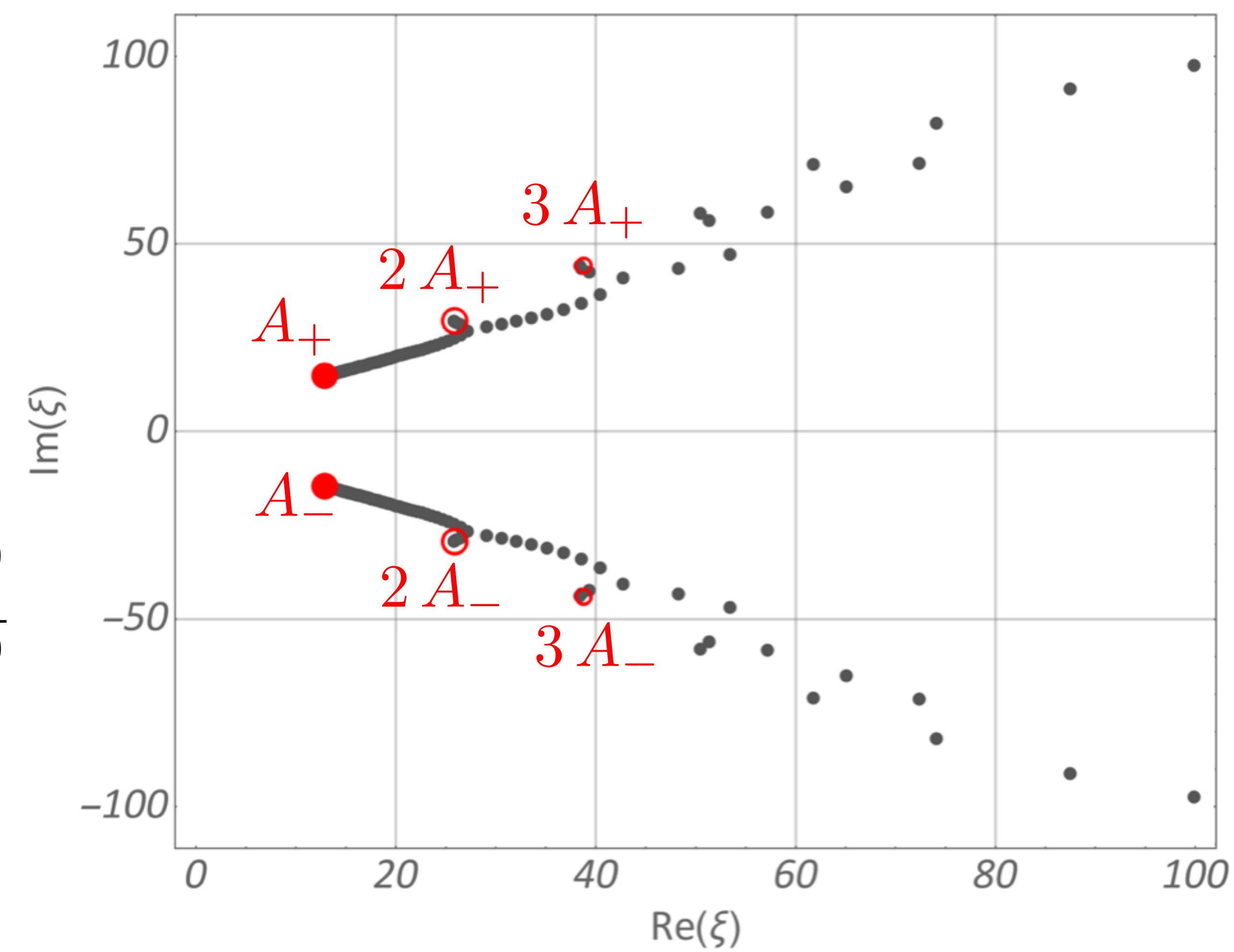
Resurgence in HJSW 1511.06358 by Aniceto & Spalinski

Long story short: we now look at a 2nd order ODE $\mathcal{A}'' + \ldots = 0$

$$\mathcal{A}(w) = \sum_{n=1}^{\infty} \frac{a_n}{w^n} + \ldots$$

$$\downarrow$$

$$B\mathcal{A}(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n!} \xi^n \approx \frac{b_0 + \ldots + b_{300}\xi^{300}}{c_0 + \ldots + c_{300}\xi^{300}}$$

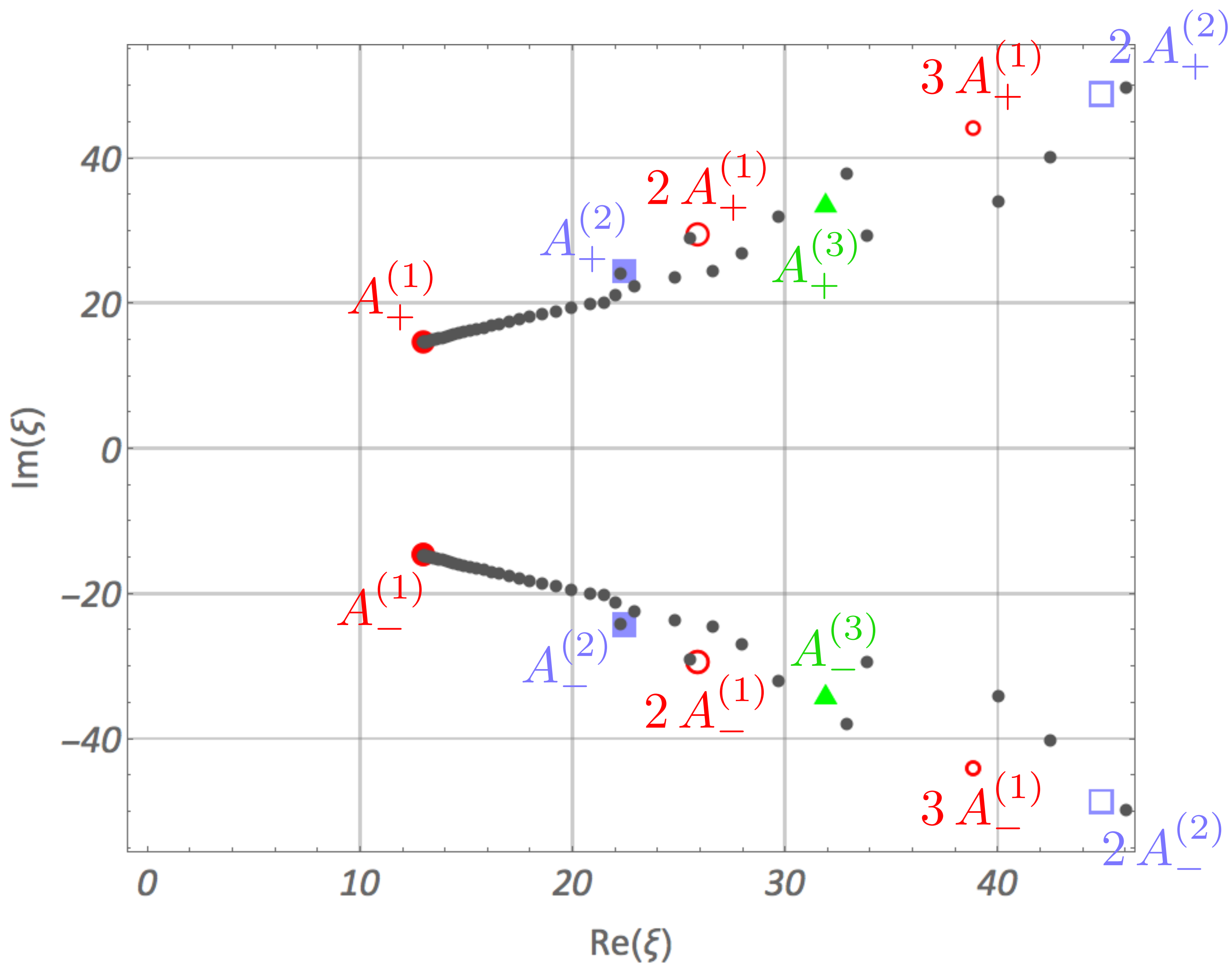


$$\mathcal{A}(w) = \sum_{n_\pm=0}^{\infty} \sigma_+^{n_+} \sigma_-^{n_-} \mathrm{e}^{-(n_+ A_+ + n_- A_-)w} w^{n_+ \beta_+ + n_- \beta_-} \Phi_{(n_+|n_-)}(w) \quad \text{with} \quad A_\pm = \frac{3}{2}(\Omega_I \pm \mathrm{i}\Omega_R)$$

2nd order EOM $\longrightarrow$ 2 real int. const. $\longrightarrow$ 2 parameter $(\sigma_\pm)$ transseries

Resurgence in holographic hydrodynamics

1302.0697 with Janik & Witaszczyk: ~240 terms (used in the plot)
1712.02772 by Casalderrey-Solana, Gushterov, Meiring: ~380 terms
1810.07130 by Aniceto, Meiring, Jankowski & Spalinski: focus on transients



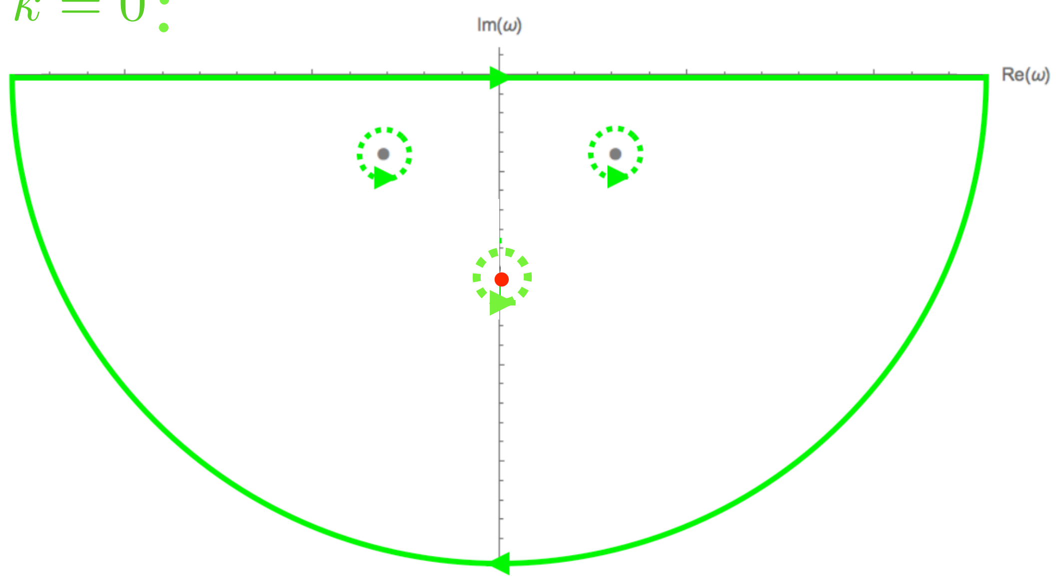
Infinitely many QNMs — infinitely many parameters in the transseries

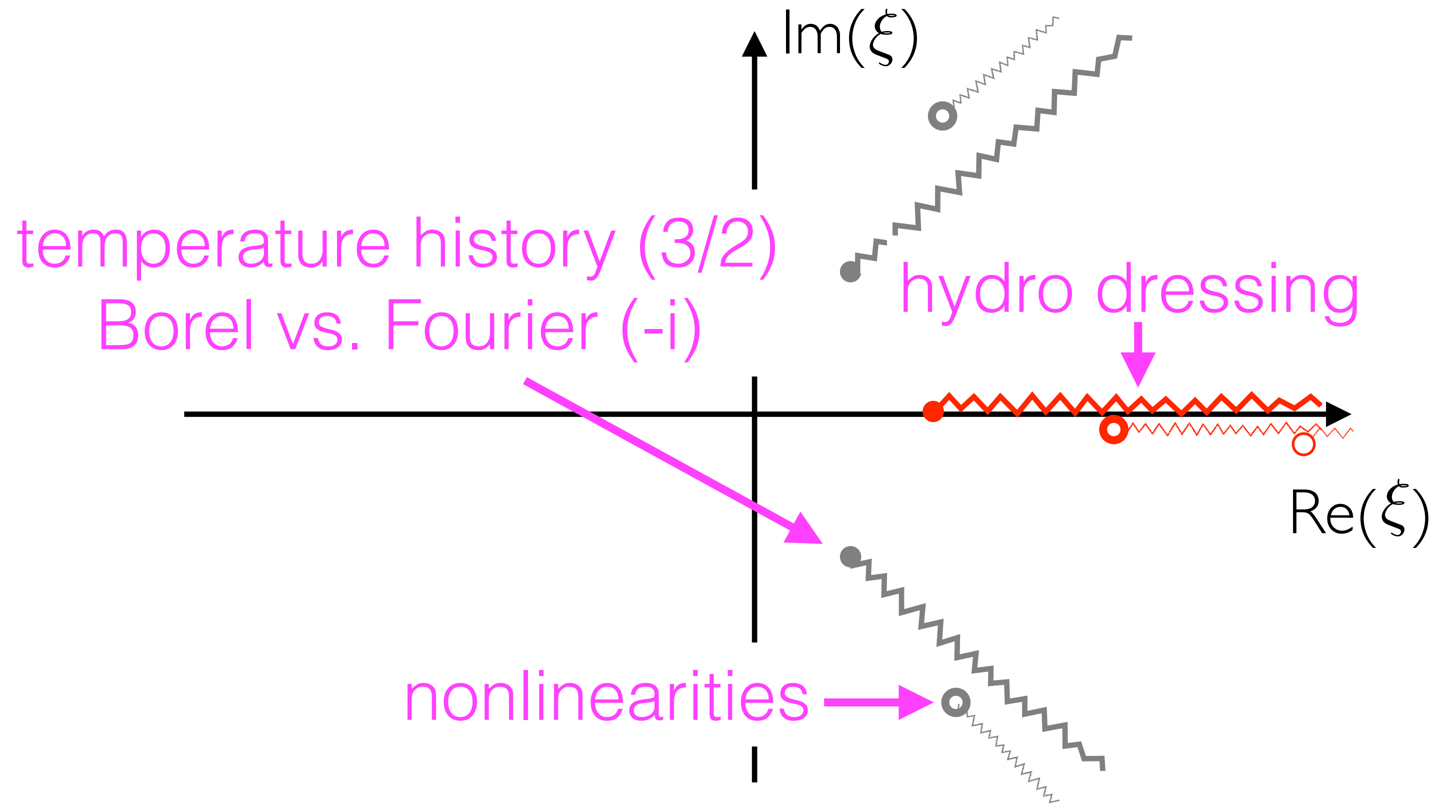
Emerging picture

Hydro gradient expansion diverges* ⟶ hydrodynamization

Transients in $G_R^{T_{\mu\nu}}(\omega, k)$ **vs.** singularities of Borel transform of hydro

$k = 0$:

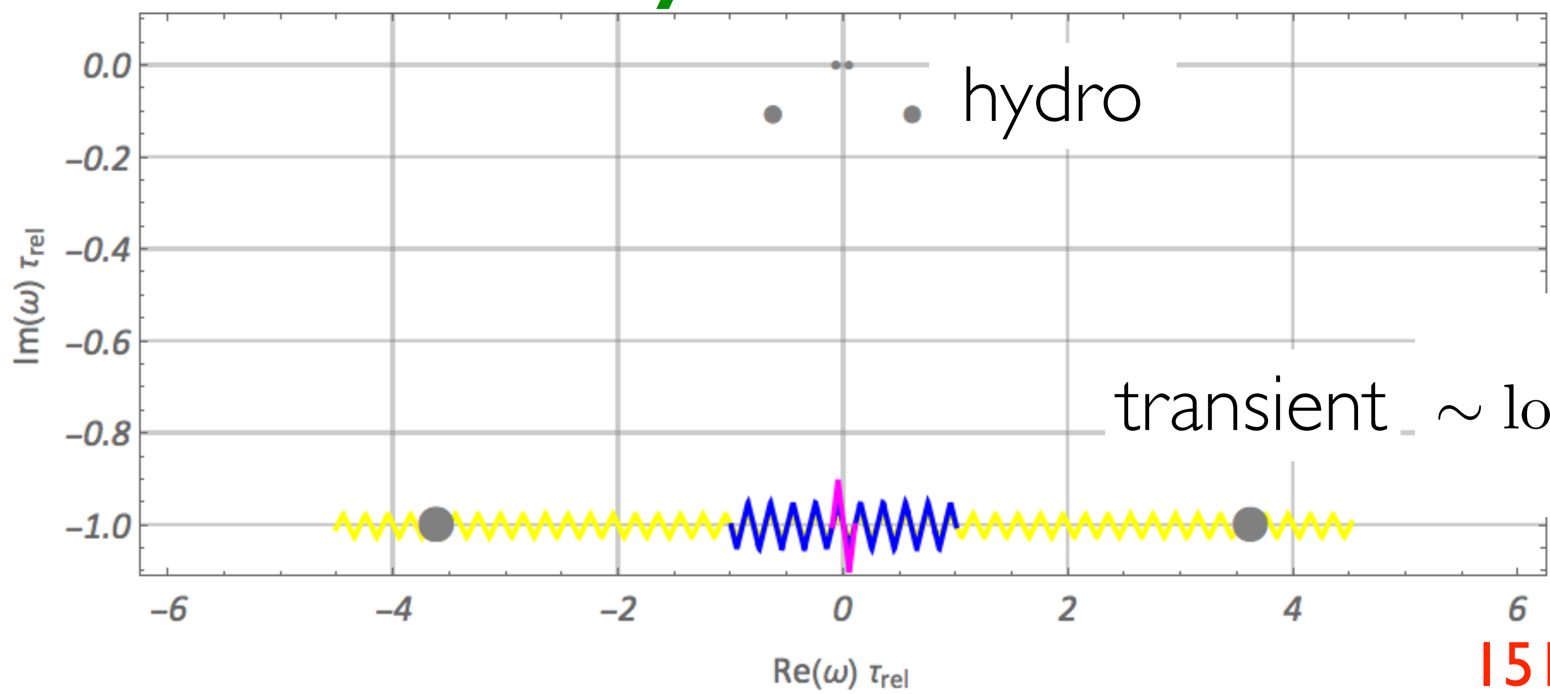




An analogy with QM:

non-equilibrium physics	**QM with**
gradient expansion in	perturbative series in
transient QNMs	instanton

RTA kinetic theory

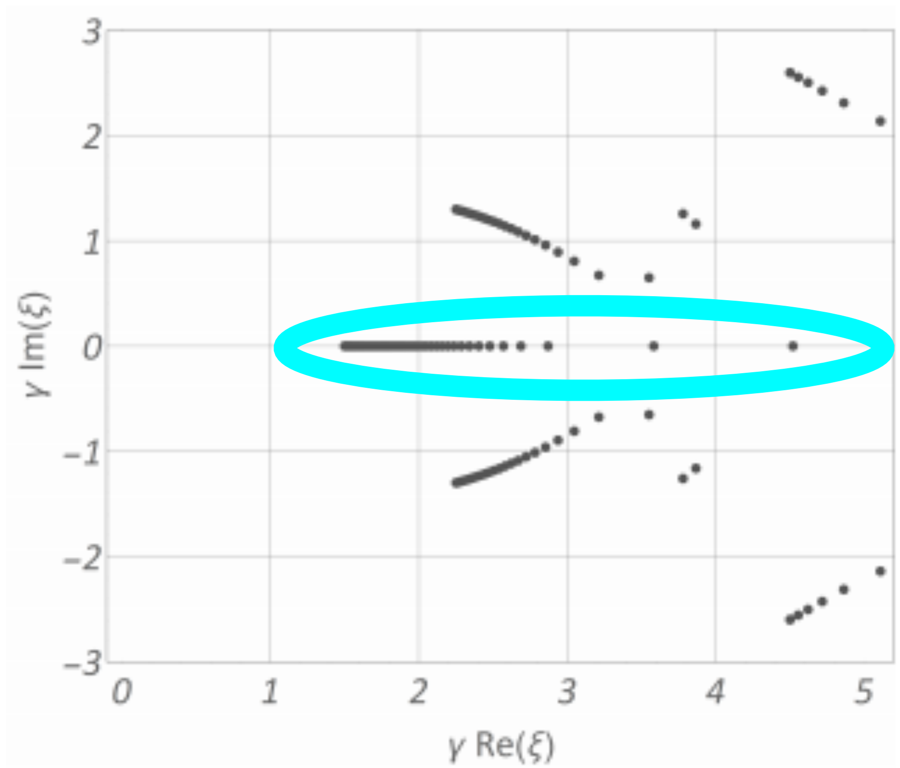


$$\sim \log \frac{\omega - k + \frac{i}{\tau_{rel}}}{\omega + k + \frac{i}{\tau_{rel}}}$$

1512.02641 by Romatschke

boost-invariant flow
Borel plane

$$\tau_{\rm rel} = \frac{\gamma}{T}$$



infinitely-many modes with the same exponential suppression and differing in their power-law behaviour

the rest of singularities is humbug

1609.04803v2 with Kurkela, Spalinski & Svensson; **1802.08225** with Svensson

Attractors

1503.07514 with Spalinski

1704.08699 by Romatschke

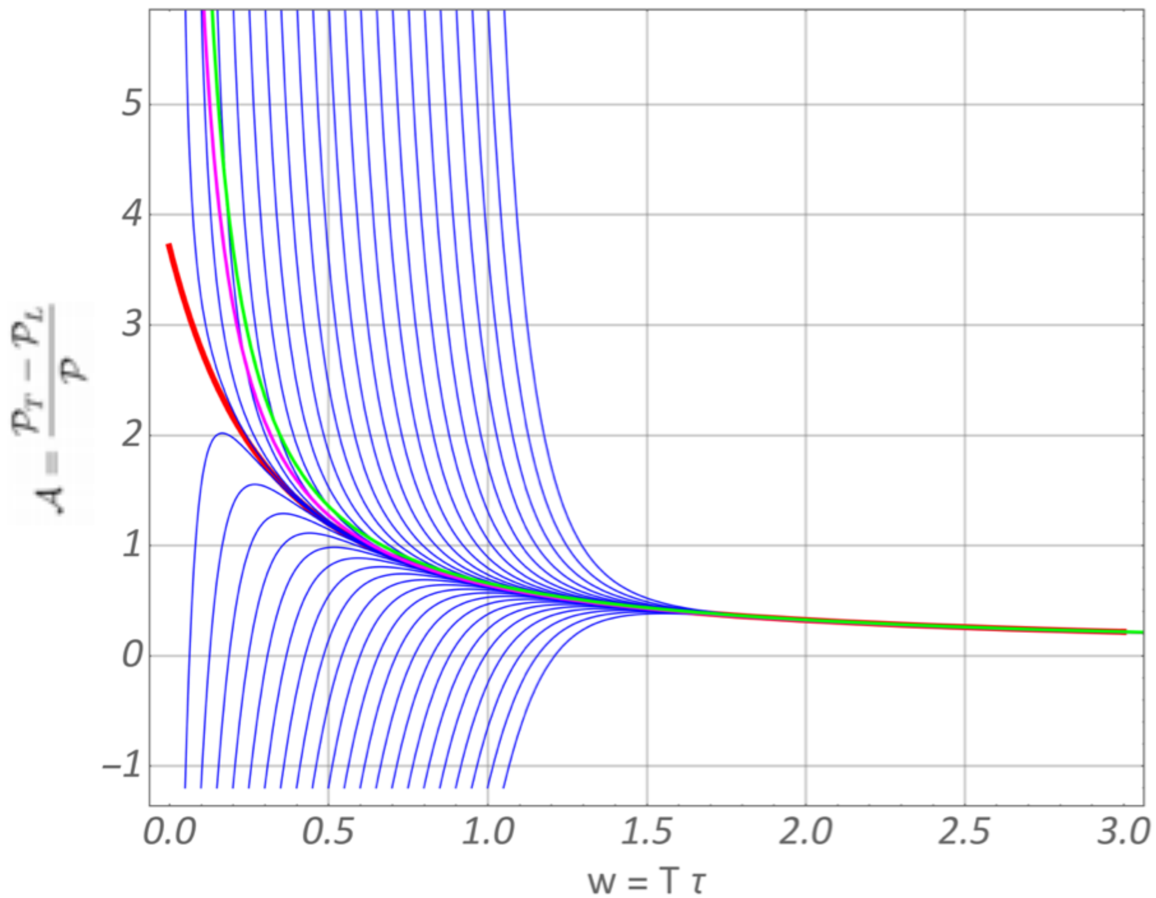
… ~15 other distinct studies of attractors …

1906.xxxxx with Jefferson, Spalinski & Svensson

(BRSSS) resummed hydrodynamics

1503.07514 with Spalinski

Idea: resummed / far-from-equilibrium hydrodynamics = attractor solutions



BRSSS:

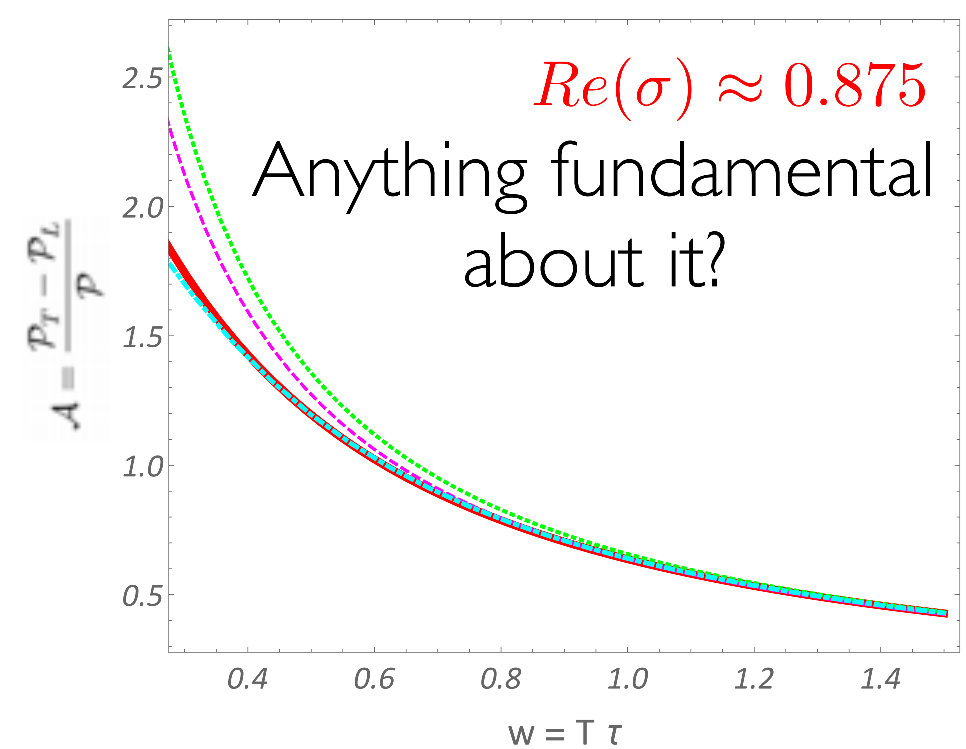
$$C_{\tau_\pi}\, w \left(1 + \frac{1}{12}\mathcal{A}\right)\mathcal{A}' + \left(\frac{1}{3}C_{\tau_\pi} + \frac{1}{8}\frac{C_{\lambda_1}}{C_\eta} w\right)\mathcal{A}^2 + \frac{3}{2} w\,\mathcal{A} - 12\, C_\eta = 0$$

„slow roll" approximation reveals an attractor solution

One can also approx. resum transseries:

$$\mathcal{A}(w) \approx \sum_{j=0}^{2} \sigma^j\, e^{-j\,A\,w}\, w^{j\,\beta}\, \Phi_{(j)}(w)$$

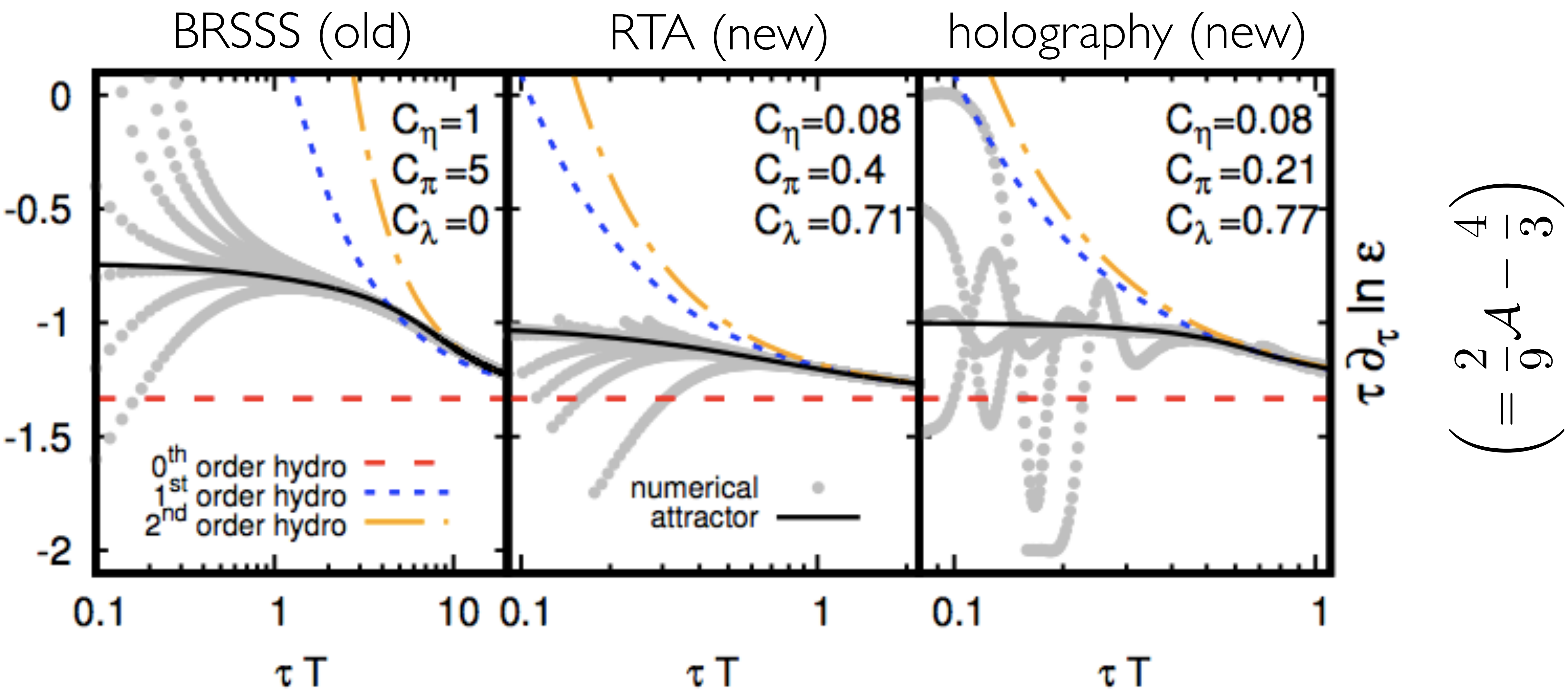
Requires 3 Borel summations



Attractor in kinetic theory and holography

1704.08699 by Romatschke (figure imported from the arXiv ver)

Idea: use the slow roll approximation to generate attractors in other theories

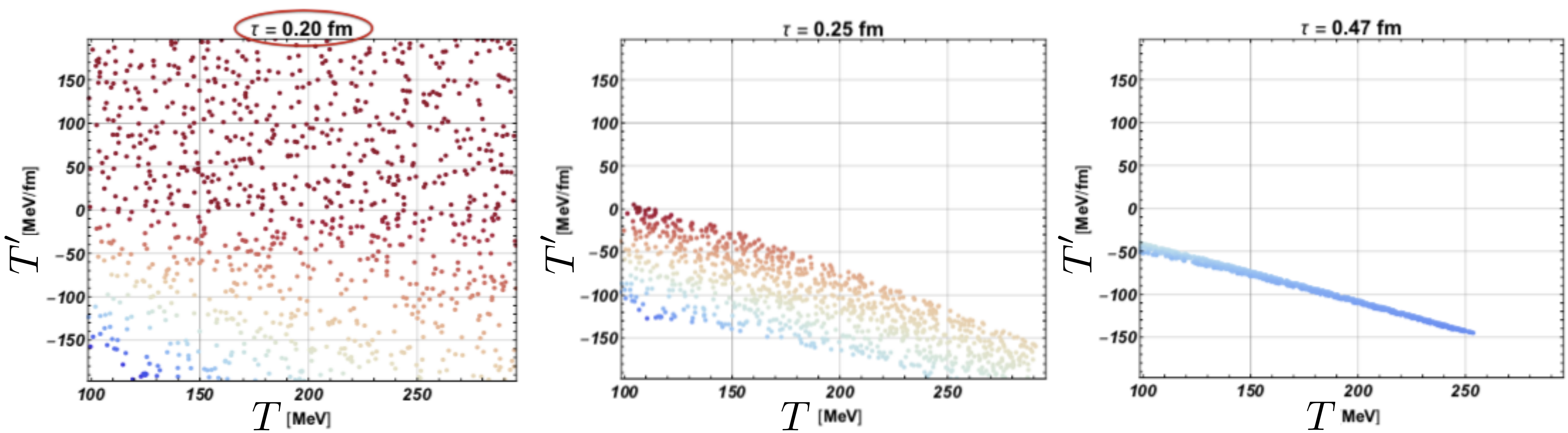


Note: centre and right are projections from infinitely-dimensional phase space

New (look at?) attractors

1906.xxxxx with Jefferson, Spalinski & Svensson

In more general situations we may not find w and $\mathcal{A}$, but we always have phase space variables in which we formulate the problem. In BRSSS: (τ, T, T')
$T' \sim \pi_{yy}$

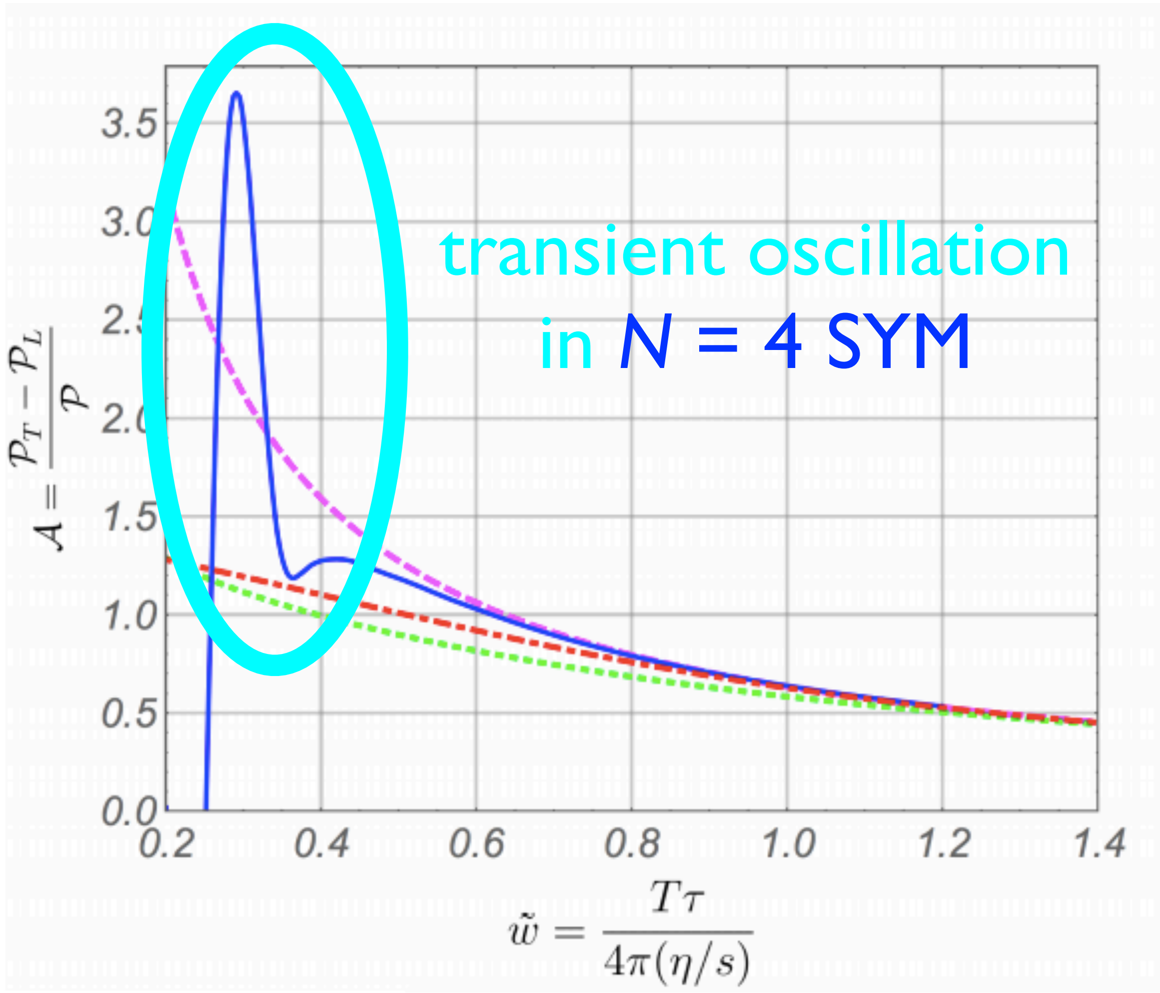


New (?) attractors: fix a set of initial conditions in phase space; dissipation is going to make it lower dimensional as time progresses; this identifies effective late-time description dependent at intermediate times on the initial set.

Executive summary and some open problems

Executive summary

What seems to control the applicability of hydrodynamics is not the gradient expansion itself, but what comes on top of it — transient modes:



As a result, applying hydro to HIC early on is not a priori crazy

Hydrodynamic attractors = universality in behaviour after transients are gone

Some open problems

What are the modes / properties of the gradient expansion if we leave our comfortable corner of theories I discussed today?
see, in particular, **1104.1586** by Kovtun, Moore & Romatschke; **1712.04376** by Kurkela & Wiedemann and **1803.00736** by Moore

In momentum space linearized hydrodynamics has a radius of convergence >0 (note e.g. $\omega_{\mathrm{BRSSS}}^{(\mathrm{exact\,shear\,hydro})} = i\,\frac{\sqrt{1-4\frac{\tau_\pi\,\eta}{T\,s}\,k^2}-1}{2\,\tau_\pi}$ or **1904.01018** by Grozdanov, Kovtun, Starinets & Tadic). What happens in the real space formulation?

Do attractors exist in some absolute sense or are an approximate and local property of phase space in theories with hydrodynamic tails?
1906.xxxxx with Jefferson, Spalinski & Svensson

Can one observe attractors in some experiment or, perhaps, remarkably quick applicability of hydrodynamics to HIC implies they have been there from day 1?