Low-x theory & Jet Quenching (LTHJ)

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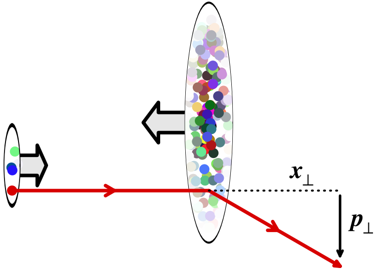
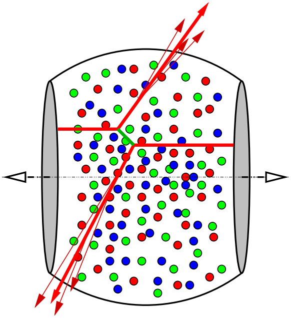


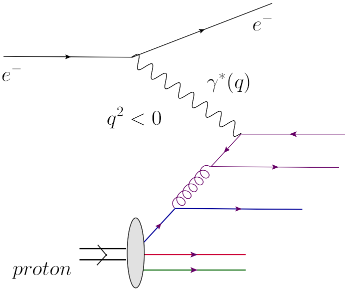


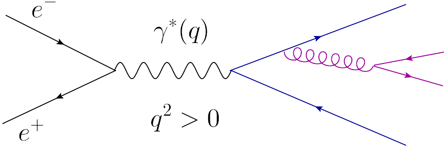
General Introduction

- One series of lectures ... but two distinct problems of physics:
  - QCD scattering at high energy (or "small x")
  - jet quenching in ultrarelativistic nucleus-nucleus collisions
- ... which are however related at a profound level:
  - they both explore the physics of high parton densities
- Weak coupling (by asymptotic freedom) but ultimately non-perturbative physics (strong non-linear phenomena)
- Profound differences in terms of kinematics, physics scenarios, observables
- ... but some common microscopic ingredients:
  - multiple scattering, quantum evolution in strong background fields
- ... and also common theoretical concepts and approximation schemes:
  - gluon saturation, momentum broadening, eikonal approximation ...

General Introduction: Parton evolution

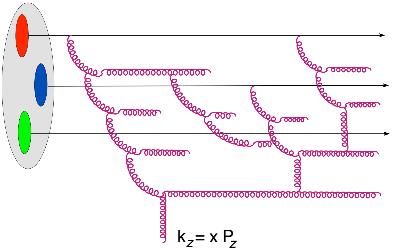
- Parton evolution (especially with increasing energy) is a key ingredient of both problems
  - space-like (initial-state) evolution: e.g. parton distributions in DIS
  - time-like (final-state) evolution: e.g. jets in $e^- e^+$ annihilation

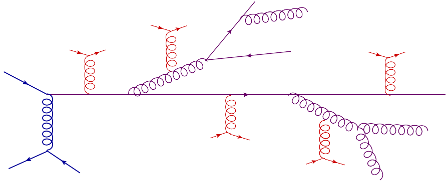




- For dilute systems, the 2 evolutions are “the same” (crossing symmetry)
  - e.g. they both involve the same DGLAP splitting functions

- Parton evolution (especially with increasing energy) is a key ingredient of both problems
  - space-like (initial-state) evolution: e.g. parton distributions in DIS
  - time-like (final-state) evolution: e.g. jets in e^-e^+ annihilation





- In the presence of partonic matter, this symmetry can be broken
  - different quantum evolutions for parton distributions and jet quenching

High energy QCD & the Color Glass Condensate

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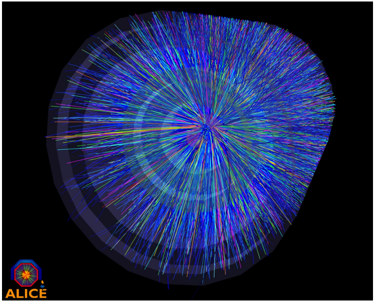
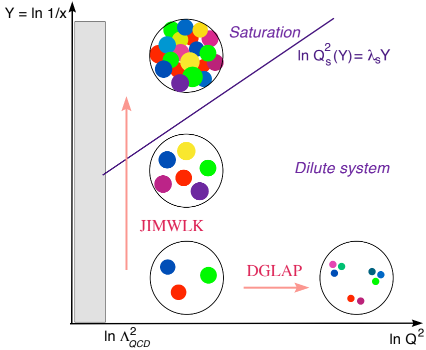




Outline

- Introduction to high-energy evolution in pQCD
  - kinematics, lifetimes, parton distributions, double-logarithmic approximation
  - gluon saturation: the general idea
- Building up the formalism
  - dipole frame, rapidities, light-cone variables
  - eikonal approximation, Wilson lines
  - dipole factorization at high energy: DIS, proton-nucleus collisions
- The Color Glass Condensate effective theory
  - McLerran-Venugopalan model as a pedagogical example
  - classical field solution (different gauges)
  - dipole scattering, saturation momentum
  - the Weiszäcker-Williams gluon distribution
  - transverse momentum broadening

Outline (cont.)

- High-energy evolution: Balitsky-Kovchegov equation
  - some details on the derivation
  - general structure and properties
  - solution (qualitatively): saturation exponent, geometric scaling
  - adding a running coupling
- Applications to DIS and to particle production in pA collisions
  - fits to the DIS structure functions at HERA
  - pA collisions: the nuclear modification factor
  - correlations in two-particle production
- Beyond the BK evolution
  - JIMWLK evolution & its Langevin reformulation
  - next-to-leading order corrections to BK: the instability
  - ... and its solution (as recently given)

Instead of references ...

- For a general and rather elementary introduction and for more references (albeit a bit outdated), you may have a look at this review paper:

arXiv.org > hep-ph > arXiv:1205.0579 Search or Artic

High Energy Physics – Phenomenology

QCD in heavy ion collisions

Edmond Iancu

(Submitted on 2 May 2012)

These lectures provide a modern introduction to selected topics in the physics of ultrarelativistic heavy ion collisions which shed light on the fundamental theory of strong interactions, the Quantum Chromodynamics. The emphasis is on the partonic forms of QCD matter which exist in the early and intermediate stages of a collision -- the colour glass condensate, the glasma, and the quark-gluon plasma -- and on the effective theories that are used for their description. These theories provide qualitative and even quantitative insight into a wealth of remarkable phenomena observed in nucleus-nucleus or deuteron-nucleus collisions at RHIC and/or the LHC, like the suppression of particle production and of azimuthal correlations at forward rapidities, the energy and centrality dependence of the multiplicities, the ridge effect, the limiting fragmentation, the jet quenching, or the dijet asymmetry.

Comments: Based on lectures presented at the 2011 European School of High-Energy Physics, 7-20 September 2011, Cheile Gradistei, Romania. 73 pages, many figures
Subjects: **High Energy Physics – Phenomenology (hep-ph)**; High Energy Physics – Experiment (hep-ex); Nuclear Theory (nucl-th)
Cite as: arXiv:1205.0579 **[hep-ph]**
(or arXiv:1205.0579v1 **[hep-ph]** for this version)

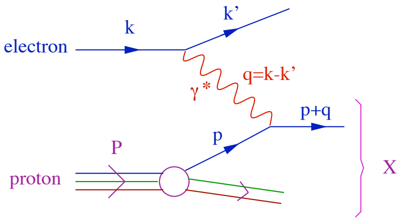
- Part of the material already introduced in the previous lectures:
  Michael Spira, Raju Venugopalan

More specific references for high-energy QCD

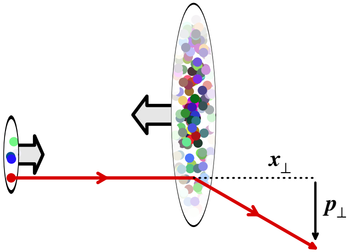
- A book: *Quantum chromodynamics at high energy*, by Yuri V. Kovchegov and Eugene Levin, 2012, 349 pp. (Cambridge Univ Press)
- A few review papers or lecture notes (not exhaustive):
  - *The Colour Glass Condensate: An Introduction*, by E. Iancu, A. Leonidov, and L. McLerran, arXiv:hep-ph/0202270
  - *The Color Glass Condensate and High Energy Scattering in QCD*, by E. Iancu and R. Venugopalan, arXiv:hep-ph/0303204
  - *High energy scattering in Quantum Chromodynamics*, by F. Gelis, T. Lappi, and R. Venugopalan, arXiv:0708.0047
  - *The Color Glass Condensate*, by F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, arXiv:1002.0333
  - *Gluon saturation and initial conditions for relativistic heavy ion collisions*, J. L. Albacete and C. Marquet, arXiv:1401.4866
- Whenever I remember, I will refer to the arXiv number

Dilute-dense scattering

- Deep inelastic scattering (DIS)



- Proton-nucleus collisions (pA)



- High-energy, or small Bjorken x

$$x \equiv \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{s} \ll 1$$

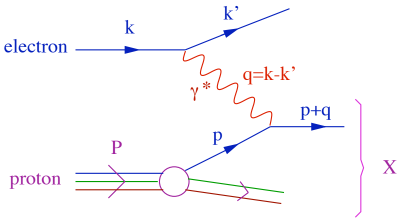
- Forward rapidity $\eta > 0$

$$x = \frac{p_\perp}{\sqrt{s}}\, \mathrm{e}^{-\eta} \ll 1$$

- x: energy fraction of the "parton" (quark or gluon) from the "target" (proton in DIS, nucleus in pA) which participates in the collision
- High energy $(s \gg Q^2, p_\perp^2)$, forward rapidity $(\eta > 0) \Longrightarrow$ small-x $(x \ll 1)$

Dilute-dense scattering

- Deep inelastic scattering (DIS)



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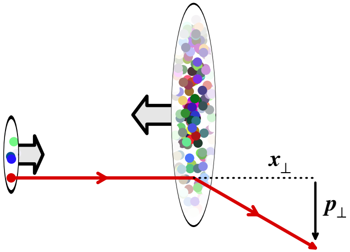

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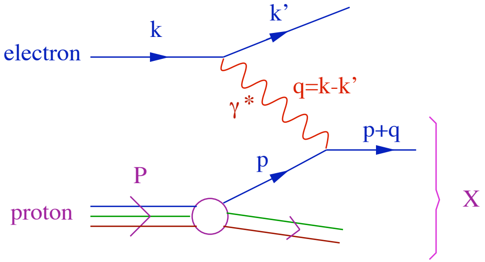
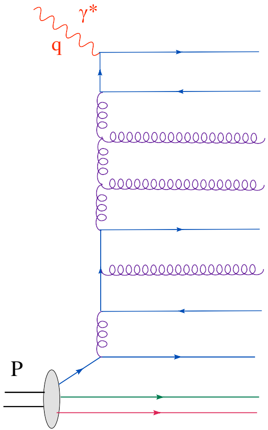
- For sufficiently small x, a hadron wavefunction is a dense gluonic system
- This is most conveniently probed via scattering with a dilute projectile (virtual photon in DIS, proton in pA)

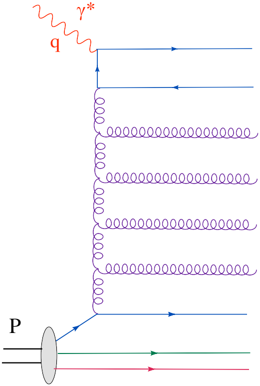
Parton evolution in QCD

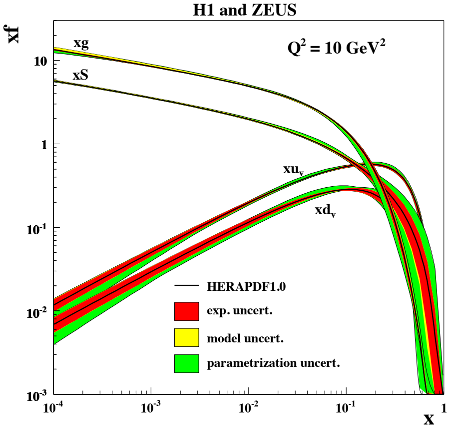




- The virtual photon γ^* couples to the (anti)quarks inside the proton
- Gluons are measured indirectly, via their effect on quark distribution
- Quantum evolution : change in the partonic content when changing the resolution scales x and Q^2, due to additional radiation

The small–x partons are mostly gluons

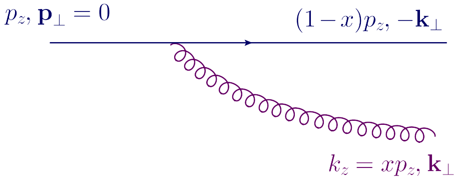




- For $x \le 0.01$ the hadron wavefunction contains mostly gluons !
- The gluon distribution is rapidly amplified by the quantum evolution with decreasing x (or increasing energy s at fixed Q^2)

Bremsstrahlung

- A quark — say, a valence quark from a proton — emits a gluon with longitudinal momentum fraction $x \leq 1$, and transverse momentum $k_\perp$



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} P_{g\leftarrow q}(x)\mathrm{d}x$$

$$P_{g\leftarrow q}(x) \equiv C_F \frac{1+(1-x)^2}{x}$$

- Logarithmic enhancement for large-$k_\perp$ emissions $(\Lambda^2_{\rm QCD} < k_\perp^2 < Q^2)$:

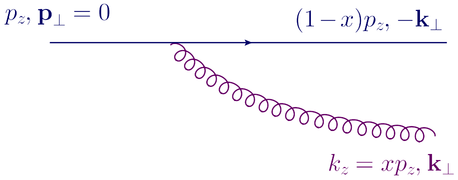
$$\int_{\Lambda^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \ln\frac{Q^2}{\Lambda^2}$$

- ... and also for soft/low-energy $(x \to 0)$ gluons: $P_{g\leftarrow q}(x) \simeq 2C_F/x$

$$\int_{x_0}^{1} \frac{\mathrm{d}x}{x} = \ln\frac{1}{x_0} \equiv Y_0$$

Bremsstrahlung

- A quark — say, a valence quark from a proton — emits a gluon with longitudinal momentum fraction $x \leq 1$, and transverse momentum $k_\perp$



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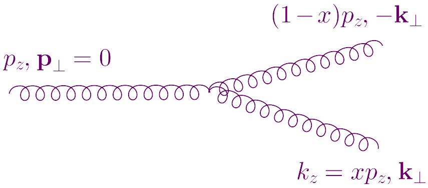
- Emissions of soft quarks are not enhanced: $\xi \equiv 1 - x \ll 1$

$$P_{q\leftarrow q}(\xi) = P_{g\leftarrow q}(x = 1-\xi) = C_F \frac{1+\xi^2}{1-\xi} \to C_F$$

Gluon splitting

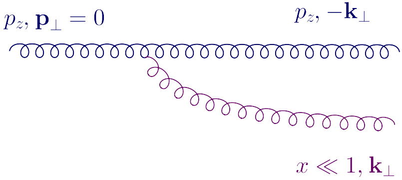
- Gluon splitting into two gluons:



$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s}{2\pi} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} P_{g\leftarrow g}(x)\mathrm{d}x$$

$$P_{g\leftarrow g}(x) \equiv 2N_c \frac{[1-x(1-x)]^2}{x(1-x)}$$

- Soft gluon emission: $x \ll 1$ (or $1-x \ll 1$, but this is treated by symmetry)

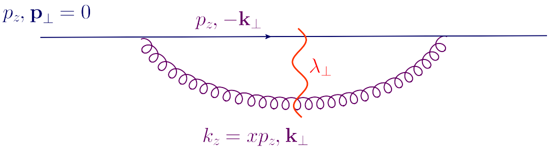


$$\mathrm{d}\mathcal{P} \simeq \frac{\alpha_s N_c}{\pi} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \frac{\mathrm{d}x}{x}$$

- Soft gluons can act as sources for even softer ones: high-energy evolution

The lifetime of a fluctuation

- An on-shell parton cannot decay into a pair of on-shell partons
  - the gluon is eventually reabsorbed: "virtual fluctuation"
- The maximal transverse separation $\sim$ gluon transverse wavelength
  - if $\Delta x_\perp > \lambda_\perp$, the quark and the gluon lose their quantum coherence and the gluon can be emitted



$$\Delta x_\perp \sim \frac{k_\perp}{k_z}\,\Delta t \lesssim \lambda_\perp \sim \frac{2}{k_\perp}$$

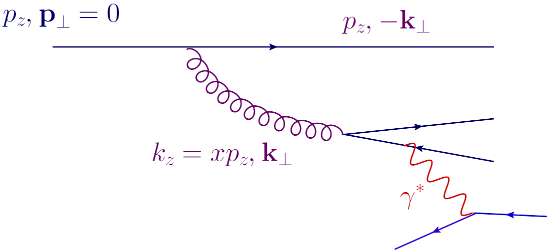
$$\Delta t \;\simeq\; \frac{2k_z}{k_\perp^2} = \frac{2xp_z}{k_\perp^2}$$

- The same estimate for Δt follows from the uncertainty principle

$$\frac{1}{\Delta t} = \Delta E \equiv \sqrt{(xp_z)^2 + k_\perp^2} + \sqrt{((1-x)p_z)^2 + k_\perp^2} - p_z \simeq \frac{k_\perp^2}{2x(1-x)p_z}$$

Transverse resolution in DIS

- Very hard fluctuations (large $k_\perp^2$) have a very short lifetime
- In DIS, the virtual photon "sees" only those fluctuations which live long enough: longer than the collision time



$$\Delta t \simeq \frac{2xp_z}{k_\perp^2}$$

$$\Delta t_{coll} \simeq \frac{1}{q_0}$$

$$x \simeq \frac{Q^2}{s} \simeq \frac{Q^2}{2p_z q_0}$$

$$\Delta t \gtrsim \Delta t_{coll} \implies k_\perp^2 \lesssim Q^2$$

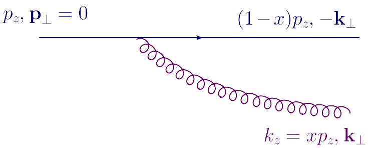
- Parton distributions: number of partons with $k_\perp^2 \lesssim Q^2$ per unit of x
    - by the uncertainty principle, such partons are localized within a transverse area $(\Delta x_\perp)^2 \sim 1/Q^2$

Gluon distribution at small x

- The gluon distribution $xG(x, Q^2)$: # of gluons with a given energy fraction x and any transverse momentum $k_\perp \lesssim Q$

$$xG(x, Q^2) = \int^Q \mathrm{d}^2\boldsymbol{k}\; x \frac{\mathrm{d}N_{\text{gluon}}}{\mathrm{d}x\mathrm{d}^2k_\perp}$$

- To leading order in α_s: single (soft) gluon emission by the quark



$$\frac{\mathrm{d}N_{\text{gluon}}}{\mathrm{d}x\mathrm{d}^2k_\perp} = \frac{\alpha_s C_F}{\pi} \frac{1}{x} \frac{1}{k_\perp^2}$$

- "unintegrated gluon distribution"

  - logarithmic sensitivity to the hard resolution scale Q^2
  - logarithmic sensitivity to the confinement scale Λ^2
  - no dependence upon energy (x)

Gluon distribution at small x

- The gluon distribution $xG(x, Q^2)$: # of gluons with a given energy fraction x and any transverse momentum $k_\perp \lesssim Q$

$$xG^{(0)}(x, Q^2) = \frac{\alpha_s C_F}{\pi} \int_{\Lambda^2}^{Q^2} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \frac{\alpha_s C_F}{\pi} \ln \frac{Q^2}{\Lambda^2}$$

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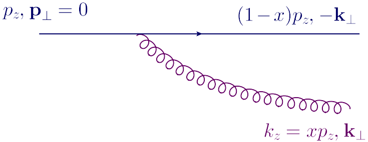

$$\frac{\mathrm{d}N_{\rm gluon}}{\mathrm{d}x \mathrm{d}^2 k_\perp} = \frac{\alpha_s C_F}{\pi} \frac{1}{x} \frac{1}{k_\perp^2}$$

- "unintegrated gluon distribution"

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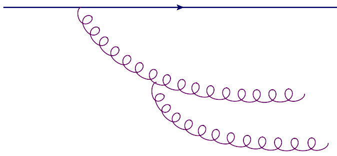
Next-to-leading order: two gluons

- The intermediate gluon $(x_1,\, k_{1\perp})$ is not measured, but acts as a source for the measured one $(x,\, k_\perp)$



$$x \ll 1,\ \Lambda^2 \ll k_\perp^2 \ll Q^2$$

$$x \ll x_1 \ll 1$$

$$\Lambda^2 \ll k_{1\perp}^2 \ll k_\perp^2$$

- The 2-gluon contribution to the gluon distribution measured at x and Q^2

$$xG^{(1)}(x,Q^2) = \frac{\alpha_s C_F}{\pi}\int_{\Lambda^2}^{Q^2}\frac{\mathrm{d}k_\perp^2}{k_\perp^2}\,\frac{\alpha_s N_c}{\pi}\int_x^1\frac{\mathrm{d}x_1}{x_1}\int_{\Lambda^2}^{k_\perp^2}\frac{\mathrm{d}k_{1\perp}^2}{k_{1\perp}^2}$$

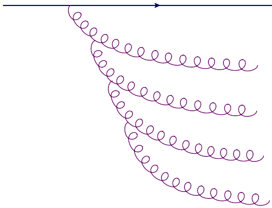
$$= xG^{(0)}(x,Q^2)\,\frac{\alpha_s N_c}{\pi}\,\frac{1}{2}\ln\frac{Q^2}{\Lambda^2}\ln\frac{1}{x}$$

- Suppressed by α_s but enhanced by a double-logarithm (energy $\times$ collinear)

The double logarithmic approximation

- When $\bar{\alpha} Y \rho \gtrsim 1 \implies$ need for all-order resummation

$$\bar{\alpha} \equiv \frac{\alpha_s N_c}{\pi}, \quad Y \equiv \ln\frac{1}{x}, \quad \rho \equiv \ln\frac{Q^2}{\Lambda^2}$$



- Strong ordering in both x (decreasing):

$$x \ll x_n \ll x_{n-1} \cdots \ll x_1 \ll 1$$

- ... and $k_\perp$ (increasing):

$$Q^2 \gg k_\perp^2 \gg k_{n\perp}^2 \cdots \gg k_{1\perp}^2 \gg \Lambda^2$$

- The lifetimes of the successive fluctuations are strongly decreasing: coherent cascade: $\Delta t_i \simeq 2 x_i p_z / k_{i\perp}^2$

$$\int_x^1 \frac{\mathrm{d}x_n}{x_n} \int_{x_n}^1 \frac{\mathrm{d}x_{n-1}}{x_{n-1}} \cdots \int_{x_2}^1 \frac{\mathrm{d}x_n}{x_n} = \frac{1}{n!} Y^n \qquad \left(\text{and similarly } \frac{1}{n!}\rho^n\right)$$

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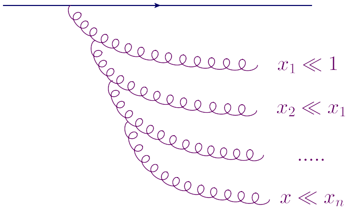

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- ... and $k_\perp$ (increasing):

$$Q^2 \gg k_\perp^2 \gg k_{n\perp}^2 \cdots \gg k_{1\perp}^2 \gg \Lambda^2$$

- After summing over cascades with any number $n \geq 0$ of intermediate gluons:

$$xG(x,Q^2) = xG^{(0)}(x,Q^2) \sum_{n\geq 0} \frac{(\bar{\alpha} Y \rho)^n}{(n!)^2} = xG^{(0)}(x,Q^2)\, \mathrm{I}_0(2\sqrt{\bar{\alpha} Y \rho})$$

- $\mathrm{I}_0(x)$: modified Bessel function of rank zero

The double logarithmic approximation

- When $\bar{\alpha} Y \rho \gtrsim 1 \implies$ need for all-order resummation

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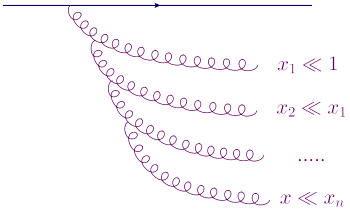

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- ... and $k_\perp$ (increasing):

$$Q^2 \gg k_\perp^2 \gg k_{n\perp}^2 \cdots \gg k_{1\perp}^2 \gg \Lambda^2$$

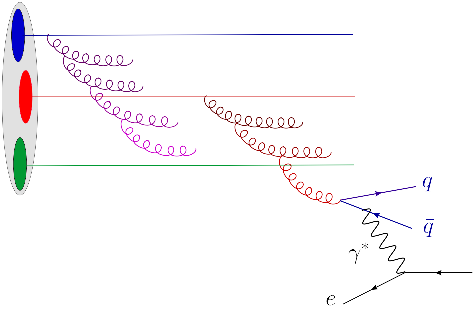
- Asymptotic behavior at small-x and large-Q^2 : $\bar{\alpha} Y \rho \gg 1$

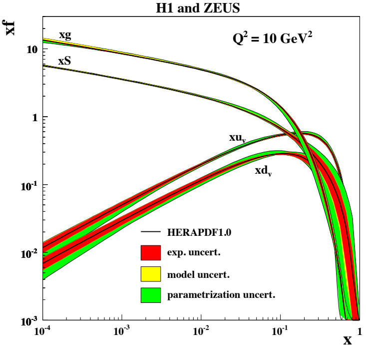
$$xG(x,Q^2) \propto \mathrm{e}^{2\sqrt{\bar{\alpha} Y \rho}} \propto \exp\left\{2\sqrt{\bar{\alpha} \ln\frac{1}{x} \ln\frac{Q^2}{\Lambda^2}}\right\}$$

- Rapid increase with both $1/x$ and Q^2: DGLAP evolution at small x

DIS @ small–x

- Rapid rise in the hadronic cross-sections with increasing energy





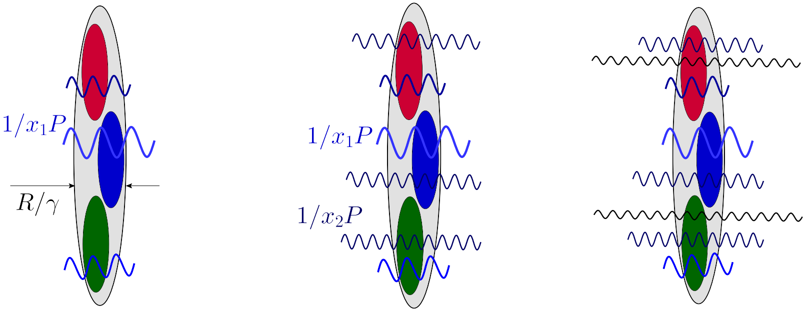
- Gluon distribution measured at HERA rises like a power of $1/x$:

$$xG(x, Q^2) \propto \frac{1}{x^\lambda} \qquad \text{with} \quad \lambda \simeq 0.2$$

- Can such a rise go on for ever ? (i.e. down to arbitrarily small x ?)

Overlapping gluons

- So far, we have assumed that the emitted gluons don't "see each other"
- Gluons which overlap with each other can interact with each other
- Uncertainty principle: A gluon with $(\boldsymbol{k}_\perp, \, k_z = xP)$ has a longitudinal extent $\Delta z \sim 1/xP$ and occupies a transverse area $\Delta x_\perp^2 \sim 1/k_\perp^2$



- small-x gluons can easily overlap in the longitudinal direction
- to actually overlap, their transverse momenta need to be small enough

Overlapping gluons

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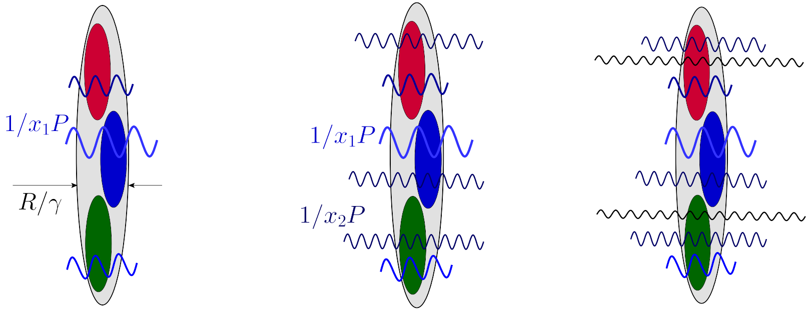

- DLA (generally, DGLAP) evolution maintains a dilute system of partons
  - rapid decrease in their transverse sizes $\Longrightarrow$ no possible overlap

Overlapping gluons

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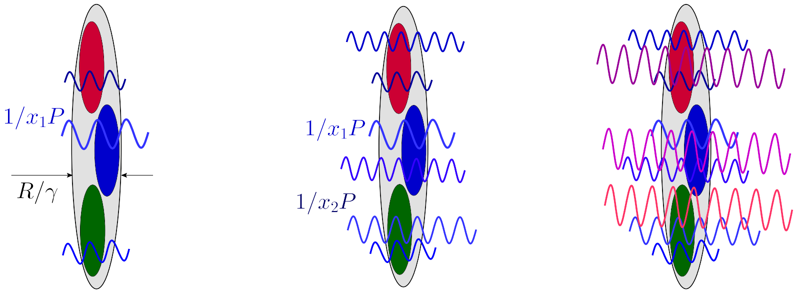


- BFKL evolution *(Balitsky, Fadin, Kuraev, Lipatov, 1974-78)*
  - decrease x at roughly fixed $k_\perp$: $\sum_n [\bar{\alpha}\ln(1/x)]^n \Longrightarrow$ increasing density
  - leading logarithmic approximation (LLA) at small x

Gluon occupancy

- BFKL evolution leads to a rapid rise in the gluon occupation number

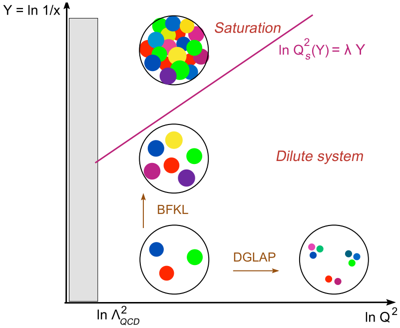
$$n(x, \boldsymbol{k}_\perp, \boldsymbol{x}_\perp) \equiv \frac{(2\pi)^3}{2(N_c^2-1)}\, x \frac{\mathrm{d}N_{\text{gluon}}}{\mathrm{d}x\, \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{x}_\perp}$$

- a simple estimate

$$n(x, Q^2) \simeq \frac{1}{Q^2} \times \frac{xG(x, Q^2)}{\pi R^2}$$

- HERA data suggest

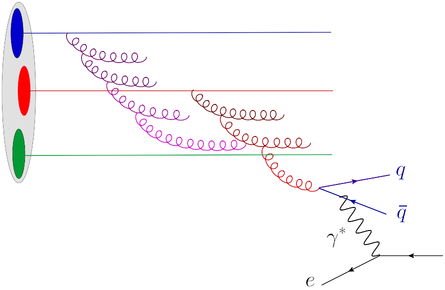
$$n(x, Q^2) \sim \frac{1}{x^\lambda} \text{ with } \lambda \simeq 0.2$$

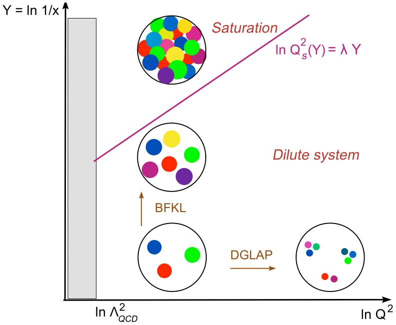


- When $n \gtrsim 1$, gluons overlap, but their interactions are still suppressed by α_s
- When $n \sim 1/\alpha_s$, gluons self interactions become of $\mathcal{O}(1)$: what happens?

Gluon saturation

- The original idea *(L. Gribov, Levin, Ryskin, 1982; Mueller, Qiu, 1987)*
  - when $n \sim 1/\alpha_s$, gluon recombination $(gg \to g)$ becomes important and equilibrates gluon splitting $(g \to gg)$





- The modern version of this idea: the non-linear BK and JIMWLK equations

  (Balitsky, 96; Kovchegov, 99; Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, and Kovner, 97–00)

The saturation momentum

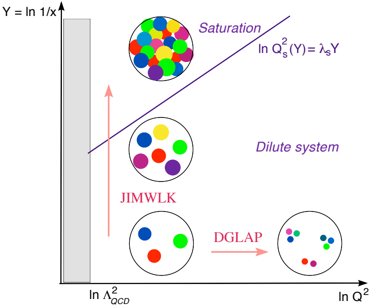
- Remember: in order to overlap, gluons must have large enough transverse sizes/small enough transverse momenta: $k_\perp^2 \lesssim Q_s^2(x)$
- $Q_s^2(x)$: the value of $k_\perp^2$ at which the occupation number is of $\mathcal{O}(1/\alpha_s)$

$$n(x,Q^2) \simeq \frac{xG(x,Q^2)}{R^2\,Q^2} \sim \frac{1}{\alpha_s} \quad \text{when} \quad Q^2 \lesssim Q_s^2(x)$$

- Parametrically $(N_g \equiv N_c^2 - 1)$

$$Q_s^2(x) \simeq \bar{\alpha}\,\frac{xG(x,Q_s^2)}{N_g R^2} \sim \frac{1}{x^{\lambda_s}}$$

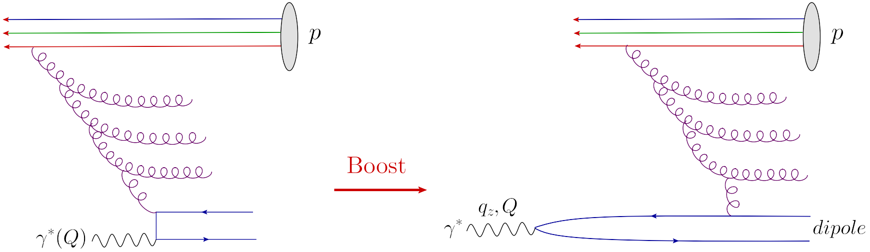
- $Q_s^2(x)$ rises with $1/x$: $\lambda_s \simeq 0.2$



- For sufficiently small x, $Q_s^2(x) \gg \Lambda^2_{\text{QCD}} \implies \alpha_s\big(Q_s^2(x)\big) \ll 1 \implies$ pQCD

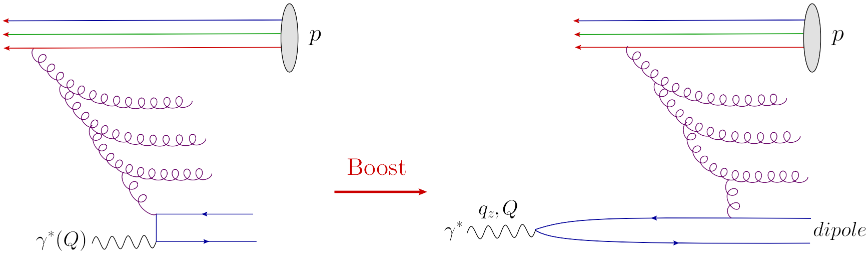
The dipole frame

- Physical picture & calculation details depend upon Lorentz frame & gauge
- Parton picture manifest in the target infinite momentum frame ($p_z \to \infty$) and in the light-cone gauge (gluons have transverse polarizations)
- Bjorken frame: $q^\mu = (q_0, \boldsymbol{q}_\perp, 0)$, $q_0 = \frac{p \cdot q}{p_0} \to 0$, $Q^2 \simeq q_\perp^2$
  - the virtual photon is a good analyser of the proton transverse structure
  - scattering counts the # of quarks with transverse size $\Delta x_\perp \sim 1/Q$



The dipole frame

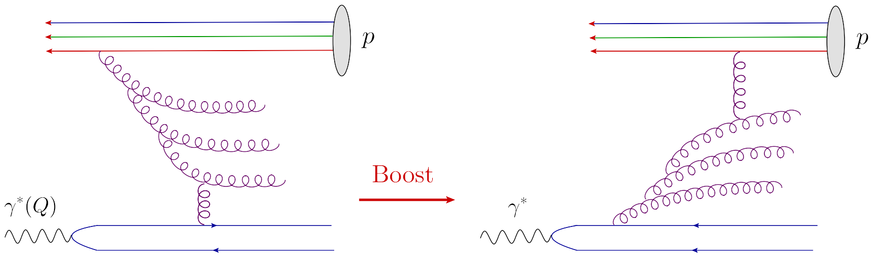
- Physical picture & calculation details depend upon Lorentz frame & gauge
- High energy/small x: boost to a frame where γ^* is energetic ($q_0 \simeq q_z \gg Q$):
  - the $q\bar{q}$ pair can now be seen as a part of the photon wavefunction
  - γ^* first fluctuates into a $q\bar{q}$ color dipole, which then scatters off the gluons from the proton wavefunction: factorization



$$x \equiv \frac{Q^2}{2p \cdot q} \ll 1 \iff \Delta t \simeq \frac{2q_z}{Q^2} \gg \frac{1}{p_z}$$

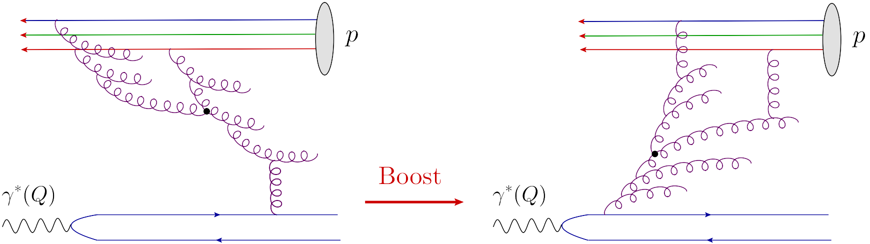
Dipole vs. proton evolution

- So far, most of the total energy is still carried by the proton ($p_z \gg q_z$)
  - the high-energy evolution (gluon emissions) associated with the proton
- By further boosting γ^*, one can transfer this evolution to the dipole
  - the virtual photon first fluctuates into a $q\bar{q}$ color dipole
  - the color dipole evolves via soft ($x = \frac{k_z}{q_z} \ll 1$) gluon emissions
  - the dressed dipole scatters off the valence quarks from the proton
  - dipole formation & evolution can be factorized from the collision



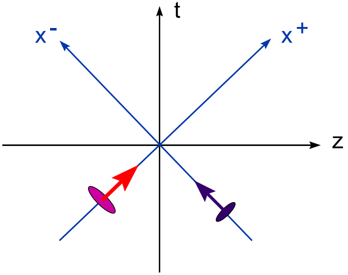
Dipole vs. proton evolution

- So far, most of the total energy is still carried by the proton ($p_z \gg q_z$)
  - the high-energy evolution (gluon emissions) associated with the proton
- The boost has particularly interesting consequences for the non-linear effects
  - proton IMF: non-linear effects in the evolution (gluon saturation)
  - dipole frame: linear evolution of the dipole + multiple scattering
  - recombination ($gg \to g$) gets mapped onto splitting ($g \to gg$)
  - gluon saturation gets mapped onto fluctuations in the BFKL evolution



Light-cone variables

- Four-momentum: $p^\mu = (p^0, p^1, p^2, p^3) \equiv (p_0, \boldsymbol{p}_\perp, p_z) = (p^+, p^-, \boldsymbol{p}_\perp)$



$$p^\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_z)$$

$$x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$$

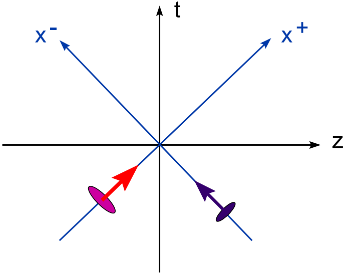
$$p \cdot x = p^+ x^- + p^- x^+ - \boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp$$

$$\mathrm{d}t\mathrm{d}z = \mathrm{d}x^+ \mathrm{d}x^-$$

- Ultrarelativistic right mover: classical trajectory $z = t$
  - $x^+ = \sqrt{2}t$ (LC time) & $x^- = 0$ (LC longitudinal coordinate)
  - "the particle sits at $x^- = 0$" (the "rest frame" of the UR particle)
- Left mover: the roles of x^+ and x^- (or p^+ and p^-) get interchanged

Light-cone variables

- Four-momentum: $p^\mu = (p^0, p^1, p^2, p^3) \equiv (p_0, \boldsymbol{p}_\perp, p_z) = (p^+, p^-, \boldsymbol{p}_\perp)$



$$p^\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_z)$$

$$x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$$

$$p \cdot x = p^+ x^- + p^- x^+ - \boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp$$

$$\mathrm{d}t\mathrm{d}z = \mathrm{d}x^+\mathrm{d}x^-$$

- On-shell particle (RM): $p^2 \equiv 2p^+p^- - p_\perp^2 = m^2$ & $p_z \simeq p_0 \gg p_\perp, m$

$$p^- = \frac{p_\perp^2 + m^2}{2p^+} \ll m_\perp \equiv \sqrt{p_\perp^2 + m^2} \ll p^+$$

  - $p^+ \simeq \sqrt{2}p_z$ (LC longitudinal momentum) & $p^- \simeq 0$ (LC energy)

- Boost-invariant def. for the longitudinal momentum fraction: $x \equiv k^+/p^+$

Rapidities

- Consider an on-shell particle: $p^\mu = (E, \boldsymbol{p}_\perp, p_z)$ with $E = \sqrt{m^2 + p_\perp^2 + p_z^2}$

$$y \equiv \frac{1}{2} \ln \frac{E + p_z}{E - p_z} = \frac{1}{2} \ln \frac{p^+}{p^-}$$

- Positive for a 'right-mover' $(p_z > 0)$ & negative for a 'left-mover' $(p_z < 0)$

$$E = m_\perp \cosh y, \quad p_z = m_\perp \sinh y, \quad p^\pm = \frac{m_\perp}{\sqrt{2}} \mathrm{e}^{\pm\eta}$$

- y transforms via a shift under a Lorentz boost along the collision axis

$$E \to \gamma(E + \beta p_z), \quad p_z \to \gamma(p_z + \beta E) \implies y \to y + \frac{1}{2} \ln \frac{1+\beta}{1-\beta}$$

  - β: boost velocity; $\gamma \equiv 1/\sqrt{1-\beta^2}$: Lorentz boost factor
  - rapidity differences $\Delta y_{ij} = y_i - y_j$ are boost invariant

Rapidities

- Consider an on-shell particle: $p^\mu = (E, \boldsymbol{p}_\perp, p_z)$ with $E = \sqrt{m^2 + p_\perp^2 + p_z^2}$

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$$E = m_\perp \cosh y, \quad p_z = m_\perp \sinh y, \quad p^\pm = \frac{m_\perp}{\sqrt{2}} \, \mathrm{e}^{\pm\eta}$$

- In the experiments, it is easier to measure angles $\Longrightarrow$ "pseudo-rapidities"

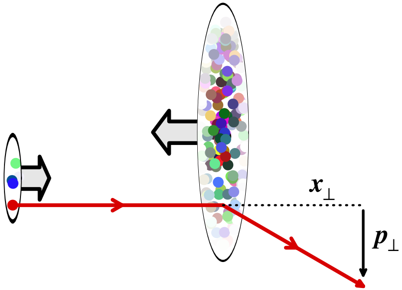
$$\eta \equiv \frac{1}{2} \ln \frac{p + p_z}{p - p_z} = - \ln \tan \frac{\theta}{2} \,, \quad \cos\theta = \frac{p_z}{p} \,, \ \sin\theta = \frac{p_\perp}{p}$$

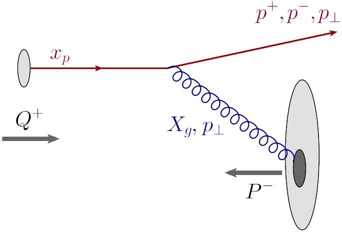
- $p \equiv |\vec{\mathbf{p}}| = \sqrt{p_\perp^2 + p_z^2} \quad \Longrightarrow \quad y = \eta$ for massless particles

- In what follows, all particles will be massless !

Particle production in pA collisions: Kinematics

- A quark initially collinear with the proton acquires a transverse momentum $p_\perp \sim Q_s$ via multiple scattering off the gluons inside the nucleus
- Formally, a $2 \to 1$ process: $qg \to q$ (in general: $qg \ldots g \to q$)



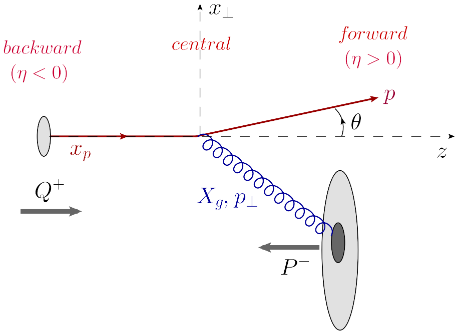


- incoming quark: $q^\mu = x_p Q^\mu = (x_p Q^+, 0, \mathbf{0}_\perp)$
- participating gluon: $k^\mu = (0, X_g P^-, \boldsymbol{p}_\perp)$
- produced quark: $p^\mu = q^\mu + k^\mu = (x_p Q^+, X_g P^-, \boldsymbol{p}_\perp)$

pA collisions: Forward production

- View the process in the COM frame of the proton-nucleon pair

$$s = (Q+P)^2 = 2Q^+P^-, \quad Q^+ = P^- = \sqrt{s/2}$$



$$p^\pm = \frac{p_\perp}{\sqrt{2}}\, e^{\pm\eta}$$

$$x_p \equiv \frac{p^+}{Q^+} = \frac{p_\perp}{\sqrt{s}}\, e^{\eta}$$

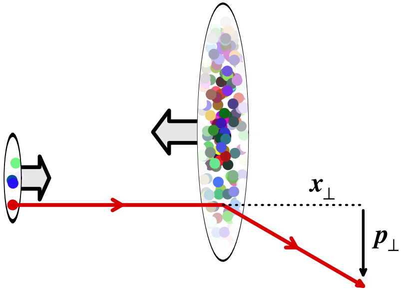
$$X_g \equiv \frac{p^-}{P^-} = \frac{p_\perp}{\sqrt{s}}\, e^{-\eta}$$

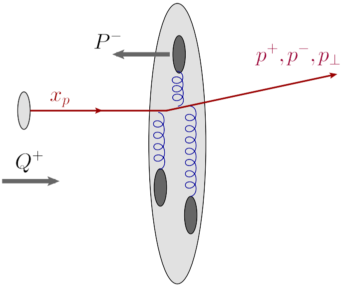
$X_g \ll x_p$ when $\eta > 0$

- $\eta = -\ln\tan(\theta/2)$: rapidity of the produced quark in COM frame
- Forward production probes gluon evolution towards small x (in the target)

Particle production in pA collisions: Kinematics

- A quark initially collinear with the proton acquires a transverse momentum $p_\perp \sim Q_s$ via multiple scattering off the gluons inside the nucleus
- How to include multiple scattering to all orders ?





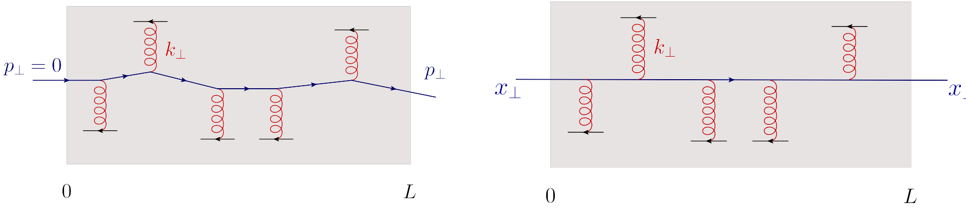
- A priori encoded in the formal definition of the S-matrix operator in QCD ...

$$\hat{S} = \mathrm{T}\, \mathrm{e}^{\mathrm{i} \int \mathrm{d}^4 x\, \mathcal{L}_{\rm int}(x)} \quad \text{with} \quad \mathcal{L}_{\rm int}(x) = j_a^\mu(x) A_\mu^a(x)$$

- ... but how to compute things in practice ?

Eikonal approximation

- A priori, these are quantum operators encoding evolution, fluctuations ....
  - $j^\mu_a(x)$: color current for partons in the projectile; e.g. $j^\mu_a = g\bar\psi\gamma^\mu t^a\psi$
  - $A^a_\mu(x)$: color field representing the gluons from the target
- At high energy the structure of the interactions gets drastically simplified
- Right-mover projectile: $j^\mu_a \propto v^\mu = \delta^{\mu +} \Longrightarrow \mathcal{L}_{\rm int}(x) = j^+_a(x) A^-_a(x)$
- Transverse coordinates are not changed by the scattering off the shockwave



  - let $L = R/\gamma$ denote the target width in a generic frame
  - transverse deviation is irrelevant if smaller than the wavelength $\lambda_\perp$

Eikonal approximation

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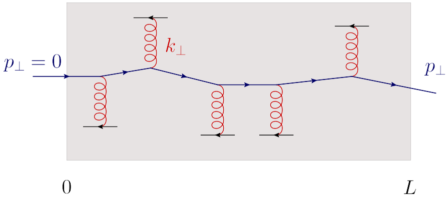
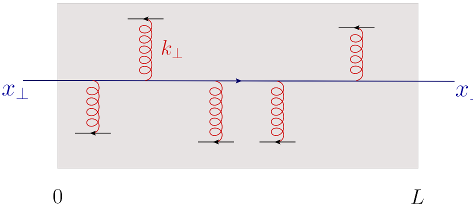




$$\Delta x_\perp \simeq \frac{p_\perp}{E} L \ll \lambda_\perp \sim \frac{1}{p_\perp} \Longrightarrow \gamma E \sim 10^6\,\text{GeV} \gg Q_s^2 R \sim 30 \div 100\,\text{GeV}$$

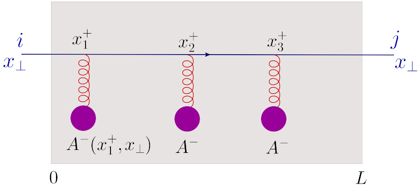
Wilson lines

- S-matrix operator: $\hat{S} = \mathrm{T}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\rm int}(x)}$ with $\mathcal{L}_{\rm int}(x) = j_a^\mu(x) A_\mu^a(x)$
- The color current density of a quark in the eikonal approximation:

$$j_a^\mu(x) \simeq g v^\mu t^a \delta(z-t)\delta^{(2)}(\boldsymbol{x}_\perp - \boldsymbol{x}_\perp^0) \simeq \delta^{\mu +} g t^a \delta(x^-)\delta^{(2)}(\boldsymbol{x}_\perp - \boldsymbol{x}_\perp^0)$$

$$\hat{S} \simeq V(\boldsymbol{x}_\perp^0) \quad \text{with} \quad V(\boldsymbol{x}_\perp) \equiv \mathrm{T}\exp\left\{ ig \int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}_\perp) t^a \right\}$$

- An exponential : multiple scattering is resummed to all orders
- A color matrix (here, in the fundamental representation)
- A unitary matrix: $V(x_\perp)V^\dagger(x_\perp) = 1 \Longrightarrow$ a rotation of the quark color state



$$\Psi_i(x_\perp) \longrightarrow V_{ji}(x_\perp)\,\Psi_i(x_\perp)$$

Wilson lines

- S-matrix operator: $\hat{S} = \mathrm{T}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\rm int}(x)}$ with $\mathcal{L}_{\rm int}(x) = j_a^\mu(x)A_\mu^a(x)$
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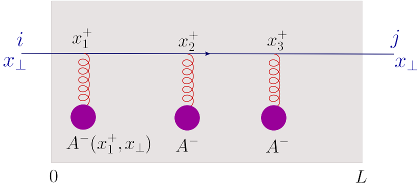
$$V_N(\boldsymbol{x}_\perp) = \mathrm{e}^{\mathrm{i}g\epsilon A_N^-}\,\mathrm{e}^{\mathrm{i}g\epsilon A_{N-1}^-}\cdots\mathrm{e}^{\mathrm{i}g\epsilon A_1^-}\,\mathrm{e}^{\mathrm{i}g\epsilon A_0^-} \qquad \left(A_n^- \equiv A_a^-(x_n^+,\boldsymbol{x}_\perp)t^a\right)$$

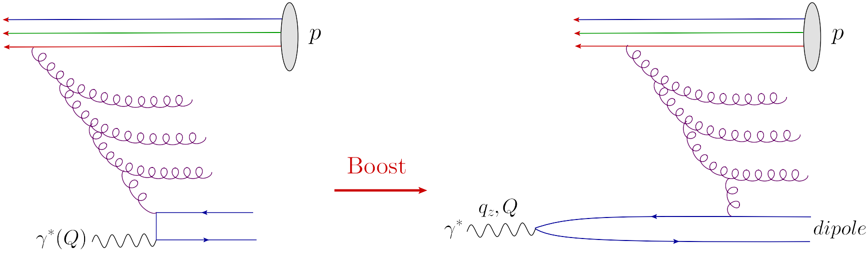
- A time-ordered exponential: color matrices do not commute with each other
- Best understood with a discretization of time: $x_n^+ = n\epsilon$, $n = 0, 1, \cdots N$
- A unitary matrix: $V(x_\perp)V^\dagger(x_\perp) = 1 \Longrightarrow$ a rotation of the quark color state



$$\Psi_i(x_\perp) \longrightarrow V_{ji}(x_\perp)\,\Psi_i(x_\perp)$$

Dipole factorization for DIS (1)

- Recall: the dipole frame $\Longrightarrow$ factorization in time

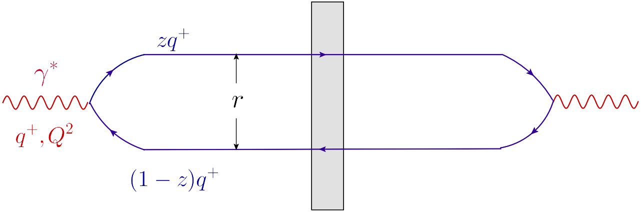


$$x \equiv \frac{Q^2}{2p \cdot q} \ll 1 \iff \Delta t \simeq \frac{2q^+}{Q^2} \gg \frac{1}{p^-}$$

- $\sigma_{\gamma^* p}$ = [probability for $\gamma^* \to q\bar{q}$] (QED) $\times$ [$\sigma_{q\bar{q}p}$] (QCD)
- The $q\bar{q}$ pair is in a color singlet state: $\frac{1}{\sqrt{3}}\big(|R\bar{R}\rangle + |B\bar{B}\rangle + |G\bar{G}\rangle\big) = |\text{dipole}\rangle$

Dipole factorization for DIS (2)

- Optical theorem: total cross-section = imaginary part of the forward scattering amplitude



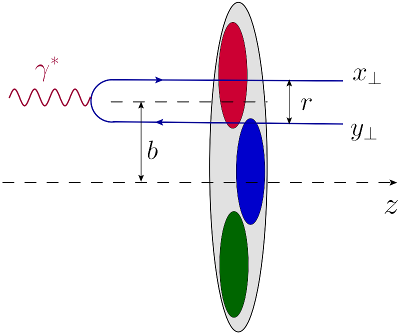
$$\sigma_{\gamma^* p}(Q^2, x) = \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \left|\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)\right|^2 \sigma_{\rm dipole}(r, x)$$

- γ^* wavefunction $\Psi_{\gamma^* \to q\bar{q}}(r, z; Q^2)$: computed in QED perturbation theory
- $r^2 \sim 1/Q^2$: dipole transverse size (dipole resolution in the transverse plane)
- The dipole cross-section $\sigma_{\rm dipole}$: encodes the QCD scattering and evolution

The dipole S-matrix

$$\sigma_{\rm dipole}(r,x) = 2\int {\rm d}^2\boldsymbol{b}\, T(\boldsymbol{r},\boldsymbol{b},x)$$

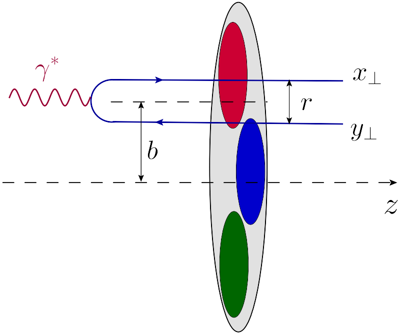
- $\boldsymbol{r} = \boldsymbol{x}_\perp - \boldsymbol{y}_\perp$: dipole size
- $\boldsymbol{b} = (\boldsymbol{x}_\perp + \boldsymbol{y}_\perp)/2$: impact parameter
- $T(\boldsymbol{r},\boldsymbol{b},x) = 1 - \langle \hat{S} \rangle$: dipole amplitude
- $T \leq 1$: unitarity bound



- 2 Wilson lines: $V(\boldsymbol{x}_\perp)$ for the quark (q) and $V^\dagger(\boldsymbol{y}_\perp)$ for the antiquark $(\bar{q})$
  - an antiquark has charge $(-g)$ and propagates backwards in time
- Color singlet: the same color states for q and $\bar{q}$ before & after the scattering
  - sum over final color states & and average over the initial ones

The dipole S-matrix

$$\sigma_{\rm dipole}(r,x) = 2\int \mathrm{d}^2\boldsymbol{b}\; T(\boldsymbol{r},\,\boldsymbol{b},x)$$



- $\boldsymbol{r} = \boldsymbol{x}_\perp - \boldsymbol{y}_\perp$: dipole size
- $\boldsymbol{b} = (\boldsymbol{x}_\perp + \boldsymbol{y}_\perp)/2$: impact parameter
- $T(\boldsymbol{r},\,\boldsymbol{b},x) = 1 - \langle\hat{S}\rangle$: dipole amplitude
- $T \le 1$: unitarity bound

$$\hat{S}_{\rm dipole}(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp) = \frac{1}{N_c}\,V^\dagger_{ij}(\boldsymbol{y}_\perp)V_{ji}(\boldsymbol{x}_\perp) \;=\; \frac{1}{N_c}\,\mathrm{tr}\big(V(\boldsymbol{x}_\perp)V^\dagger(\boldsymbol{y}_\perp)\big)$$

- $\langle\hat{S}\rangle$: average over the color fields A^-_a in the target: CGC effective theory

$$\langle S_{\boldsymbol{xy}}\rangle \;=\; \int [\mathcal{D}A^-]\; W[A^-]\;\frac{1}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)[A^-]$$

- $W[A^-]$: functional probability distribution (gauge-invariant)

The single scattering approximation (1)

- Expanding the exponential in the Wilson lines: multiple scattering series

$$V(\boldsymbol{x}_\perp) = 1 + ig\int \mathrm{d}x^+ A_a^-(x^+,\boldsymbol{x}_\perp)t^a$$
$$-\frac{g^2}{2}\int \mathrm{d}x^+\int \mathrm{d}y^+\left[\theta(x^+-y^+)t^a t^b + \theta(y^+-x^+)t^b t^a\right]A_a^-(x^+)A_a^-(y^+) + \cdots$$

- Dipole S-matrix in the 2-gluon exchange approximation: single scattering

$$\hat{S}_{\rm dipole}(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp) = 1 - \frac{g^2}{4N_c}\left[A_a^-(\boldsymbol{x}_\perp) - A_a^-(\boldsymbol{y}_\perp)\right]^2 + \mathcal{O}(g^3)$$

$$A_a^-(\boldsymbol{x}_\perp) \equiv \int \mathrm{d}x^+ A_a^-(x^+,\boldsymbol{x}_\perp), \quad \mathrm{tr}\, t^a = 0, \quad \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$

- N.B. Color non-commutativity becomes important at $\mathcal{O}(g^3)$ and higher

$$A_a^-(\boldsymbol{x}_\perp) - A_a^-(\boldsymbol{y}_\perp) \simeq (x_\perp^i - y_\perp^i)\frac{\partial}{\partial b^i}A_a^-(\boldsymbol{b}_\perp) = r^i F_a^{i-}(\boldsymbol{b}_\perp)$$

The single scattering approximation (1)

- Expanding the exponential in the Wilson lines: multiple scattering series

$$V(\boldsymbol{x}_\perp) = 1 + ig\int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}_\perp)t^a$$
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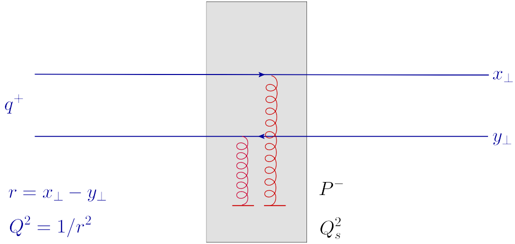
- N.B. Color non-commutativity becomes important at $\mathcal{O}(g^3)$ and higher
- After averaging over A^- $\Longrightarrow$ the gluon distribution in the target

$$\int \mathrm{d}^2\boldsymbol{b}\, \langle F_a^{i-}(\boldsymbol{x})\, F_a^{i-}(\boldsymbol{y})\rangle \propto xG(x, Q^2 = 1/r^2)$$

The single scattering approximation (2)

- The dipole is a direct probe of the gluon distribution in the hadronic target
- Remember: a QED dipole couples to the electric field $\vec{E}$: $V(\boldsymbol{r}) = gr^i E^i$
- A ultrarelativistic color dipole couples to the chromo-electric field F_a^{i-}

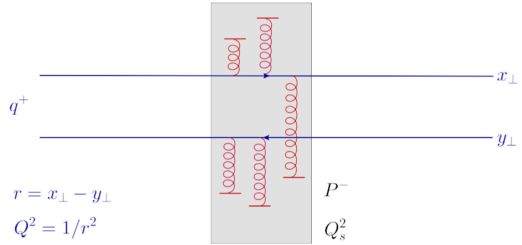


$$\sigma_{\rm dipole}(r,x) \simeq 2\pi^2\alpha_s r^2 \,\frac{C_F}{N_g}\, xG(x,1/r^2) \simeq 2\pi R^2 \left[r^2 Q_s^2(x)\right]$$

- Single scattering is valid so long as $T_0(r,x) \simeq r^2 Q_s^2(x) \ll 1$
- The scattering amplitude vanishes like r^2: color transparency

Multiple scattering

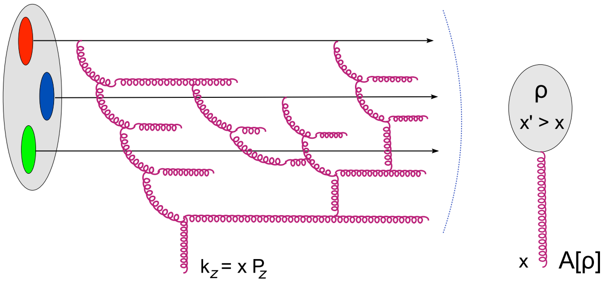
- A hadronic cross section cannot growth with E "much faster" than the geometric cross-section
  - $\sigma(E) \leq 2\pi R^2 \ln^2(E/M)$: Froissart bound
- Single scattering is a good approximation only so long as $rQ_s(x) \ll 1$



- When $r \gtrsim 1/Q_s(x)$, the scattering probes a dense gluon distribution
  - multiple scattering should become important
- 'Duality': gluon saturation in the target $\longleftrightarrow$ unitarization $T \sim 1$

Color Glass Condensate

- Small–x gluons : classical color fields $A_a^\mu[\rho]$ radiated by a frozen distribution ρ_a of color charges representing partons with $x' \gg x$
  - lifetimes are strongly ordered by Lorentz time dilation: $\Delta t = 2xP_z/k_\perp^2$



- $W_Y[\rho]$: CGC weight function, built via renormalization group ("JIMWLK")
  - successively integrating out gluons in layers of x': high-energy evolution
  - initial condition at $x' \sim 0.1 \div 0.01$: McLerran-Venugopalan model

The McLerran-Venugopalan model (1994)

- A model for the gluon distribution in a large nucleus $(A \gg 1)$ at not that small values of x $(x \sim 0.01)$ $\Rightarrow$ quantum evolution can be neglected

- A large nucleus: a collection of $A \times N_c \simeq 600$ valence quarks acting as independent color sources $\Rightarrow$ A Gaussian CGC weight function

- If the nucleus is a ultrarelativistic left mover $\Rightarrow$ $J_a^\mu(x) \simeq \delta^{\mu -} \rho_a(x^+, \boldsymbol{x})$
  - independent of x^- (LC time for a left-mover) by Lorentz time dilation
  - color charge density ρ_a localized near $x^+ = 0$ by Lorentz contraction

$$W_0[\rho] = \exp\left\{ - \int \mathrm{d}x^+ \mathrm{d}^2\boldsymbol{x}\, \frac{\rho_a(x^+, \boldsymbol{x})\rho_a(x^+, \boldsymbol{x})}{2\lambda(x^+, \boldsymbol{x})} \right\}$$

- $\lambda(x^+, \boldsymbol{x})$: density of color charge squared (for one quark: $(gt^a)^2 = g^2 C_F$)

$$\int \mathrm{d}x^+ \mathrm{d}^2\boldsymbol{x}\, \lambda(x^+, \boldsymbol{x}) \,=\, \frac{AN_c\, g^2 C_F}{N_c^2 - 1} \,\equiv\, \pi R_A^2 \mu^2$$

- Small-x observables (e.g. dipole scattering) cannot discriminate the local structure in x^+ $\Rightarrow$ only sensitive to $\mu^2 \propto A^{1/3}$

The color field of a shockwave

- Solution to the classical Yang-Mills equations:

$$D_\nu^{ab} F_b^{\mu\nu}(x) = \delta^{\mu -} \rho_a(x^+, \boldsymbol{x})$$

  - $D_\nu^{ab} = \partial_\nu \delta^{ab} - g f^{abc} A_\nu^c, \; F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g f^{abc} A_b^\mu A_c^\nu$

- The solution is particularly simple due to the special structure of the current:

$$F_a^{ij} = 0, \quad A_a^+ = 0, \quad A_a^i, \, A_a^- : \quad \text{independent of } x^-$$

- Covariant gauge: $\partial_\mu A_a^\mu = \partial_i A_a^i = 0 \Rightarrow A_a^i = 0 \Rightarrow$ just a Coulomb field A_a^-

$$-\nabla_\perp^2 A_a^-(x) \;=\; \rho_a(x^+, \boldsymbol{x}) \; : \quad \text{linear equation, local in } x^+$$

$$A_a^-(x^+, \boldsymbol{x}) = \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} \, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, \frac{\rho_a(x^+, \boldsymbol{k})}{k_\perp^2} \; = \int \frac{\mathrm{d}^2\boldsymbol{y}}{4\pi} \, \ln \frac{1}{(\boldsymbol{x}-\boldsymbol{y})^2 \Lambda^2} \, \rho_a(x^+, \boldsymbol{y})$$

  - Λ: "infrared cutoff" introduced by confinement

- The only non-zero field strength components: $F_a^{i-}(x^+, \boldsymbol{x}) = \partial^i A_a^-$

Gluons & the light-cone gauge

- Valence quarks and their Coulomb fields, localized in the longitudinal direction (x^+) ... but where are the gluons ?!

- They only "live" in the nucleus light-cone gauge $A^-_a = 0$: the would-be large component is forced to vanish $\Longrightarrow$ transverse components A^i_a

$$D_i^{ab} F_b^{i-}(x) = -D_i^{ab} \frac{\partial A_b^i}{\partial x^+} = \rho_a(x^+, \boldsymbol{x})$$

  - two non-zero components A^1_a and A^2_a, which however obey $F^{ij}_a = 0$
  - non-locality in $x^+ \Longrightarrow$ delocalization

- The weak field ("QED") case first: linear equation, Fourier transform

$$A_a^i(x^+, \boldsymbol{x}) = \int \frac{\mathrm{d}k^- \mathrm{d}^2\boldsymbol{k}}{(2\pi)^3} \, \mathrm{e}^{-ik^-x^+ + i\boldsymbol{k}\cdot\boldsymbol{x}} \, \frac{-k^i}{k^- + i\epsilon} \, \frac{\rho_a(k^-, \boldsymbol{k})}{k_\perp^2}$$

  - $\partial^- \equiv \partial/\partial x^+ \to 1/k^- \Longrightarrow$ axial pole at $k^- = 0$
  - "retarded"/"advanced" prescription $\pm i\epsilon$: physically irrelevant

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$$D_i^{ab} F_b^{i-}(x) \;=\; -D_i^{ab}\,\frac{\partial A_b^i}{\partial x^+} \;=\; \rho_a(x^+,\boldsymbol{x})$$

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$$\mathcal{A}_a^i(x^+,\boldsymbol{x}) \;=\; \int \mathrm{d}y^+\Theta(x^+ - y^+)\int\frac{\mathrm{d}^2\boldsymbol{y}}{2\pi}\,\frac{\boldsymbol{x}^i - \boldsymbol{y}^i}{(\boldsymbol{x}-\boldsymbol{y})^2}\,\rho_a(y^+,\boldsymbol{y})$$

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  - non-locality in $x^+ \Longrightarrow$ delocalization

- The weak field ("QED") case first: linear equation, Fourier transform
- Color charge (valence quarks) localized near $y^+ = 0$: $\rho_a(y^+, \boldsymbol{y}) \simeq \delta(y^+) \rho_a(\boldsymbol{y})$

$$\mathcal{A}_a^i(x^+, \boldsymbol{x}) \simeq \Theta(x^+) \int \frac{\mathrm{d}^2\boldsymbol{y}}{2\pi} \frac{\boldsymbol{x}^i - \boldsymbol{y}^i}{(\boldsymbol{x} - \boldsymbol{y})^2} \rho_a(\boldsymbol{y})$$

  - Weiszäcker-Williams field: quasi-real gluons $\Longrightarrow$ "partons"
  - modes with a given momentum k^- are delocalized over $\Delta x^+ \sim 1/k^-$

Gluons & the light-cone gauge

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- They only "live" in the nucleus light-cone gauge $A_a^- = 0$: the would-be large component is forced to vanish $\Longrightarrow$ transverse components A_a^i

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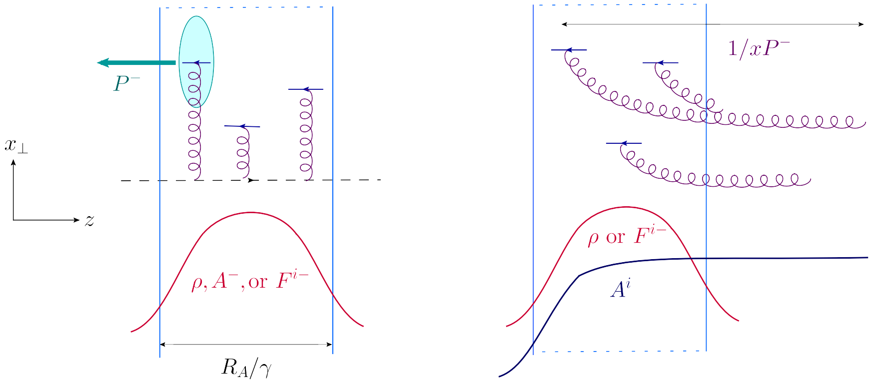
  - two non-zero components A_a^1 and A_a^2, which however obey $F_a^{ij} = 0$
  - non-locality in $x^+ \Longrightarrow$ delocalization

- The non-linear Yang-Mills equation can be analytically solved too
  - via a gauge rotation from the COV gauge, where $A_a^\mu = \delta^{\mu-} A_a^-$

$$\mathcal{A}_a^i(x^+, \boldsymbol{x}) t^a \simeq \Theta(x^+) \frac{i}{g} V^\dagger(\boldsymbol{x}) \partial^i V(\boldsymbol{x}), \qquad V(\boldsymbol{x}) \equiv \mathrm{T} e^{ig \int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}_\perp) t^a}$$

  - N.B.: LC-gauge field $\mathcal{A}^i$ related to the color charge ρ in the COV gauge

- Covariant gauge: instantaneous Coulomb exchanges between the projectile and the target; eikonal coupling j^+A^-; no saturation.
- LC gauge $A^- = 0$: delocalized gluons which overlap in z (or x^+); non-linear effects (gluon saturation); non-eikonal coupling to the projectile: j^iA^i.

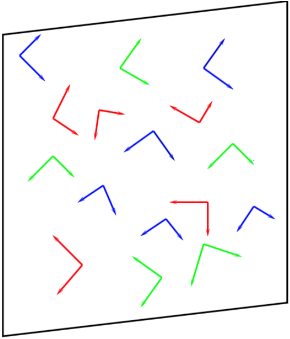


The Weiszäcker-Williams fields

- The only non-trivial field strength component: F_a^{i-} (in any gauge)
- Non-zero transverse components for the electric and the magnetic fields

$$F_a^{i+} = 0 \;\Rightarrow\; F^{i0} = -F^{i3}, \qquad F^{+-} = 0 \;\Rightarrow\; E^3 = B^3 = 0$$

$$F^{i\pm} = \frac{1}{\sqrt{2}}\big(F^{i0} \pm F^{i3}\big) \;\Longrightarrow\; \begin{cases} E^1 \equiv -F^{10} = F^{13} \equiv B^2 \\ E^2 \equiv -F^{20} = F^{23} \equiv -B^1 \end{cases}$$



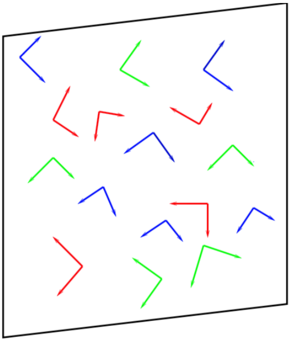
$$\boldsymbol{E}_a \perp \boldsymbol{B}_a \perp \boldsymbol{z}$$

$$\boldsymbol{E}_\perp \cdot \boldsymbol{B}_\perp = 0\,, \quad |\boldsymbol{E}_\perp| = |\boldsymbol{B}_\perp| \sim \frac{1}{g}$$

- transverse polarizations
- chromo-electromagnetic waves
- Lorentz contraction: $\propto \delta(x^+)$

The Weiszäcker-Williams fields

- The only non-trivial field strength component: F_a^{i-} (in any gauge)
- Non-zero transverse components for the electric and the magnetic fields
- Similar to plane waves describing free photons/gluons: "equivalent photons"
- Fields vary over a distance $\sim 1/Q_s \Longrightarrow$ gluons typically have $k_\perp \sim Q_s$
- Fields have strength $\sim 1/g \Longrightarrow$ gluons have occupation numbers $\sim 1/\alpha_s$



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- transverse polarizations
- chromo-electromagnetic waves
- Lorentz contraction: $\propto \delta(x^+)$

The gluon distribution

- Gluon distribution $xG(x, Q^2)$: # of gluons with a given longitudinal momentum fraction x and transverse momenta $k_\perp \leq Q$ ... in the LC gauge

$$xG(x,Q^2) = \int \mathrm{d}^2\boldsymbol{k}_\perp \Theta(Q^2 - k_\perp^2) \int \mathrm{d}^2\boldsymbol{b}_\perp \; k^- \frac{\mathrm{d}^2 N_{gluon}}{\mathrm{d}k^- \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{b}_\perp}\Big|_{k^-=xP^-}$$

- Occupation number in phase-space $(\boldsymbol{r} = \boldsymbol{x}_\perp - \boldsymbol{y}_\perp,\, \boldsymbol{b} = (\boldsymbol{x}_\perp + \boldsymbol{y}_\perp)/2)$:

$$n(x, \boldsymbol{k}_\perp, \boldsymbol{b}_\perp) \;=\; \frac{1}{N_c^2 - 1} \int_{x^+, y^+} \int_{\boldsymbol{r}_\perp} \mathrm{e}^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp} \langle F_a^{i-}(x^+, \boldsymbol{x}_\perp) F_a^{i-}(y^+, \boldsymbol{y}_\perp)\rangle\Big|_{A^-=0}$$

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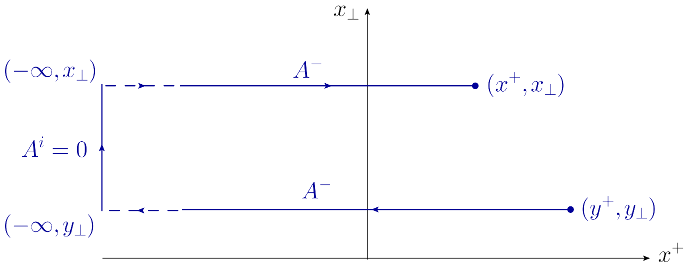
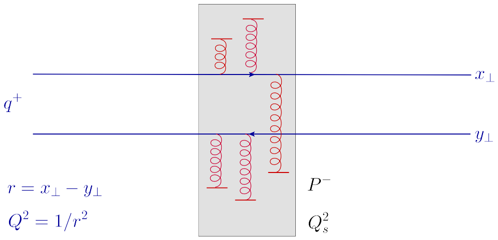
$$n(x, \boldsymbol{k}_\perp, \boldsymbol{b}_\perp) \; = \; \frac{1}{N_c^2 - 1} \int_{x^+, y^+} \int_{\boldsymbol{r}_\perp} \mathrm{e}^{-i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp} \langle F_a^{i-}(x^+, \boldsymbol{x}_\perp) W_\gamma^{ab}(x,y) F_b^{i-}(y^+, \boldsymbol{y}_\perp)\rangle$$



Exploring multiple scattering in the CGC

- A relatively simple observable: a quark-antiquark "color dipole"



$$S(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\boldsymbol{x})V^\dagger(\boldsymbol{y})\right]\right\rangle$$

- A Gaussian Ansatz for the gluon distribution: MV model for a large nucleus

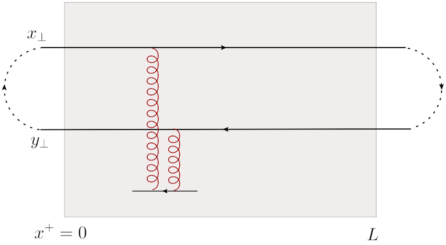
$$\langle \rho_a(x^+, \boldsymbol{x})\rho_b(y^+, \boldsymbol{y})\rangle = \delta^{ab}\delta(x^+ - y^+)\delta^{(2)}(\boldsymbol{x} - \boldsymbol{y})\lambda(x^+)$$

- Classical Yang-Mills solution covariant gauge: $-\nabla_\perp^2 A_a^-(x) = \rho_a(x^+, \boldsymbol{x})$

- Explicit analytic results, which allow for a transparent physical picture

Dipole scattering in the MV model

- Independent scatterings $\Longrightarrow$ the multiple scattering series exponentiates



$$S(r) = \mathrm{e}^{-T_0(r)}$$

$$T_0(r) = \frac{g^2}{4N_c}\left\langle \left[A_a^-(\boldsymbol{x}_\perp) - A_a^-(\boldsymbol{y}_\perp)\right]^2 \right\rangle$$

$$A_a^-(\boldsymbol{x}) \equiv \int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x})$$

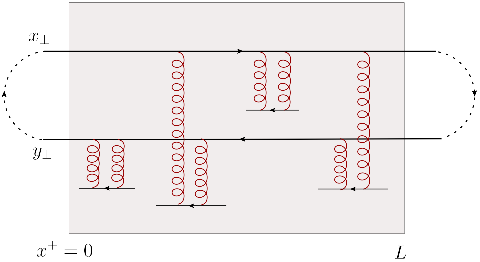
$$\langle A_a^-(\boldsymbol{x}) A_b^-(\boldsymbol{y})\rangle = \delta^{ab}\mu^2 \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\,\frac{\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{k_\perp^4}\,, \qquad \mu^2 \equiv \int \mathrm{d}x^+ \lambda(x^+) = \frac{g^2 A}{2\pi R_A^2}$$

- A factor $1/k_\perp^2$ for each gluon exchange (Coulomb scattering)

$$T_0(r) = g^2 C_F \int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\,\frac{\mu^2}{k_\perp^4}\left[1 - \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\right] \simeq \frac{\alpha_s C_F}{4}\, r^2\mu^2 \ln\frac{1}{r^2\Lambda^2}$$

Dipole scattering in the MV model

- Independent scatterings $\Longrightarrow$ the multiple scattering series exponentiates



$$S(r) = \mathrm{e}^{-T_0(r)}$$

$$S(r) = \exp\left\{-\frac{r^2 Q_{0A}^2}{4}\ln\frac{1}{r^2\Lambda^2}\right\}$$

$$Q_{0A}^2 \equiv \alpha_s C_F \mu^2 = \frac{2\alpha_s^2 C_F A^{1/3}}{R^2}$$

- The dipole scatters off all the quarks within an area $\sim r^2$ around its impact parameter $b_\perp \Longrightarrow$ a tube with length $L = R_A/\gamma$, with $R_A = RA^{1/3}$

- $T_0(r) \propto r^2$: color transparency (cancellation between $q\bar{q}$ and qq, or $\bar{q}\bar{q}$)

- $\ln(1/r^2\Lambda^2)$: gluon exchanges within the range $r < \Delta x_\perp < 1/\Lambda$

Dipole scattering in the MV model (cont.)

$$S(r) = \mathrm{e}^{-T_0(r)} = \exp\left\{-\frac{r^2 Q_{0A}^2}{4}\ln\frac{1}{r^2\Lambda^2}\right\} \equiv 1 - T(r)$$

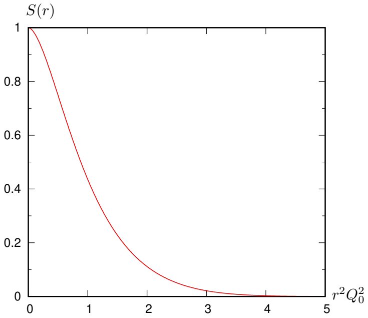
- $S(r)$: Probability for the $q\bar{q}$ pair to survive in a color singlet ("dipole") state
- $S(r) \to 1$ (i.e. $T(r) \to 0$) when $r \to 0$: color transparency
- Small enough dipole $\Longrightarrow$ weak scattering: $T(r) \simeq T_0(r) \ll 1$
- Scattering becomes strong, i.e. $T(r) \sim \mathcal{O}(1)$, when $T_0(r) \sim \mathcal{O}(1)$
- Saturation momentum Q_s: conventionally defined as $T_0(r) = 1$ for $r = \frac{2}{Q_s}$

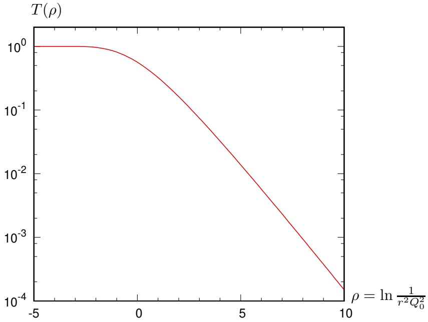
$$Q_s^2(A) \,=\, Q_{0A}^2\,\ln\frac{Q_s^2(A)}{4\Lambda^2}\,\propto\, A^{1/3}\ln A^{1/3}$$

- Gluon saturation in the nucleus manifests as multiple scattering for the probe
  - typical scale for the onset of non-linear physics

The saturation front (MV model)

- Left: the dipole S-matrix as a function of $r^2Q_0^2$

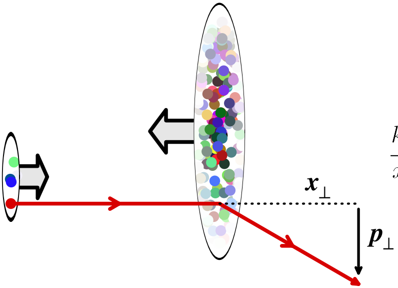


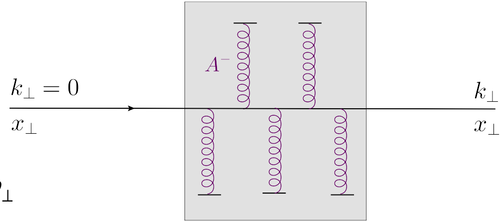


- Right: the dipole amplitude $T \equiv 1 - S$ as a function of $\rho \equiv \ln(1/r^2Q_0^2)$
  - small dipole $r \ll 1/Q_s \Longrightarrow$ large values for ρ : $T \simeq T_0 \sim \mathrm{e}^{-\rho}$
  - large dipole $r \gtrsim 1/Q_s \Longrightarrow$ negative ρ : $T = 1$

Transverse momentum broadening

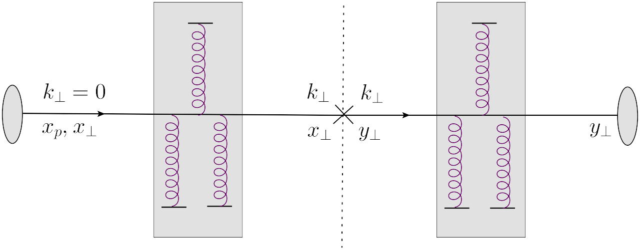
- A quark initially collinear with the proton acquires a transverse momentum $k_\perp \sim Q_s$ via multiple scattering off the gluons inside the nucleus
  - a random process leading to a distribution in the final momentum $\boldsymbol{k}_\perp$





- Transverse momentum broadening can be studied in the eikonal approx.
  - fixed transverse coordinate, but the transverse momentum can vary
  - a quark Wilson line $V(\boldsymbol{x}_\perp)$ built with the target field A_a^-

Dipole picture for pA collisions



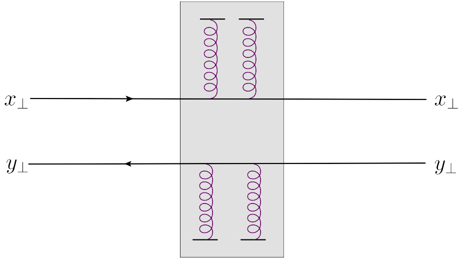
Amplitude: $$\mathcal{M}_{ij}(\boldsymbol{k}_\perp) \equiv \int \mathrm{d}^2\boldsymbol{x}_\perp \, \mathrm{e}^{-i\boldsymbol{x}_\perp\cdot\boldsymbol{k}_\perp} \, V_{ij}(\boldsymbol{x}_\perp)$$

Cross-section: $$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}_\perp} \simeq x_p q(x_p, Q^2) \, \frac{1}{N_c} \left\langle \sum_{ij} |\mathcal{M}_{ij}(\boldsymbol{k}_\perp)|^2 \right\rangle_{X_g}$$

- $x_p q(x_p, Q^2)$: quark distribution in the proton for $Q^2 \sim k_\perp^2$ and $x_p = \frac{k_\perp}{\sqrt{s}} \, \mathrm{e}^{\eta}$
- Average over the target: unintegrated gluon distribution at $X_g = \frac{k_\perp}{\sqrt{s}} \, \mathrm{e}^{-\eta}$

Dipole picture for pA collisions

- Two Wilson lines at different transverse coordinates, traced over color
- Equivalently: elastic S-matrix for a $q\bar{q}$ color dipole (here, a fictitious dipole)



$$S(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; X_g) \equiv \frac{1}{N_c} \left\langle \mathrm{tr}\left[V(\boldsymbol{x}_\perp)V^\dagger(\boldsymbol{y}_\perp)\right]\right\rangle_{X_g}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{k}} \simeq x_p q(x_p) \int \frac{\mathrm{d}^2\boldsymbol{x}\mathrm{d}^2\boldsymbol{y}}{(2\pi)^2} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}} \, S(\boldsymbol{x}, \boldsymbol{y}; X_g)$$

- Fourier transform $\mathcal{S}(\boldsymbol{k}, X_g)$ of the dipole S-matrix

Momentum broadening in the MV model (1)

- Consider an incoming quark for simplicity and use MV model for $S(r)$

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\ \mathrm{e}^{-\frac{1}{4}r^2 Q_{0A}^2 \ln \frac{1}{r^2\Lambda^2}}$$

- Two interesting situations which allow for simple results:

- Typical values $k_\perp \sim Q_s$, as transferred by multiple scattering
  - integral cut off at $r \sim 1/Q_s$ by the S-matrix $S(r)$
  - one can replace $4/r^2 \to Q_s^2$ within the argument of the log

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{1}{\pi Q_s^2(A)}\, \mathrm{e}^{-k_\perp^2/Q_s^2(A)}$$

  - a Gaussian distribution: random walk in $\boldsymbol{k}$

$$\langle k_\perp^2 \rangle \equiv \int \mathrm{d}^2\boldsymbol{k}\, k_\perp^2\, \frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = Q_s^2(A) \propto L = RA^{1/3}$$

Momentum broadening in the MV model (1)

- Consider an incoming quark for simplicity and use MV model for $S(r)$

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{e}^{-\frac{1}{4}r^2 Q_{0A}^2 \ln\frac{1}{r^2\Lambda^2}}$$

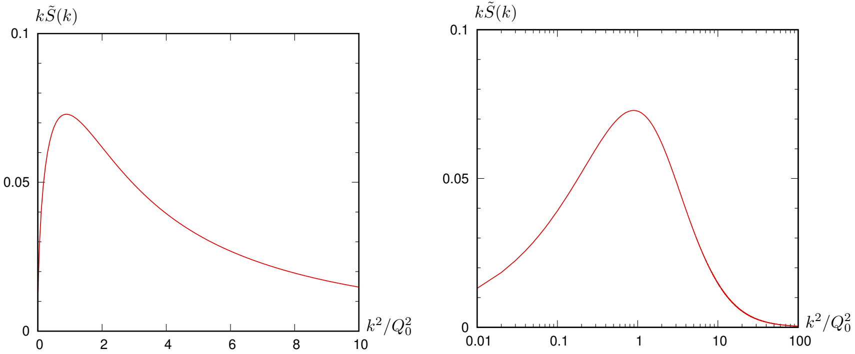
- Two interesting situations which allow for simple results:
- Large values $k_\perp \gg Q_s$, as given by a single hard scattering
  - integral cut off at $r \sim 1/k_\perp$ by the exponential
  - $rQ_s \ll 1 \implies$ one can expand $S \simeq 1 - T_0$ (one scattering)

$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{k}} \simeq \frac{Q_{0A}^2}{\pi k_\perp^4}$$

  - power-law tail at high $k_\perp$: Rutherford scattering

The "dipole" gluon distribution

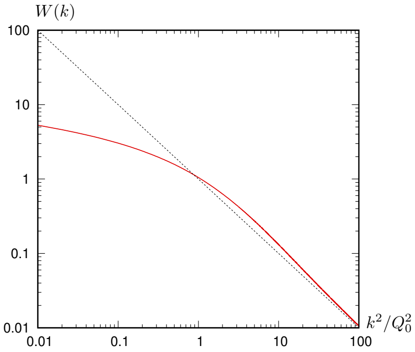
- Left: the Fourier transform $k_\perp \mathcal{S}(k_\perp)$
  - the probability distribution for $k_\perp$
  - "the dipole unintegrated gluon distribution"



- Right: the same function, but in logarithmic units
  - peaked at $k \simeq Q_s$, power-law tail at $k \gg Q_s$

The Weiszäcker-Williams gluon distribution

- Left: the gluon occupation number $n(k_\perp) = W(k_\perp)/\alpha_s N_c$
  - "the Weiszäcker-Williams unintegrated gluon distribution"
  - $W(k_\perp) \simeq \ln \frac{Q_s^2}{k_\perp^2}$ when $k \lesssim Q_s$, $W(k_\perp) \simeq \frac{Q_0^2}{k_\perp^2}$ at high $k_\perp$



- Different structure at saturation, but same physical picture
  - the bulk of the gluon distribution has $k_\perp \sim Q_s$

High-energy factorization for pA ("hybrid")

- After scattering, the quark must "fragment into hadrons" : $D_{h/q}(z, \mu^2)$

$$\frac{\mathrm{d}\sigma_h}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{p}} = \int \frac{\mathrm{d}z}{z^2}\, x_p q(x_p, \mu^2) \left[\int_{\boldsymbol{x},\boldsymbol{y}} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{k}}\; S(\boldsymbol{x}, \boldsymbol{y}; X_g)\right] D_{h/q}(z, \mu^2)$$

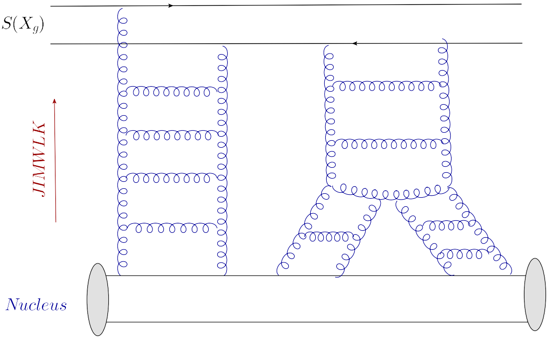
- There is also a gluon-initiated channel, albeit less important when $x_p \sim \mathcal{O}(1)$
- Collinear factorization for the incoming proton/outgoing hadron
  - DGLAP evolution for quark distribution/ fragmentation
- High-energy (CGC) factorization for the quark-nucleus scattering
  - JIMWLK (BK) for target gluon distribution (dipole S-matrix)
- Natural, but non-trivial already at leading order: currently proven at NLO

(Kovchegov and Tuchin, 2002; Dumitru, Hayashigaki, and Jalilian-Marian, 2005)
(Mueller and Munier, 2012; Chirilli, Xiao, and Yuan, 2012)

High-energy evolution: from the target ...

$$S(\boldsymbol{x},\boldsymbol{y};X_g)\equiv\langle\hat{S}_{\boldsymbol{xy}}\rangle_{X_g}=\int[DA^-]\,W_{X_g}[A^-]\,\frac{1}{N_c}\mathrm{tr}\big(V_{\boldsymbol{x}}V_{\boldsymbol{y}}^\dagger\big)$$

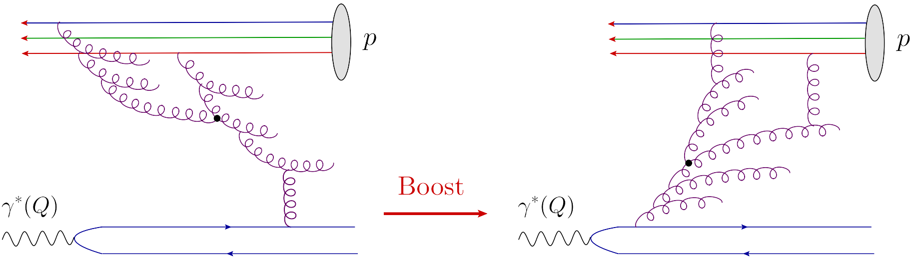
- JIMWLK: functional equation for the CGC weight function $W_{X_g}[A^-]$



- gluon emissions with smaller and smaller $X = p^-/P^-$, down to X_g
- non-linear effects in both the evolution (gluon saturation) and the collision (multiple scattering)

... to the projectile

- Balitsky-Kovchegov equation for the dipole S-matrix $S(\boldsymbol{x}, \boldsymbol{y}; X_g)$

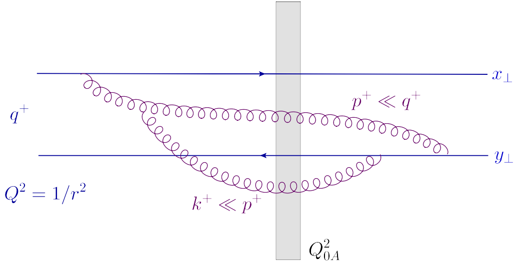


  - gluon emissions with smaller and smaller $x = p^+/q^+$, down to X_g
  - non-linear effects in the collision but not also in the evolution
  - important fluctuations in the evolution of the dipole: dilute system

- Physical picture & calculation in the LC gauge of the RM dipole: $A_a^+ = 0$

Dipole evolution

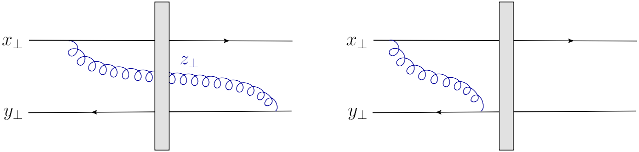
- Probability $\sim \alpha_s \ln(1/x)$ for emitting a soft $(x \ll 1)$ gluon
  - $x \equiv k^+/q^+$: longitudinal momentum fraction for a right mover
- Gluons must be emitted and reabsorbed within the dipole (color neutrality)



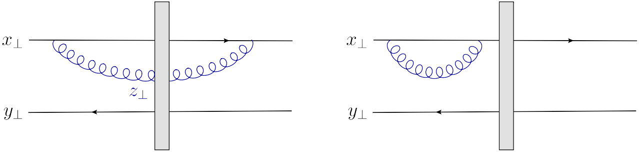
$$x_2 = \frac{k^+}{q^+} \ll x_1 = \frac{p^+}{q^+} \ll 1\,, \qquad Q^2_{0A} \ll p_\perp^2,\, k_\perp^2 \ll Q^2$$

- Strong ordering in x, no ordering in $k_\perp$ $\Rightarrow$ decreasing lifetimes: $\Delta x^+ \sim \frac{2xq^+}{k_\perp^2}$
- Leading logarithmic approx: resum $(\bar{\alpha}Y)^n$ with $Y = \ln(1/x_{\min})$

- To construct the evolution equation, it is enough to look at the first emission
- The gluon can be exchanged between the quark and the antiquark



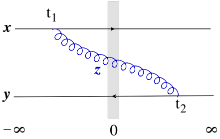
- ... or be emitted and reabsorbed by a same fermion ("self-energy graph")



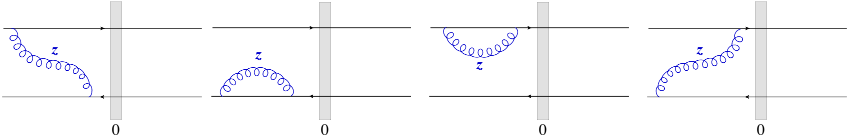
- In both cases, the gluon may cross the shockwave ("real contributions")
- ... or not ! "Virtual corrections", or "evolution in the initial/final state"

One step in the BK evolution (2)

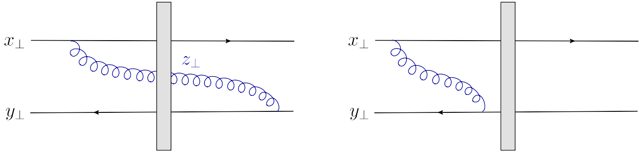
- And of course there are several possible permutations of the gluon vertices
- 'Real contributions': the soft gluon can interact with the shockwave
  - the system which scatters: a 3-parton system ($q\bar{q}g$)



- 'Virtual contributions': only the original $q\bar{q}$ dipole interacts



- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$



$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$
$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V_{\boldsymbol{y}}^\dagger\big)\right\rangle_Y$$

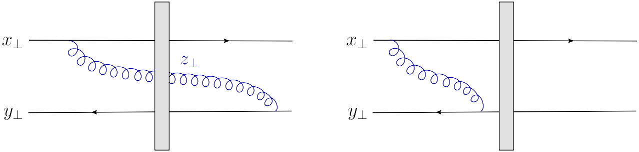
- 'Real' term: the gluon emitted at $\boldsymbol{x}$ hits the shockwave a $\boldsymbol{z}$

$$g\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\quad : \quad \text{amplitude for gluon emission and propagation from } \boldsymbol{x} \text{ to } \boldsymbol{z}$$

- A "gluon" in the LC gauge: the Weizsäcker-Williams field of the quark

"Exchange" graphs in more detail

- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$



$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$
$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)\right\rangle_Y$$

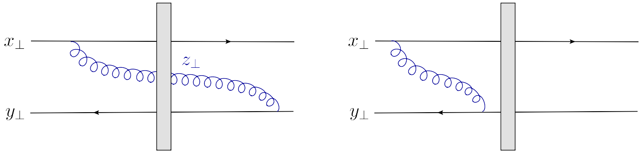
- 'Real' term: the gluon emitted at $\boldsymbol{x}$ hits the shockwave a $\boldsymbol{z}$

$$A^i_a(\boldsymbol{z}) = g t^a \int_{xq^+}^{q^+} \frac{\mathrm{d}k^+}{2\pi}\,\frac{1}{k^+}\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\,\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{z}-\boldsymbol{x})}\,\frac{k^i}{k_\perp^2} = g t^a \mathrm{d}Y\,\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}$$

- Integral over k^+ generates the log enhancement: $\ln\frac{1}{x} = \mathrm{d}Y$

"Exchange" graphs in more detail

- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$



$$\mathrm{d}_1 S_Y(\boldsymbol{x}, \boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

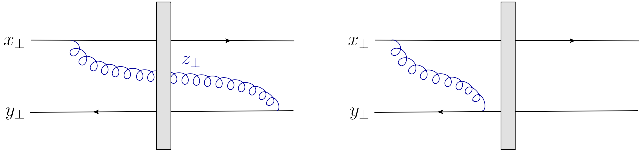
$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V_{\boldsymbol{y}}^\dagger\big)\right\rangle_Y$$

- 'Real' term: the gluon emitted at $\boldsymbol{x}$ hits the shockwave a $\boldsymbol{z}$
- Wilson line for the gluon at $\boldsymbol{z}$ in adjoint representation: $(T^a)_{bc} = if^{abc}$

$$\tilde{V}(\boldsymbol{z}) = \mathrm{T}\exp\left\{ig\int \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}_\perp)T^a\right\}$$

"Exchange" graphs in more detail

- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$



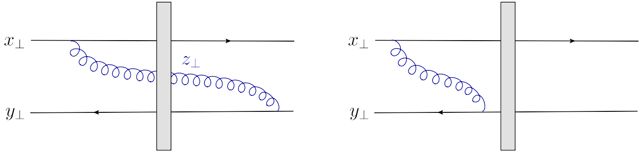
$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$
$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)\right\rangle_Y$$

- 'Virtual' term: the gluon is emitted at $\boldsymbol{x}$ and reabsorbed at $\boldsymbol{y}$, either before, or after, the scattering:

$$\frac{1}{N_c}\,\mathrm{tr}\big(t^a t^a V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big) = \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V^\dagger_{\boldsymbol{y}}\big)$$

"Exchange" graphs in more detail

- Small step in rapidity: $\alpha_s \mathrm{d}Y \ll 1$

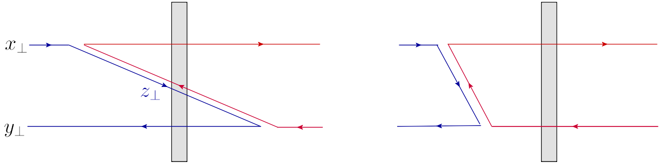


$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) = -\frac{\alpha_s}{\pi^2}\,\mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}$$
$$\left\langle \tilde{V}_{ab}(\boldsymbol{z})\,\frac{1}{N_c}\,\mathrm{tr}\Big(V(\boldsymbol{x})t^b\,V^\dagger(\boldsymbol{y})t^a\Big) - \frac{C_F}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}V_{\boldsymbol{y}}^\dagger\big)\right\rangle_Y$$

- Not a closed equation: evolution couples $S_{q\bar{q}}(\boldsymbol{x},\boldsymbol{y})$ to $S_{q\bar{q}g}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$
  - not a surprise: one additional gluon that is measured by the scattering
- A closed evolution equation is obtained in the multi-color limit $N_c \to \infty$

BK evolution at large N_c

- At large N_c, a gluon can be replaced by a quark-antiquark pair
  - gluon emission by a dipole $\approx$ dipole splitting into 2 dipoles



$$\mathrm{d}_1 S_Y(\boldsymbol{x},\boldsymbol{y}) \simeq -\frac{\alpha_s N_c}{2\pi^2}\mathrm{d}Y\int_{\boldsymbol{z}}\frac{(\boldsymbol{x}-\boldsymbol{z})^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\frac{(\boldsymbol{y}-\boldsymbol{z})^i}{(\boldsymbol{y}-\boldsymbol{z})^2}\Big\{S_Y(\boldsymbol{x},\boldsymbol{z})\,S_Y(\boldsymbol{z},\boldsymbol{y})-S_Y(\boldsymbol{x},\boldsymbol{y})\Big\}$$

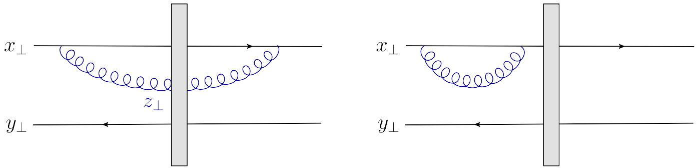
- At large N_c, expectation values of colorless operators factorize

$$\left\langle\frac{\mathrm{tr}\big(V_{\boldsymbol{x}}V_{\boldsymbol{z}}^\dagger\big)}{N_c}\,\frac{\mathrm{tr}\big(V_{\boldsymbol{z}}V_{\boldsymbol{y}}^\dagger\big)}{N_c}\right\rangle_Y \simeq S_Y(\boldsymbol{x},\boldsymbol{z})\,S_Y(\boldsymbol{z},\boldsymbol{y})$$

- Finite-N_c corrections are suppressed as $1/N_c^2 \lesssim 10\%$ for $N_c = 3$

Self-energy graphs

- Similar manipulations for the self-energy graphs
  - emission and reabsorption by the same quark
  - opposite (positive) sign, since the same charge at both vertices



  - replace $\boldsymbol{y} \to \boldsymbol{x}$ in the emission kernel

$$\mathrm{d}_2 S_Y(\boldsymbol{x},\boldsymbol{y}) \simeq \frac{\alpha_s N_c}{2\pi^2}\,\mathrm{d}Y \int_{\boldsymbol{z}} \frac{1}{(\boldsymbol{x}-\boldsymbol{z})^2}\left\{S_Y(\boldsymbol{x},\boldsymbol{z})\,S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\right\}$$

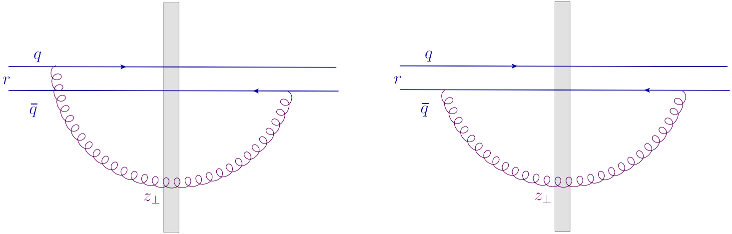
The BK equation *(Balitsky, '96; Kovchegov, '99)*

$$\frac{\partial S_Y(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\big]$$

- Dipole kernel: BFKL kernel in the dipole picture *(Al Mueller, 1990)*

$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2$$

- Cancellations between large-distance contributions from "exchange" ($q\bar{q}$) and "self-energy" (qq or $\bar{q}\bar{q}$) graphs, by color neutrality



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- Cancellations between large-distance contributions from "exchange" ($q\bar{q}$) and "self-energy" (qq or $\bar{q}\bar{q}$) graphs, by color neutrality
  - rapid decrease of the emission probability at large $z_\perp$:

$$\mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

  - color transparency: $\mathcal{M}_{\boldsymbol{xyz}} \propto (\boldsymbol{x}-\boldsymbol{y})^2 = r^2$
  - a zero-size "dipole" cannot emit, as it has zero charge

The BK equation *(Balitsky, '96; Kovchegov, '99)*

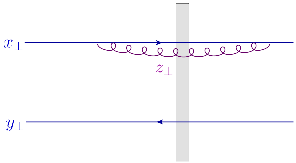
$$\frac{\partial S_Y(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) - S_Y(\boldsymbol{x},\boldsymbol{y})\big]$$

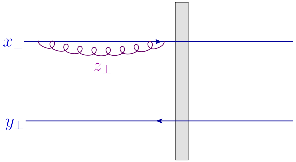
- Dipole kernel: BFKL kernel in the dipole picture *(Al Mueller, 1990)*

$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2$$

- Short-distance poles ($\boldsymbol{z}=\boldsymbol{x}$ or $\boldsymbol{z}=\boldsymbol{y}$) cancel between 'real' and 'virtual'

$$\boldsymbol{z}\to\boldsymbol{x} \implies S_Y(\boldsymbol{x},\boldsymbol{z})S_Y(\boldsymbol{z},\boldsymbol{y}) \to \mathbb{I}\times S_Y(\boldsymbol{x},\boldsymbol{y})$$





BFKL & Unitarity

- Non-linear generalization of the BFKL equation for $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\right]$$

- Non-linear term T^2: the simultaneous scattering of both daughter dipoles

BFKL & Unitarity

- Non-linear generalization of the BFKL equation for $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}}\right]$$

- When scattering is weak, $T \ll 1$, one recovers the linear BFKL equation
  - conformal symmetry: $\boldsymbol{x} \to a\boldsymbol{x} \Rrightarrow \int \mathrm{d}^2\boldsymbol{z}\mathcal{M}_{\boldsymbol{xyz}} =$ invariant
  - pure powers $r^{2\gamma}$ are eigenfunctions of the BFKL kernel:

$$\mathcal{K}_{\rm BFKL} \otimes r^{2\gamma} = \bar{\alpha}\chi(\gamma)r^{2\gamma} \qquad \text{for any } 0 < \gamma < 1$$

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \qquad \psi(\gamma) = \mathrm{d}\ln\Gamma(\gamma)/\mathrm{d}\gamma$$

  - a basis of exact solutions: $T_\gamma(r, Y) \propto r^{2\gamma}\, \mathrm{e}^{\bar{\alpha}\chi(\gamma)Y}$
  - general solution: superposition in γ (Mellin transform)
  - exponential increase with Y leading to unitarity violation

BFKL & Unitarity

- Non-linear generalization of the BFKL equation for $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\right]$$

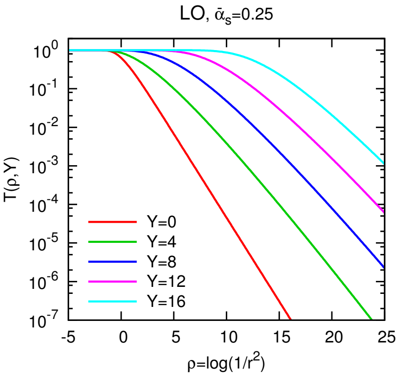
- The non-linear term in BK restores unitarity: $T(r, Y) \leq 1$ for any r and Y
  - $T = 0$ (no scattering) and $T = 1$ (total absorption) are fixed points
- Saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 1/Q_s(Y)$
  - $Q_s(Y)$ increases rapidly with Y due to the BFKL dynamics

$$T_{\text{BFKL}}\left(r = \frac{1}{Q_s}, Y\right) \sim \left(\frac{Q_0^2}{Q_s^2}\right)^{\gamma} \mathrm{e}^{\bar{\alpha}\chi(\gamma)Y} = 0.5 \quad \Longrightarrow \quad Q_s^2(Y) \simeq Q_0^2 \, \mathrm{e}^{\bar{\alpha}\frac{\chi(\gamma)}{\gamma}Y}$$

  - Mellin superposition selects $\gamma_s \simeq 0.63 \Longrightarrow \lambda_s = \bar{\alpha}\frac{\chi(\gamma_s)}{\gamma_s} \simeq 4.88\bar{\alpha}$

The saturation front

- Numerical solutions to BK with initial condition from the MV model
- Logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2) \Longrightarrow$ large $\rho \leftrightarrow$ small r



$$T(r, Y=0) = 1 - \mathrm{e}^{-\frac{r^2 Q_0^2}{4} \ln \frac{1}{r^2 \Lambda^2}}$$

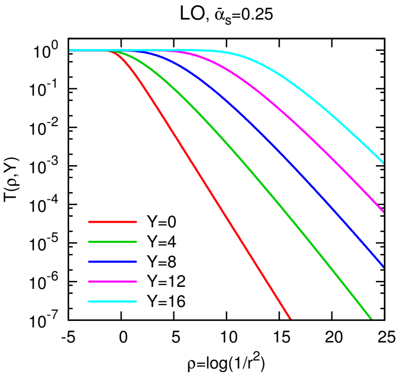
$$T(\rho_s(Y), Y) = 0.5 \quad \text{for} \quad \rho_s(Y) = \lambda_s Y$$

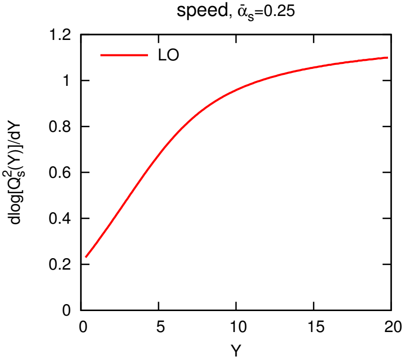
$$T(\rho, Y) \simeq \begin{cases} \mathrm{e}^{-\gamma_s(\rho-\rho_s)}\, \mathrm{e}^{-\frac{(\rho-\rho_s)^2}{2\beta_s \bar{\alpha} Y}} & (\rho > \rho_s) \\ 1 & (\rho \lesssim \rho_s) \end{cases}$$

- Geometric scaling: $T(r, Y) \simeq \left(r^2 Q_s^2(Y)\right)^{\gamma_s}$ when $\rho - \rho_s \ll \sqrt{2\beta_s \bar{\alpha} Y}$

The saturation front

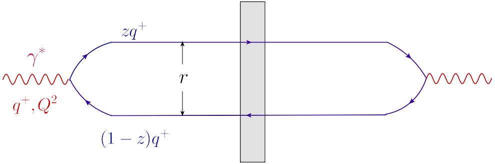
- Numerical solutions to BK with initial condition from the MV model
- Logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2) \Longrightarrow$ large $\rho \leftrightarrow$ small r





- Saturation exponent $\lambda_s \equiv \frac{d\rho_s}{dY} \simeq 4.88\bar{\alpha}$: the speed of the saturation front

- Already before BK equation: fits to small-x DIS using the idea of saturation
  - dipole factorization + saturation models for the dipole cross-section

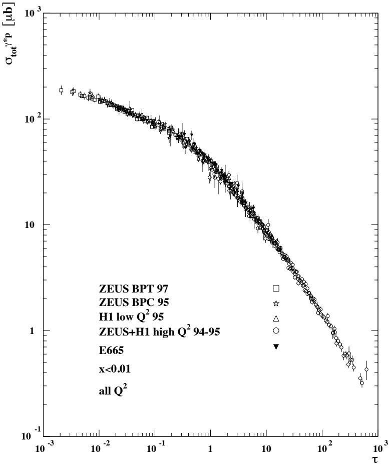


$$\sigma_{\gamma^* p}(Q^2, x) = \int_{r,z} \left|\Psi_{\gamma^*}(r, z; Q^2)\right|^2 \sigma_{\rm dip}(r, x)$$

- GBW model (Golec-Biernat, Wüsthoff, '99)

$$\sigma_{\rm dip}(r, x) = \sigma_0 \left[1 - {\rm e}^{-r^2 Q_s^2(x)}\right], \quad Q_s^2(x) \propto \frac{1}{x^\lambda}$$

- "MV model with ad-hoc evolution in Q_s"



Saturation models for HERA

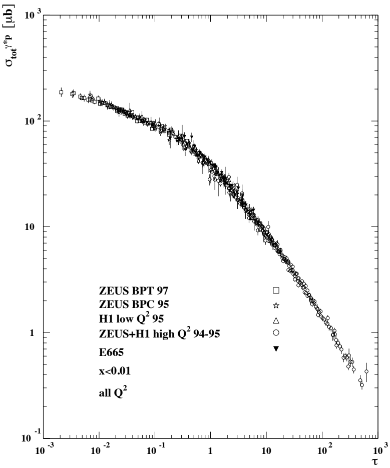
- Already before BK equation: fits to small-x DIS using the idea of saturation
  - dipole factorization + saturation models for the dipole cross-section

$$\sigma_{\rm dip}(r,x) = \sigma_0\left[1 - {\rm e}^{-r^2 Q_s^2(x)}\right], \quad Q_s^2(x) \propto \frac{1}{x^\lambda}$$

- built-in geometric scaling
- good fit at $x \leq 0.01$ despite simplicity
- data clearly prefer a small value for the saturation exponent: $\lambda \simeq 0.3$
- inspired the search for geometric scaling

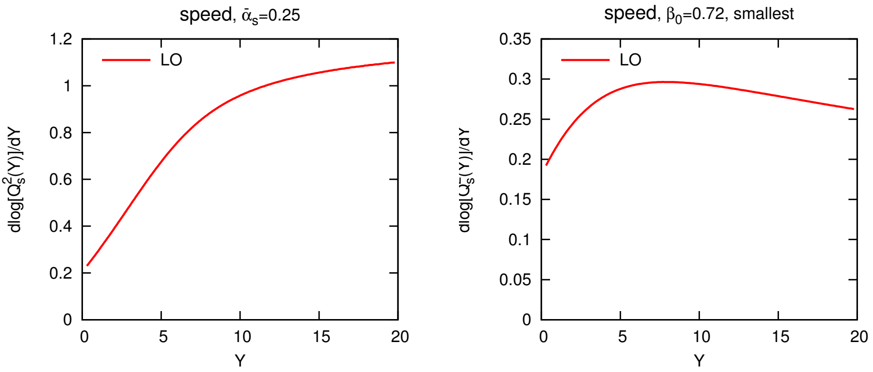
(Staśto, Golec-Biernat, Kwieciński, 2000)

$\sigma(x, Q^2)$ vs. $\tau \equiv Q^2/Q_s^2(x)$



Adding running coupling: rcBK

- BK naturally explains geometric scaling, but $\lambda_s \simeq 4.88\bar{\alpha} \sim 1$ is way too large
- Using a running coupling dramatically slows down the evolution
  - $\lambda_s \simeq 0.3$ in good agreement with the data



- Rather successful phenomenology based on rcBK (see below)

- BK naturally explains geometric scaling, but $\lambda_s \simeq 4.88\bar{\alpha} \sim 1$ is way too large
- Using a running coupling dramatically slows down the evolution
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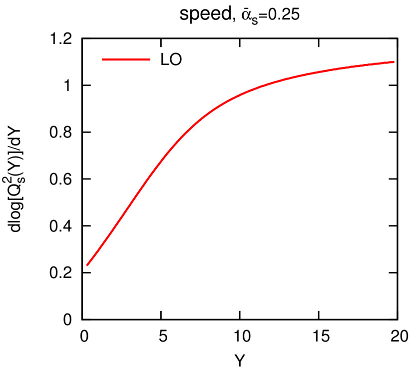
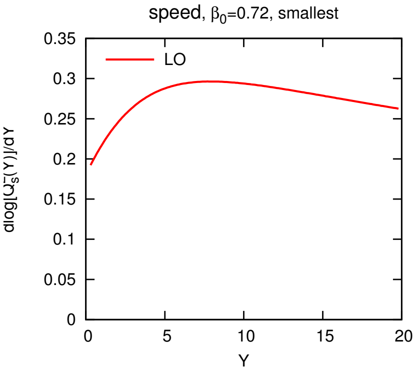
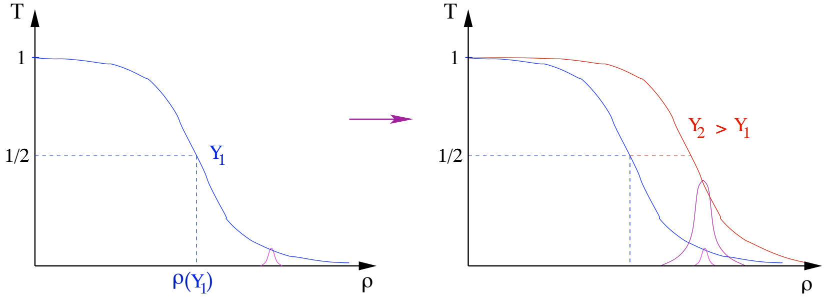




- But why should the effect of the running be so important ?!
  - the running is a next-to-leading order effect and is only logarithmic

Pulled front

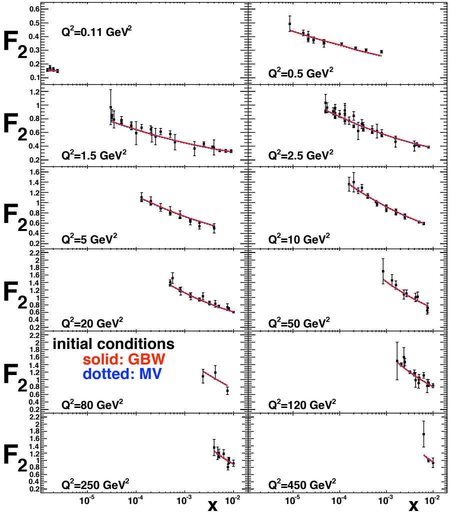
- The saturation front is pulled by the BFKL growth in the dilute tail
  - this is why one can compute λ_s from BFKL + saturation boundary
  - deep connexion to "reaction-diffusion problem" in statistical physics *(Munier and Peschanski, 2003; Iancu, Mueller and Munier, 2004)*
- The scale for the running coupling is Q_s and increases exponentially with $\bar{\alpha}Y$



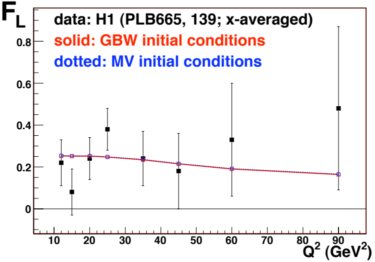
$$\alpha_s(Q_s^2) = \frac{1}{\beta_0 \ln \frac{Q_s^2}{\Lambda^2}} = \frac{1}{\beta_0(\rho_s(Y) + \rho_0)} \simeq \frac{1}{\beta_0 \lambda_s Y} \quad : \quad \text{decreasing with } Y$$

rcBK fit to F_2 at HERA (+ prediction for F_L)

(Albacete et al, hep-ph/09021112)



BK + running coupling



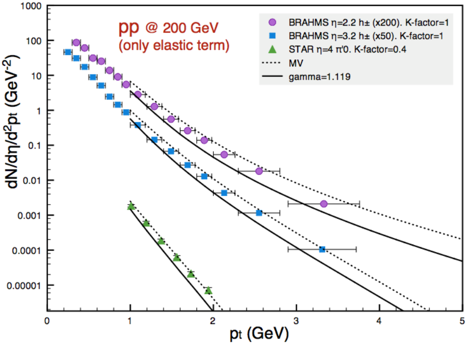
3 parameters (R, Q^2_{s0}, C)

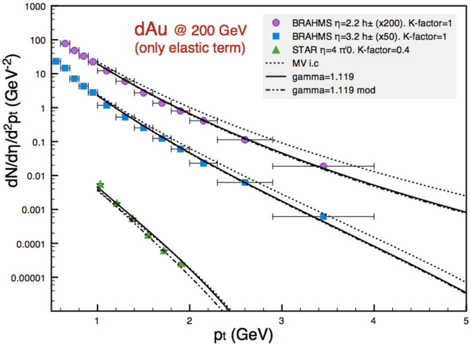
847 data points, χ^2/d.o.f. $\simeq 1$

rcBK fit to forward particle production at RHIC

(Albacete, Dumitru, Fujii, Nara, arXiv:1209:2001)

- Fit parameters: initial condition for the rcBK equation + K-factors

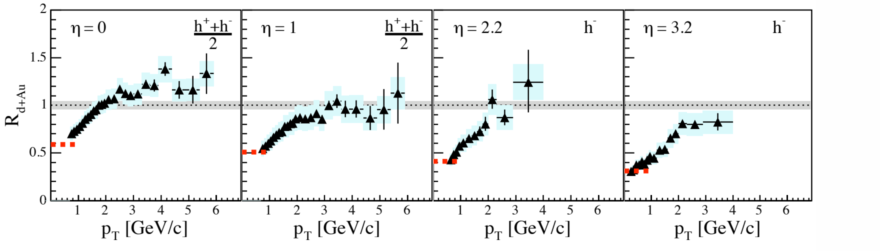




$$\left.\frac{\mathrm{d}N_h}{\mathrm{d}\eta\,\mathrm{d}^2\boldsymbol{k}}\right|_{\mathrm{LO}} = K_h \int_{x_p}^{1} \frac{\mathrm{d}z}{z^2}\,\frac{x_p}{z} q\left(\frac{x_p}{z}\right) \mathcal{S}\left(\frac{\boldsymbol{k}}{z}, X_g\right) D_{h/q}(z)$$

The nuclear modification factor

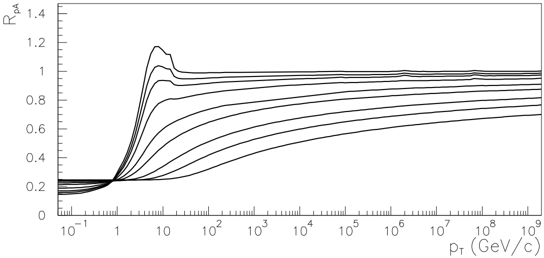
$$R_{pA} \equiv \frac{1}{A}\,\frac{\mathrm{d}\sigma_{pA}/\mathrm{d}^2p_\perp \mathrm{d}\eta}{\mathrm{d}\sigma_{pp}/\mathrm{d}^2p_\perp \mathrm{d}\eta}$$



- It would be equal to one if pA = incoherent superposition of pp collisions
  - any deviation from unity is a signature of nuclear (high density) effects
- At RHIC: $R_{\mathrm{d+Au}}$, hence $A \to 2A$ with $A = 197$
  - central rapidity ($\eta \simeq 0$) and $p_\perp \gtrsim 2$ GeV: $R_{\mathrm{d+Au}} > 1$ ("Cronin peak")
  - forward rapidity ($\eta > 1$) : the peak disappears when $\eta \gtrsim 1$
  - larger forward rapidities ($\eta \gtrsim 3$): $R_{\mathrm{d+Au}} < 1$ ("suppresion")

Consistent with the CGC

$$R_{pA} \equiv \frac{1}{A^{1/3}} \frac{\mathrm{d}N_{pA}/\mathrm{d}^2p_\perp \mathrm{d}\eta}{\mathrm{d}N_{pp}/\mathrm{d}^2p_\perp \mathrm{d}\eta} = \frac{1}{A^{1/3}} \frac{\mathcal{S}_A(p_\perp, X_g)}{\mathcal{S}_p(p_\perp, X_g)}$$



$$\mathcal{S}(p_\perp, X_g) = \int_{\boldsymbol{r}} \mathrm{e}^{-\mathrm{i}\boldsymbol{r}\cdot\boldsymbol{p}}\, S(\boldsymbol{r}, X_g)$$

$$X_g = \frac{p_\perp}{\sqrt{s}}\, \mathrm{e}^{-\eta}$$

$\eta = 0$, 0.05, 0.1, 0.2, 0.4, 0.6, 1, 1.4 and 2 *(BK equation: Albacete et al, 2003)*

- Use BK equation for $S(\boldsymbol{r}, X_g)$ with initial condition from the MV model
- Exactly the same features as in the RHIC data !
- What is the underlying physical picture ?

Midrapidity: the Cronin peak

- d+Au collisions at RHIC: $\sqrt{s} = 200$ GeV, $p_\perp \sim 2$ GeV and $\eta \approx 0$
  - $x_1 = x_2 \simeq 0.01 \Longrightarrow$ little evolution, the proton is still dilute
  - nucleus: incoherent superposition of valence quarks (MV model)

$$\frac{\mathcal{S}(p_\perp)}{4\pi} \simeq \begin{cases} \dfrac{1}{Q_s^2(A)}\, \mathrm{e}^{-\frac{p_\perp^2}{Q_s^2(A)}}, & \text{for the nucleus} \\[2ex] \dfrac{Q_{0p}^2}{p_\perp^4}, & \text{for the proton} \end{cases}$$

  - remember the distinction between the two scales $Q_s^2(A)$ and Q_{0A}^2 :

$$Q_s^2(A) = Q_{0A}^2 \ln\frac{Q_s^2(A)}{\Lambda^2}, \qquad Q_{0A}^2 = A^{1/3} Q_{0p}^2$$

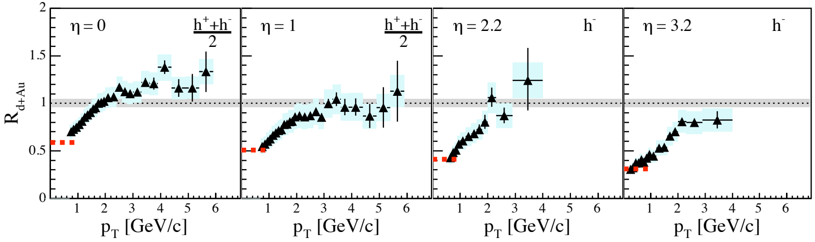
$$R_{pA} = \frac{1}{A^{1/3}} \frac{\mathcal{S}_A(p_\perp)}{\mathcal{S}_p(p_\perp)} \simeq \ln\frac{Q_s^2(A)}{\Lambda^2} \times \left[\frac{p_\perp^2}{Q_s^2(A)}\right]^2 \mathrm{e}^{-\frac{p_\perp^2}{Q_s^2(A)}}$$

- A peak at $p_\perp^2 = 2Q_s^2(A)$ with height $\ln\left[Q_s^2(A)/\Lambda^2\right] > 1$

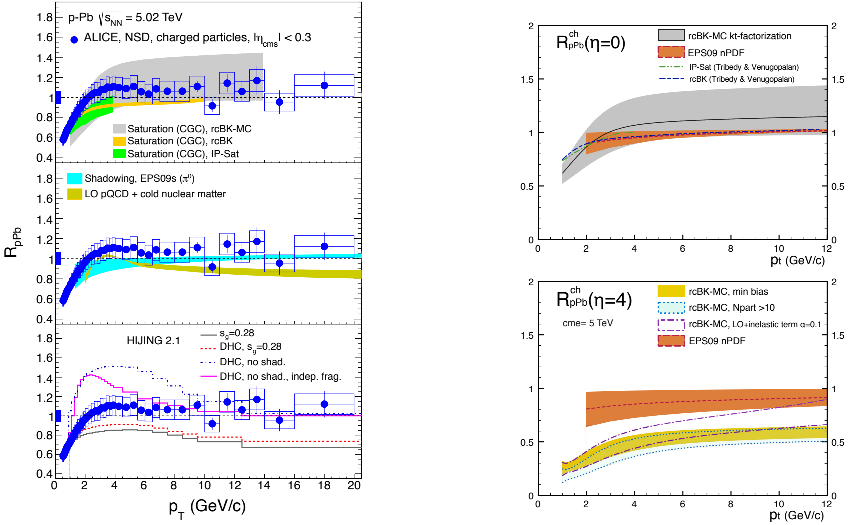
Forward rapidities: R_{pA} suppression

- Why is the Cronin peak washed out when increasing η (decreasing X_g) ?
- The gluon distribution in the proton rises faster than that in the nucleus
  - growth driven by BFKL dynamics in the dilute tail at $p_\perp > Q_s$
  - the logarithmic phase-space $\rho = \ln(p_\perp^2/Q_s^2)$ is larger for the proton than for the nucleus, since $Q_{0p} < Q_{0A}$

$$\rho_p = \ln \frac{p_\perp}{Q_{0p}^2} > \rho_A = \ln \frac{p_\perp}{Q_{0A}^2} \quad \text{since} \quad Q_{0A}^2 = A^{1/3} Q_{0p}^2$$

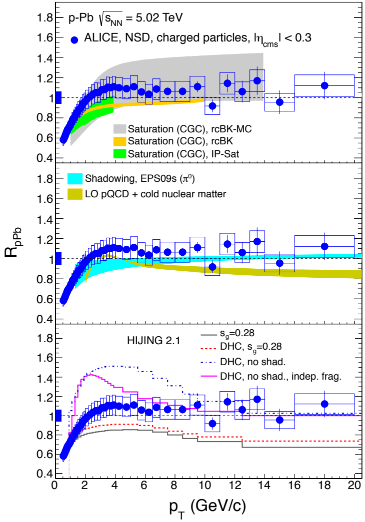


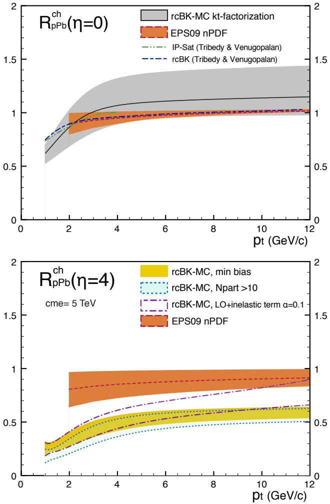
R_{p+Pb} at the LHC for central rapidities



- Midrapidity ($\eta \simeq 0$) at the LHC is like $\eta \sim 2$ at RHIC: $x_1 \sim x_2 \sim 10^{-3}$
- Cronin peak and small evolution compensate each other: $R_{pA} \simeq 1$

$R_{\mathrm{p+Pb}}$ at the LHC for central rapidities





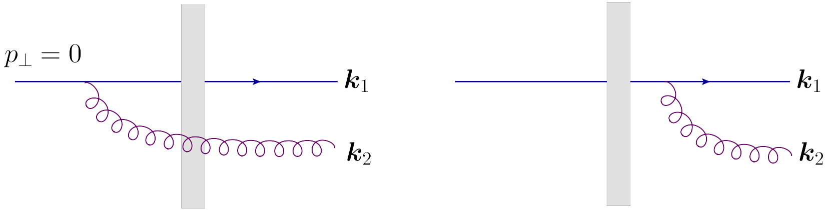
- Various models could be differentiated by going to forward rapidities
- This could be measured e.g. by LHCb (large η & semi-hard $p_\perp$)

Beyond the BK equation

- BK eq.: relatively simple and phenomenologically successful ... but ...
  1. large N_c approximation
  2. leading order in pQCD + running coupling corrections
  3. restricted to a simple projectile: the color dipole
- How serious are these limitations in practice ?
  - large N_c approximation works surprisingly well ☺
  - ... but the leading order pQCD approx is surprisingly bad ☹
  - the running coupling spectacularly improves the phenomenology ☺
  - ... but this cannot be the end of the story: other NLO corrections ☹
  - multiparticle production probes Wilson line correlations which are more complicated than just a dipole ☹
- Upgrading to JIMWLK equation solves problems 1. and 3.
- A full NLO calculation is needed to clarify problem 2.

Two particle production in pA collisions

- Forward rapidities: the 2 measured hadrons $\sim$ partons from the proton
- The collinear quark radiates a gluon prior to, or after, the scattering

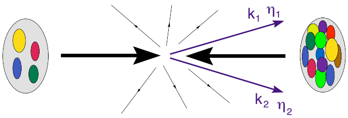


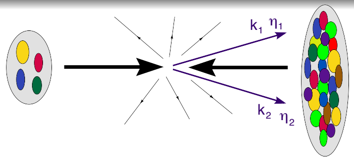
- Up to four Wilson lines in the cross–section
- At large N_c, this factorizes into color dipoles and quadrupoles

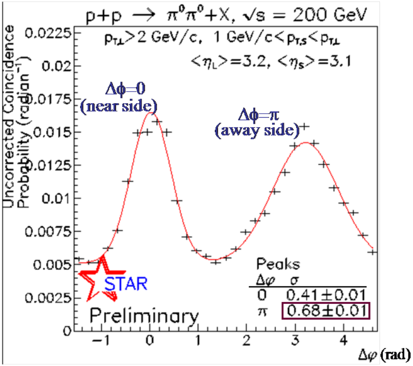
$$\left\langle Q_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\right\rangle_Y = \frac{1}{N_c}\left\langle \mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^\dagger_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4})\right\rangle_Y$$

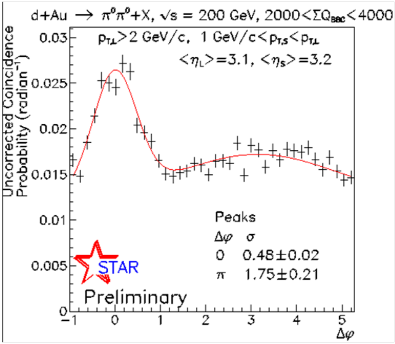
- In the absence of scattering, the final particles would propagate back to back in the transverse plane: $\boldsymbol{k}_1 + \boldsymbol{k}_2 \simeq 0$

Di–hadron correlations at RHIC: p+p vs. d+Au



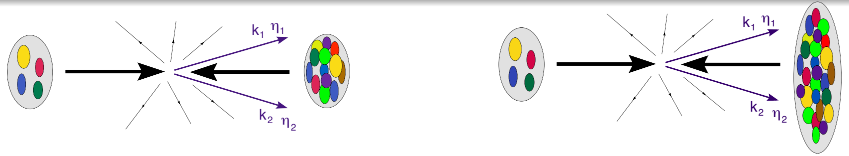


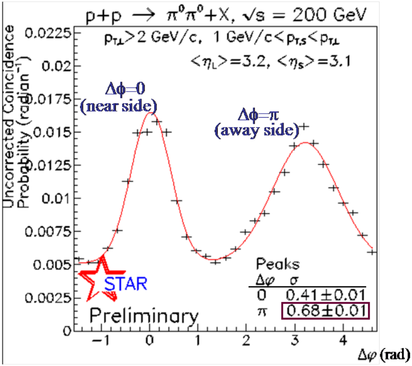


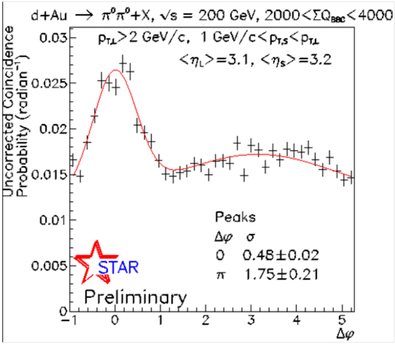


- Significant broadening even in pp collisions: recoil in jet fragmentation
- The broadening in d+Au is considerably stronger than that in pp

Di–hadron correlations at RHIC: p+p vs. d+Au







- The scattering transfers an overall momentum $|\boldsymbol{k}_1 + \boldsymbol{k}_2| \simeq Q_s(X_g)$
- Predicted by the CGC *(Marquet, 2007; Albacete and Marquet, 2010)*

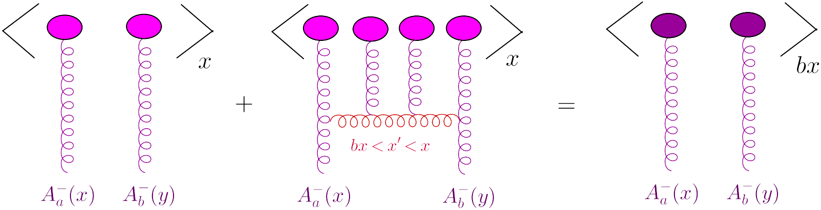
JIMWLK evolution

(Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, and Kovner, 97–00)

- Small-x evolution of the gluon distribution in the dense target: saturation
  - change in classical color charge ρ_a (color field A_a^-) and its correlations
  - evolution of the CGC weight function(al) $W_Y[\rho]$ ($Y = \ln(1/x)$)
- ρ_a: color charge density of the "fast" partonic sources with $x' > x$
  - valence quarks + soft gluons with $1 \gg x' \gg x$
  - frozen in some random configuration by Lorentz time dilation
  - gauge-invariant, functional, probability distribution $W_Y[\rho]$
- With decreasing x, new quantum fluctuations are "frozen" and must be included in ρ_a (hence, in $W_Y[\rho]$)
- $W_Y[\rho]$ is built by integrating out soft gluon fluctuations in layers of x
- Initial condition at low energy ($x_0 \sim 0.01$): MV model (valence quarks)

JIMWLK evolution

- One step in the quantum evolution $\Rightarrow$ JIMWLK "Hamiltonian" ("time" $= Y$)
  - $x \to bx$ with $b \ll 1$, but such that $\mathrm{d}Y \equiv \bar{\alpha} \ln \frac{1}{b} \ll 1$ as well
  - small additional color charge $\delta\rho_a$ in the strong background of $A^-[\rho]$
  - for a fixed background A^-: renormalization of the 1-point and 2-point functions of ρ: $\langle \delta\rho_a \rangle_A$ and $\langle \delta\rho_a \delta\rho_b \rangle_A$



- The quantum gluon can scatter of the strong color fields generated in previous steps $\Longrightarrow$ non-linear evolution

JIMWLK evolution

- One step in the quantum evolution $\Rightarrow$ JIMWLK "Hamiltonian" ("time" $=Y$)
  - $x \to bx$ with $b \ll 1$, but such that $\mathrm{d}Y \equiv \bar{\alpha}\ln\frac{1}{b} \ll 1$ as well
  - small additional color charge $\delta\rho_a$ in the strong background of $A^-[\rho]$
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- A functional differential equation for the non-linear evolution of $W_Y[\rho]$:

$$\frac{\partial W_Y[\rho]}{\partial Y} = H_{\rm JIMWLK}\left[\rho, \frac{\delta}{\delta\rho}\right] W_Y[\rho]$$

- A 2nd order functional-derivative operator: Fokker-Planck equation

$$H_{\rm JIMWLK}\left[\rho, \frac{\delta}{\delta\rho}\right] = \frac{1}{2}\int_{\boldsymbol{x},\boldsymbol{y}} \frac{\delta}{\delta\rho_a(\boldsymbol{x})}\,\chi_{ab}(\boldsymbol{x},\boldsymbol{y})[\rho]\,\frac{\delta}{\delta\rho_b(\boldsymbol{y})}$$

- The "diffusion" kernel $\chi[\rho]$ depends upon ρ via Wilson lines

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- BK (Balitsky) equations are obtained after an integration by parts:

$$\frac{\partial}{\partial Y}\langle \hat{S} \rangle_Y = \int [\mathcal{D}\rho]\, (H W_Y[\rho])\, \hat{S}[\rho] = \int [\mathcal{D}\rho]\, W_Y[\rho]\, (H \hat{S}[\rho]) = \langle H \hat{S} \rangle_Y$$

- For finite N_c: an infinite hierarchy of coupled differential equations ☹
- JIMWLK equation can be numerically solved for $N_c = 3$ ☺

Fokker-Planck vs. Langevin

- A Fokker-Planck differential equation for a probability distribution can often be rewritten as a Langevin (stochastic) equation for the quantity undergoing the stochastic process

- Simplest example: Brownian motion in 1-dim in Langevin form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \nu(t), \quad \langle \nu(t) \rangle = 0, \quad \langle \nu(t)\nu(t') \rangle = D\delta(t-t')$$

- Formal notations, well defined only with a discretization of time: $t_n = n\epsilon$

$$\frac{x_n - x_{n-1}}{\epsilon} = \nu_n, \qquad \langle \nu_n \nu_m \rangle = D\,\frac{1}{\varepsilon}\,\delta_{mn}$$

$$\langle \Delta x_n \rangle = 0, \quad \langle (\Delta x_n)^2 \rangle = \epsilon^2 \langle \nu_n^2 \rangle = D\epsilon \quad \Longrightarrow \quad \frac{\mathrm{d}\langle x^2 \rangle}{\mathrm{d}t} = D$$

- Not a differentiable process: $\Delta x_n = \epsilon \nu_n \propto \sqrt{\epsilon}$

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- Equivalently: FP equation for the probability $P(x,t)$ with $P(x,0) = \delta(x)$

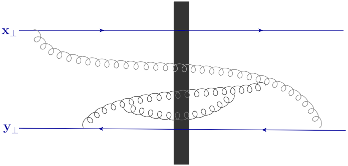
$$\frac{\partial P}{\partial t} = \frac{D}{2}\frac{\partial^2 P}{\partial x^2} \quad \Longrightarrow \quad P(x,t) = \frac{1}{\sqrt{2\pi D t}}\,\mathrm{e}^{-\frac{x^2}{2Dt}}$$

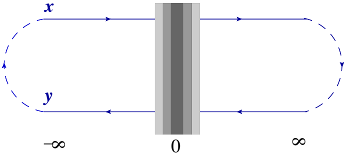
- Genuine differential equation, a.k.a. Poisson equation (heat transfer)

- JIMWLK equation has the mathematical structure of a (functional) FP equation and admits an equivalent representation as a Langevin process in the functional space of Wilson lines.

JIMWLK evolution in Langevin form (1)

- Useful to compare projectile (dipole) and target (nucleus) evolutions

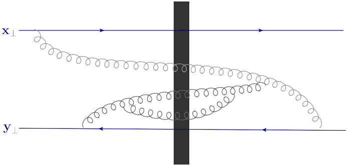


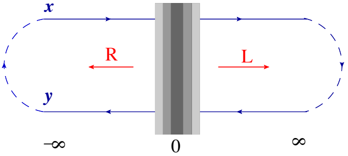


  - projectile: gluon emissions closer and closer to the target
  - target: color charges further and further away from the valence quarks
- Uncertainty principle: decreasing $x = k^-/P^- \leftrightarrow$ increasing $\Delta x^+ \sim 1/k^-$
- JIMWLK evolution builds the color charge distribution in layers of x^+
- New sources are one-loop quantum fluctuations
  - random variables with a Gaussian distribution
  - can equivalently be represented as a Gaussian noise
- A Langevin equation: random walk in the space of the Wilson lines

JIMWLK in Langevin form *(Blaizot, E.I., Weigert, '03)*

- Discretize the rapidity interval: $Y = n\epsilon$, $\epsilon \equiv \ln(1/b)$





$$V_{\boldsymbol{x}}(n\varepsilon + \varepsilon) \,=\, \exp\big(i\varepsilon\alpha^a_{L\boldsymbol{x}}t^a\big)\, V_{\boldsymbol{x}}(n\varepsilon)\, \exp\big(-i\varepsilon\alpha^b_{R\boldsymbol{x}}t^b\big)$$

- $\alpha^a_{R,L}$: the change δA^-_a at larger negative (R) or positive (L) values of x^+

$$\alpha^a_{L\boldsymbol{x}} \,=\, g\int_{\boldsymbol{z}} \frac{x^i - z^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\nu^{ia}_{\boldsymbol{z}}\,, \qquad \alpha^a_{R\boldsymbol{x}} \,=\, g\int_{\boldsymbol{z}} \frac{x^i - z^i}{(\boldsymbol{x}-\boldsymbol{z})^2}\,\tilde{V}^{ab}_{\boldsymbol{z}}\,\nu^{ib}_{\boldsymbol{z}}$$

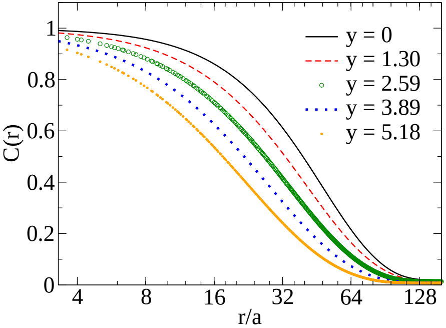
- Noise ν^a: random color charge of the newly emitted gluon

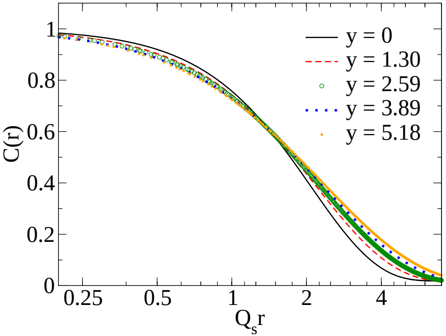
$$\langle \nu^{ia}_{\boldsymbol{x}}(m\varepsilon)\nu^{jb}_{\boldsymbol{y}}(n\varepsilon)\rangle = \frac{1}{\varepsilon}\,\delta_{mn}\delta^{ij}\delta^{ab}\delta_{\boldsymbol{xy}}$$

- Well suited for numerics: 2D lattice *(Weigert and Rummukainen, '03)*

Solving JIMWLK via Langevin

- Several numerical implementations *(Weigert and Rummukainen, '03) Lappi (2011); Schenke et al (since 2012); Roiesnel (2016)*
- Here: the lattice calculation of the dipole S-matrix par T. Lappi (2011)

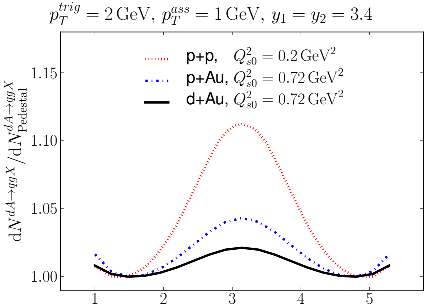


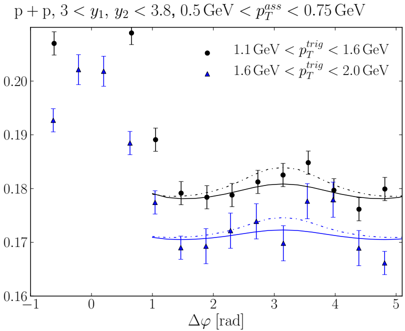


- $C(r) \equiv S(r, Y)$ as a function of r and of $rQ_s(Y) \Longrightarrow$ geometric scaling

The mean field approximation

- Gaussian Ansatz for $W_Y[\rho]$: "MV model with Y-dependent 2-point function"
  - all Wilson lines correlators (quadrupole etc) can be related to the dipole S-matrix, as obtained by solving the BK equation





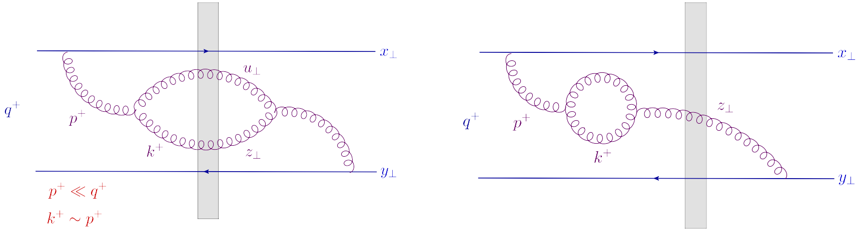
- Left: different combinations projectile–target

  (Lappi and Mäntysaari, 2012; see also Stasto, Xiao, Yuan, 2011)

- Right: comparison with RHIC data (PHENIX, 2012)

Next-to-leading order

- Any effect of $\mathcal{O}(\bar{\alpha}^2 Y) \Longrightarrow \mathcal{O}(\bar{\alpha})$ correction to the r.h.s. of BK eq.



- The prototype: two successive, soft, emissions, with similar longitudinal momentum fractions: $p^+ \sim k^+ \ll q^+$
- Exact kinematics (full QCD vertices, as opposed to eikonal)
- Typically: two transverse momentum convolutions: $\boldsymbol{u}_\perp, \boldsymbol{z}_\perp$
- New color structures, up to 3 dipoles at large N_c
- NLO BFKL: *Fadin, Lipatov, Camici, Ciafaloni ... 95-98*

BK equation at NLO *Balitsky, Chirilli (arXiv:0710.4330)*

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar\alpha}{2\pi}\int d^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar\alpha\left[\bar b\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar b\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left. + \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

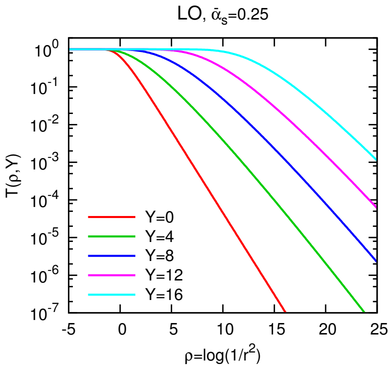
$$+\,\frac{\bar\alpha^2}{8\pi^2}\int\frac{d^2\boldsymbol{u}\,d^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

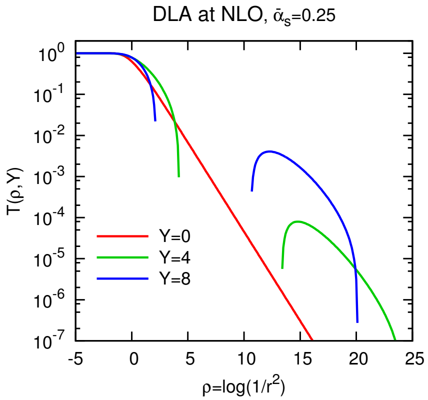
$$\Bigg\{-2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- green : leading-order (LO) terms
- violet : running coupling corrections ($\bar b = (11N_c - 2N_{\rm f})/12N_c$)
- blue : single collinear logarithm (DGLAP)
- red : double collinear logarithm : troublesome !

NLO : unstable numerical solutions





- Left: leading-order BK
- Right: LO BK + the double collinear logarithm alone
- Similar conclusion from full NLO BK *(Lappi, Mäntysaari, arXiv:1502.02400)*
- The source of instability: the double collinear logarithm

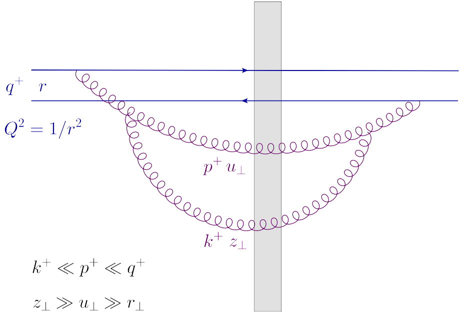
The double (anti)collinear logarithms

- Important in the "hard-to-soft" evolution: relatively large daughter dipoles

$$-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \simeq -\frac{1}{2}\ln^2\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2} \quad \text{if} \quad |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

- Generated by integrating out one gluon (at $\boldsymbol{u}$) whose size is intermediate:

$$|\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \simeq |\boldsymbol{z}-\boldsymbol{u}| \gg |\boldsymbol{u}-\boldsymbol{x}| \simeq |\boldsymbol{u}-\boldsymbol{y}| \gg r$$



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- Keeping just the double anti-collinear logarithms (notation: $|\boldsymbol{z}-\boldsymbol{x}| \to z$):

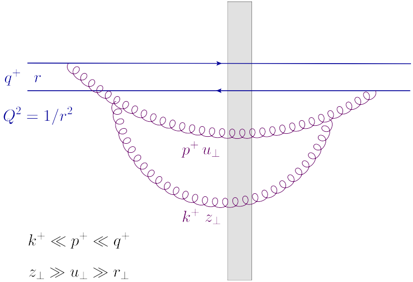
$$\frac{\partial T(Y,r)}{\partial Y} = \bar{\alpha}\int_{r^2}^{1/Q_0^2} \mathrm{d}z^2\,\frac{r^2}{z^4}\left\{1-\frac{\bar{\alpha}}{2}\ln^2\frac{z^2}{r^2}\right\}T(Y,z)$$

- The upper limit: $z = 1/Q_0$ with Q_0 the target saturation scale at low energy
- The r.h.s. becomes negative if $r^2 Q_0^2$ is small enough
- The typical situation for dilute-dense scattering at high-energy

$$\frac{1}{r^2} \sim Q_s^2(Y) = Q_0^2\,\mathrm{e}^{\lambda_s Y} \gg Q_0^2$$

Time ordering

- Successive emissions are ordered in k^+, by construction
- They should be also ordered in lifetimes ... but this condition is not enforced in perturbation theory and may be violated



- lifetime of a gluon fluctuation:

$$\Delta t_p \simeq \frac{2p^+}{p_\perp^2} \sim p^+ u_\perp^2$$

- time-ordering condition:

$$\Delta t_p \sim p^+ u_\perp^2 \; > \; \Delta t_k \sim k^+ z_\perp^2$$

- violated when $z_\perp$ is large enough

- The correct time-ordering is eventually restored via quantum corrections, but only order-by-order
- The loop corrections restoring TO are enhanced by double collinear logs

Time ordering

- Integrate out the harder gluon $(p^+, u_\perp)$ to double-log accuracy:
- Without time-ordering (usual perturbation theory)

$$\bar{\alpha}\int_{k^+}^{q^+}\frac{\mathrm{d}p^+}{p^+}\int_{r^2}^{z^2}\frac{\mathrm{d}u^2}{u^2} = \bar{\alpha}\Delta Y \ln\frac{z^2}{r^2}\,, \qquad \Delta Y \equiv \ln\frac{q^+}{k^+}$$

  - $\mathcal{O}(\bar{\alpha}\Delta Y)$: one step in the leading-order evolution

- After also enforcing time-ordering:

$$\bar{\alpha}\int_{k^+}^{q^+}\frac{\mathrm{d}p^+}{p^+}\int_{r^2}^{z^2}\frac{\mathrm{d}u^2}{u^2}\,\Theta(p^+u^2 - k^+z^2) = \bar{\alpha}\Delta Y \ln\frac{z^2}{r^2} - \frac{\bar{\alpha}}{2}\ln^2\frac{z^2}{r^2}$$

  - the additional term does not count for the LO evolution (no ΔY)
  - it contributes to the NLO evolution after the integration over $(k^+, \boldsymbol{z})$

$$\bar{\alpha}\int_{q_0^+}^{q^+}\frac{\mathrm{d}k^+}{k^+}\int_{r^2}^{1/Q_0^2}\frac{\mathrm{d}z^2}{z^2}\left(-\frac{\bar{\alpha}}{2}\ln^2\frac{z^2}{r^2}\right) = \bar{\alpha}Y\rho\times\left(-\frac{\bar{\alpha}}{6}\rho^2\right), \quad \rho \equiv \ln\frac{1}{r^2Q_0^2}$$

Resumming the double collinear logs

- Different pieces generated by TO are formally treated in different orders
  - an infinite series of terms $\propto (\alpha\rho^2)^n$, with $n \geq 1$ and alternating signs
  - ill-defined perturbative expansion (non-convergent truncations)
- This whole series can be resummed by enforcing TO within LO BK eq.
  - modified ("collinearly improved") version of the BK equation

(G. Beuf, 2014; E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, 2015)

- The evolution becomes stable ☺ ... but it still lacks predictivity ☹
  - strong dependence upon the precise choice for a resummation scheme
  - a remnant of the strong instability of the original perturbative expansion for the evolution with Y
- Recent solution: reformulate the evolution in terms of the "target rapidity" η

$$\eta \equiv Y - \rho \ = \ln \frac{q^+}{q_0^+} - \ln \frac{Q^2}{Q_0^2}$$

BK evolution in η

(Ducloué, E.I., Mueller, Soyez, and Triantafyllopoulos, arXiv:1902.06637)

- Why is this "simple" change of variables so useful ?

$$\eta = \ln \frac{q^+}{q_0^+} - \ln \frac{Q^2}{Q_0^2} = \ln \frac{\tau_q}{\tau_0}, \quad \tau_q = \frac{2q^+}{Q^2}, \quad \tau_0 = \frac{2q_0^+}{Q_0^2}$$

  - ordering in $\eta \Longleftrightarrow$ ordering in lifetimes
  - the proper time-ordering is automatically satisfied ☺

- Why is η the "target rapidity" ?

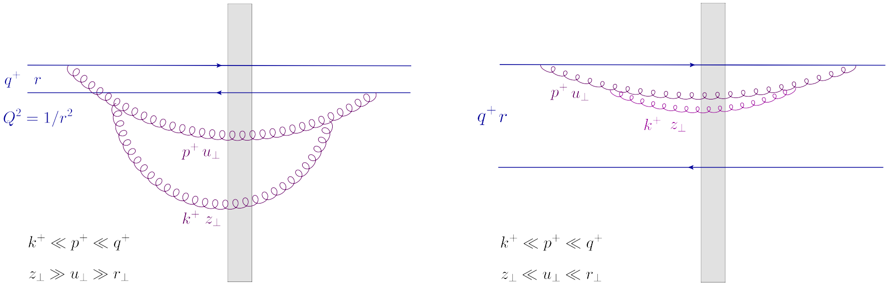
$$\eta = \ln \frac{\tau_q}{\tau_0} = \ln \frac{q_0^-}{q^-} = \ln \frac{2q^+ q_0^-}{Q^2} = \ln \frac{s}{Q^2} = \ln \frac{1}{x_{\rm Bj}}$$

  - ordering w.r.t. longitudinal momentum fraction of the struck parton in the target: Bjorken x (the right variable for the parton picture)

- Why not evolve directly the target (rather than changing variables ?)
  - NLO corrections available only for the evolution of the dilute projectile

Collinear resummations in η

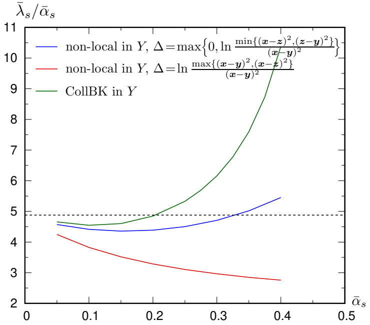
- The change of variables $Y = \eta + \rho$ replaces the (infinite series of) anti-collinear double-logs by collinear ones

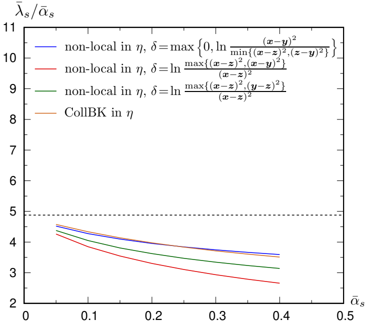


- The collinear double-logs can be resummed to all orders by enforcing ordering in k^+ (now, $k^+ \equiv k_\perp^2/2k^-$)

- Resummation not unique, but results only weakly scheme dependent
    - "soft-to-hard" evolution: atypical for the dilute-dense problem at hand

- $\bar{\lambda}_s \equiv \frac{\mathrm{d}\ln Q_s^2}{\mathrm{d}\eta}$: the speed of the saturation front in η
- Recall: LO result $\bar{\lambda}_s \simeq 4.88\bar{\alpha}$ (way too large)
- Collinear resummations at fixed coupling : the ratio $\bar{\lambda}_s/\bar{\alpha}$ against $\bar{\alpha}$

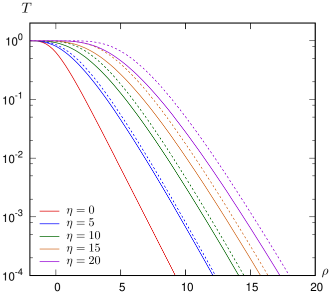


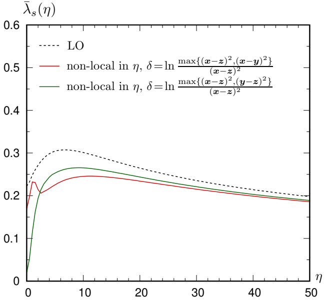


- Left: resummations in Y: strong scheme dependence, no clear pattern
- Right: resummations in η: only weak scheme dependence, significant reduction $(20 \div 30\%)$ w.r.t. LO

BK evolution in η: running coupling

- But a reduction of only $20 \div 30\%$ w.r.t. the LO is clearly insufficient !
- Recall: phenomenology requires $\bar{\lambda}_s \simeq 0.20 \div 0.25$
- The main reduction comes from the use of a running coupling
  - below: $\bar{\alpha}(r_{\min})$ where $r_{\min} = \min\{|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{x}-\boldsymbol{z}|, |\boldsymbol{y}-\boldsymbol{z}|\}$





- Left: saturation fronts in η: collBK (full lines) vs. LO BK (dashed)
- Right: saturation exponent: $\bar{\lambda}_s \simeq 0.2$ at large η ☺