Non-equilibrium QCD dynamics on the lattice

Sören Schlichting | Universität Bielefeld

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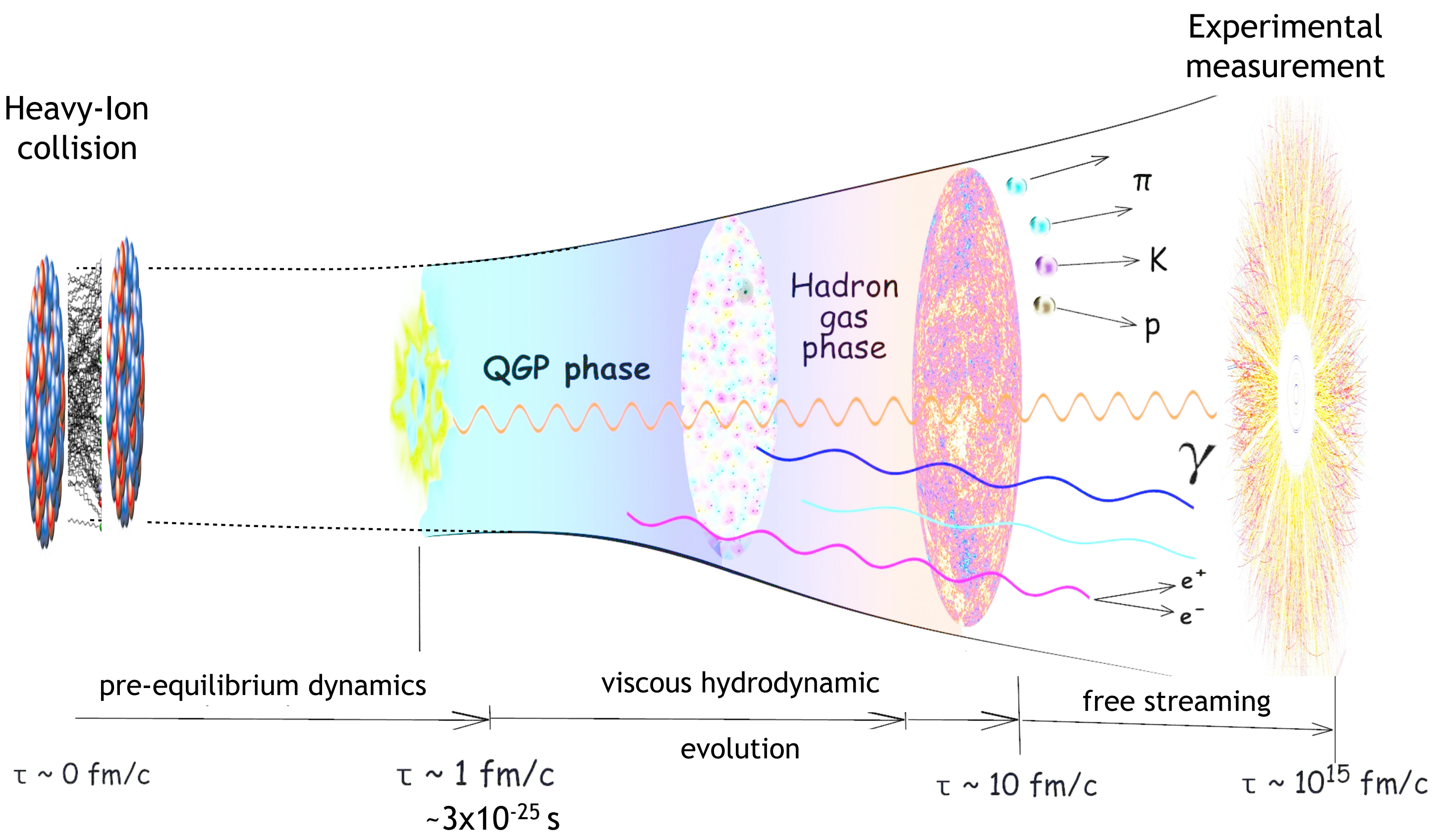
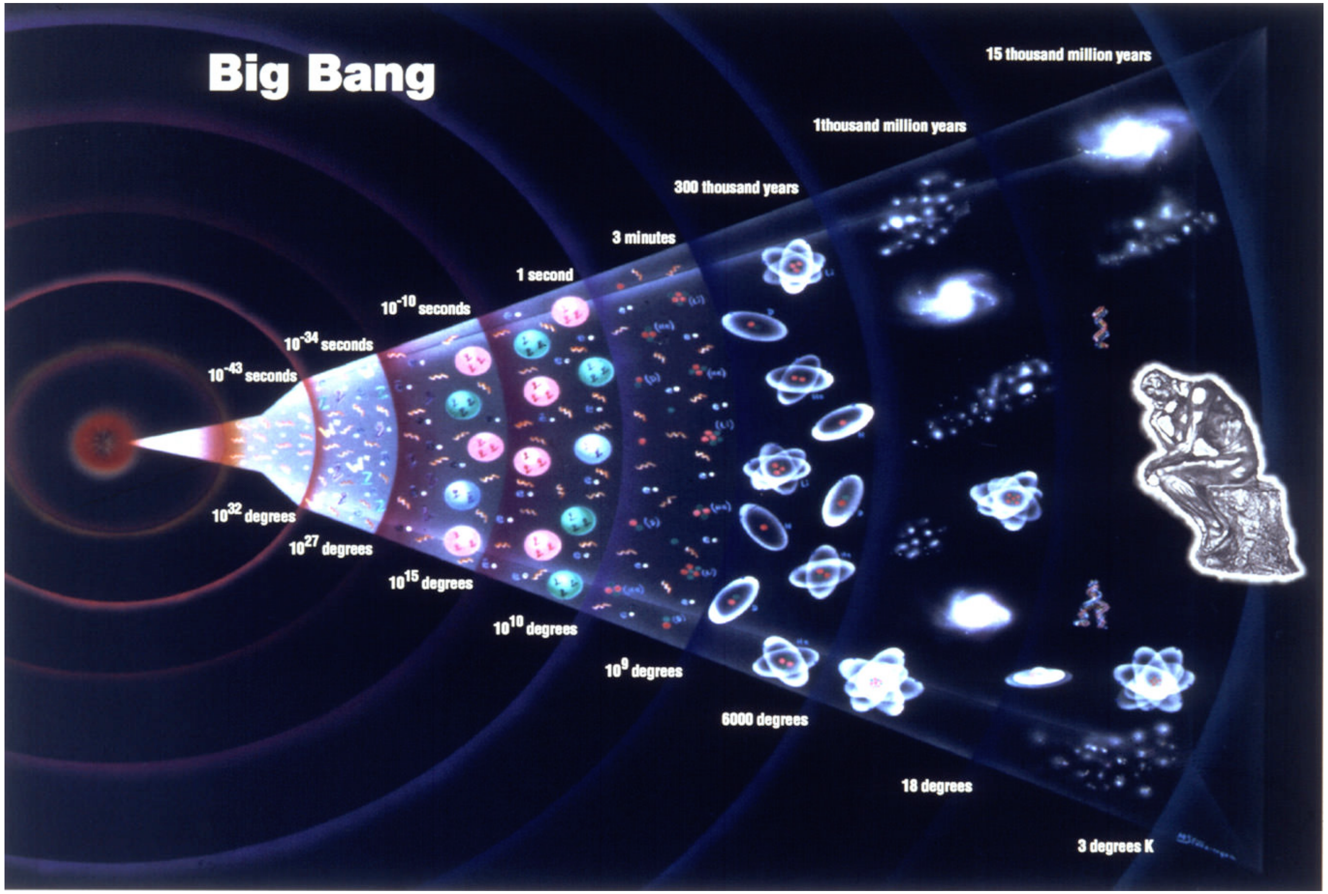
Strong-interaction matter under extreme conditions

Heavy-Ion collisions



Challenge to develop ab-initio description of HICs from QCD

Cosmology



Challenge to understand how matter as we know it is created from initial quantum fluctuations

Introduction & Motivation

We want to describe non-equilibrium QFT to study
Heavy-Ion collisions, Cosmology, Cold Atom experiments, …

What can we do? No exact methods a la lattice QCD

Need to choose most suitable among a variety of well established non-equilibrium methods

microscopic descriptions:

- semi-classical descriptions
- real-time effective actions
- kinetic theory
- holography

macroscopic descriptions:

- hydrodynamics

Different non-equilibrium methods have different range of validity

=> possibly need multiple methods to describe physics

Outline

Basics of non-equilibrium QFT

Classical-statistical lattice techniques

Non-equilibrium phenomena & thermalization processes in scalar and gauge theories

Non-thermal fixed points & connections to turbulence

New directions in non-equilibrium QFT

Some useful references include

Introduction to nonequilibrium quantum field theory
Juergen Berges (Heidelberg U.). Sep 2004. 131 pp.
Published in **AIP Conf.Proc. 739 (2004) no.1, 3-62**

Universal attractor in a highly occupied non-Abelian plasma
Juergen Berges (U. Heidelberg, ITP & Darmstadt, EMMI), Kirill Boguslavski (U. Heidelberg, ITP), Soeren Schlichting (U. Heidelberg, ITP & Brookhaven), Raju Venugopalan (Brookhaven).
Nov 12, 2013. 45 pp.
Published in **Phys.Rev. D89 (2014) no.11, 114007**

Non-equilibrium quantum mechanics (QM)

General question of interest

Schroedinger picture

Initial state — Measurement

Density matrix: $\hat{\rho}(t=0)$ — time t — Operator: $\hat{O}$

Evolution of density matrix governed by von-Neumann equation

$$i\hbar\,\partial_t\hat{\rho}(t) = \left[\hat{H},\hat{\rho}(t)\right] \qquad \hat{\rho}(t) = e^{-i\hat{H}t/\hbar}\hat{\rho}(t=0)e^{+i\hat{H}t/\hbar}$$

Expectation value O(t) of operator calculated as

$$O(t) = \mathrm{tr}\left[\hat{\rho}(t)\hat{O}\right]$$

Not practical for interacting QFTs (construction of Hilbert space, exponential growth of number of states,…)

Non-equilibrium path integral

General strategy to avoid use of operator formalism in QFT
is to introduce path integral

Consider for definiteness a scalar field theory

$$S_{\rm cl}[\varphi] = \int_{t,\mathbf{x}} \left\{ \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^4 - \frac{\lambda}{4!}\varphi^4 \right\}$$

By insertion of complete sets of states $\hat{\phi}(x)|\varphi\rangle = \varphi(x)|\varphi\rangle$

we can express $O(t) = \mathrm{tr}\left[\rho_0 e^{+i\hat{H}t/\hbar}\hat{O}e^{-i\hat{H}t/\hbar}\right]$ as

$$O(t) = \int [d\varphi]_0^+ [d\varphi]_0^- \langle \varphi_0^+ |\, \hat{\rho}_0 \,| \varphi_0^- \rangle \langle \varphi_0^- |\, e^{+i\hat{H}t/\hbar}\, \hat{O}\, e^{-i\hat{H}t/\hbar} \,| \varphi_0^+ \rangle$$

$$O(t) = \int [d\varphi]_0^+ [d\varphi]_0^- [d\varphi]_f^+ [d\varphi]_f^- \left\langle \varphi_0^+ \middle| \hat{\rho}_0 \middle| \varphi_0^- \right\rangle \left\langle \varphi_0^- \middle| e^{+i\hat{H}t/\hbar} \middle| \varphi_f^- \right\rangle \left\langle \varphi_f^- \middle| \hat{O} \middle| \varphi_f^+ \right\rangle \left\langle \varphi_f^+ \middle| e^{-i\hat{H}t/\hbar} \middle| \varphi_0^+ \right\rangle$$

Non-equilibrium path integral

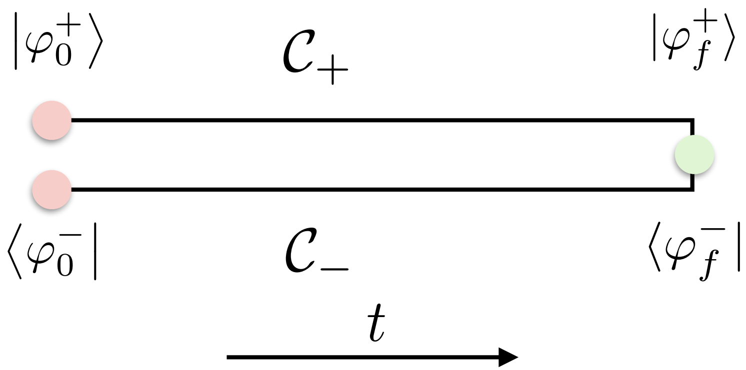
Expectation value of operator determined by

$$O(t) = \int [d\varphi]_0^+ [d\varphi]_0^- [d\varphi]_f^+ [d\varphi]_f^- \underbrace{\left\langle \varphi_0^+ \middle| \hat{\rho}_0 \middle| \varphi_0^- \right\rangle}_{\text{initial state matrix element}} \underbrace{\left\langle \varphi_0^- \middle| e^{+i\hat{H}t/\hbar} \middle| \varphi_f^- \right\rangle}_{\text{cc transition amplitude}} \underbrace{\left\langle \varphi_f^- \middle| \hat{O} \middle| \varphi_f^+ \right\rangle}_{\text{operator matrix element}} \underbrace{\left\langle \varphi_f^+ \middle| e^{-i\hat{H}t/\hbar} \middle| \varphi_0^+ \right\rangle}_{\text{direct transition amplitude}}$$

e.g. if φ_0/φ_f eigenstate of density operator/observable

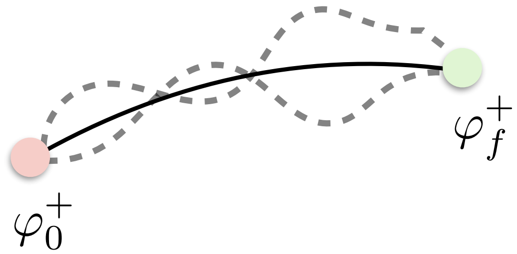
O(t) ≃ probability of initial state φ_0 x probability for transition $\varphi_0 \rightarrow \varphi_f$ x value of observable in state φ_f

Visualization on the Schwinger-Keldysh contour



Non-equilibrium path integral

Standard construction of path integral
for transition amplitudes



$$\left\langle \varphi_f^+ \middle| e^{+i\hat{H}t/\hbar} \middle| \varphi_0^+ \right\rangle = \int_{\varphi_0^+}^{\varphi_f^+} D\varphi^+ e^{+\frac{i}{\hbar} S_{\mathrm{cl}}[\varphi^+]}$$

completes derivation of non-equilibrium path integral

$$O(t) = \underbrace{\int [d\varphi_0^+][d\varphi_0^-] \langle \varphi_0^+ | \hat{\rho}_0 | \varphi_0^- \rangle}_{\text{initial state}} \underbrace{\int [d\varphi_f^+][d\varphi_f^-] \langle \varphi_f^- | \hat{O} | \varphi_f^+ \rangle}_{\text{measurement}} \underbrace{\int_{\varphi_0^+}^{\varphi_f^+} D\varphi^+ \int_{\varphi_0^-}^{\varphi_f^-} D\varphi^- e^{\frac{i}{\hbar}\left(S_{\mathrm{cl}}[\varphi^+] - S_{\mathrm{cl}}[\varphi^-]\right)}}_{\text{non-equilibrium dynamics}}$$

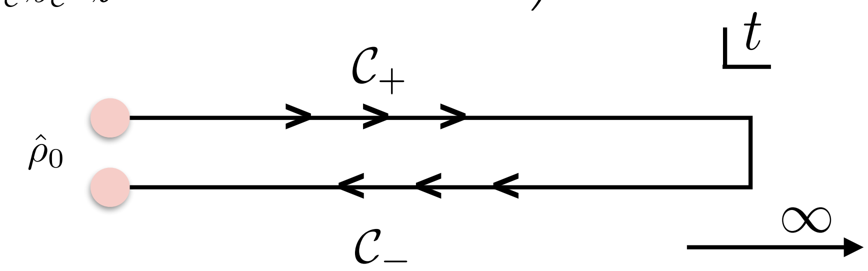
where Dφ is integral over all possible paths connecting beginning and endpoints

Non-equilibrium path integral

Since typically not interested in one specific observable, better to consider non-equilibrium generating functional

$$Z[J,R,\hat\rho_0] = \mathrm{tr}\left(\hat\rho_0\,\mathcal{T}_\mathcal{C}\exp\left[\frac{i}{\hbar}\Big(\int_{x^0_\mathcal{C},\mathbf{x}} J(x)\hat\phi_H(x) + \frac{1}{2}\int_{x^0_\mathcal{C},y^0_\mathcal{C}\mathbf{x},\mathbf{y}} \hat\phi_H(x)R(x,y)\hat\phi_H(y)\Big)\right]\right)$$

where $\mathcal{T}_\mathcal{C}$ denotes path ordering along Schwinger-Keldysh contour



$$Z[J,R,\hat\rho_0] = \underbrace{\int [d\varphi_0^+][d\varphi_0^-]\langle\varphi_0^+|\hat\rho_0|\varphi_0^-\rangle}_{\text{initial state}}\underbrace{\int_{\varphi_0^+}^{\varphi_0^-} D\varphi\; e^{\frac{i}{\hbar}\left[S_\mathcal{C}[\varphi]+\int_{x_\mathcal{C}} J(x)\varphi(x)+\frac{1}{2}\int_{x_\mathcal{C},y_\mathcal{C}}\varphi(x)R(x,y)\varphi(y)\right]}}_{\text{non-equilibrium dynamics}}$$

generic observables can be constructed as usual by functional differentiation wrt the sources

Non-equilibrium path integral

Question: How to evaluate non-equilibrium path integral

$$Z[J,R,\hat{\rho}_0] = \underbrace{\int [d\varphi_0^+][d\varphi_0^-]\langle\varphi_0^+|\hat{\rho}_0|\varphi_0^-\rangle}_{\text{initial state}} \underbrace{\int_{\varphi_0^+}^{\varphi_0^-} D\varphi\; e^{\frac{i}{\hbar}\left[S_{\mathcal{C}}[\varphi]+\int_{x_{\mathcal{C}}} J(x)\varphi(x)+\frac{1}{2}\int_{x_{\mathcal{C}},y_{\mathcal{C}}}\varphi(x)R(x,y)\varphi(y)\right]}}_{\text{non-equilibrium dynamics}}$$

Direct numerical evaluation essentially impossible due to severe sign problem e^{iS} (no purely real representation)

-> some ideas to be discussed at the end of lectures

Generally need to find suitable approximations based on expansion parameter, e.g.

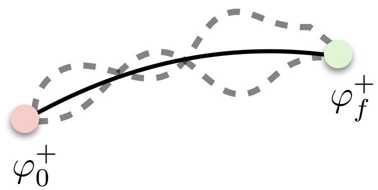
- semi-classical expansions ($\hbar$)
- weak-coupling expansions (λ)

Classical aspects of non-equilibrium quantum fields

Quantum field theory:

$$O(t) = \underbrace{\int [d\varphi_0^+][d\varphi_0^-]\langle\varphi_0^+|\hat{\rho}_0|\varphi_0^-\rangle}_{\text{initial state}} \underbrace{\int [d\varphi_f^+][d\varphi_f^-]\langle\varphi_f^-|\hat{O}|\varphi_f^+\rangle}_{\text{measurement}} \underbrace{\int_{\varphi_0^+}^{\varphi_f^+} D\varphi^+ \int_{\varphi_0^-}^{\varphi_f^-} D\varphi^- e^{\frac{i}{\hbar}\left(S_{\rm cl}[\varphi^+]-S_{\rm cl}[\varphi^-]\right)}}_{\text{non-equilibrium dynamics}}$$

for each initial/final state φ_0/φ_f contribution from all paths separately for direct and complex conjugate amplitude



Classical field theory:

classical equation of motion

$$\frac{\delta S_{\rm cl}[\phi]}{\delta\phi} = 0 \quad <=> \quad \partial_\mu\partial^\mu\phi + m^2\phi + \frac{\lambda}{6}\phi^3 = 0$$

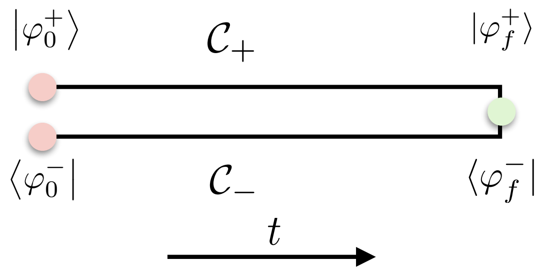
gives unique trajectory for each initial/final state ϕ_0/ϕ_f

Classical aspects of non-equilibrium quantum fields

Quantum field theory:

$$O(t) = \int [d\varphi_0^+][d\varphi_0^-]\langle\varphi_0^+|\hat\rho_0|\varphi_0^-\rangle \int [d\varphi_f^+][d\varphi_f^-]\langle\varphi_f^-|\hat O|\varphi_f^+\rangle \int_{\varphi_0^+}^{\varphi_f^+} D\varphi^+ \int_{\varphi_0^-}^{\varphi_f^-} D\varphi^- e^{\frac{i}{\hbar}\left(S_{\rm cl}[\varphi^+]-S_{\rm cl}[\varphi^-]\right)}$$

Change of basis/Keldysh-rotation



$$\varphi = \frac{\varphi^+ + \varphi^-}{2} \qquad \chi = \frac{\varphi^+ - \varphi^-}{\hbar}$$

difference of action becomes

$$\frac{i}{\hbar}\left(S_{\rm cl}[\varphi^+]-S_{\rm cl}[\varphi^-]\right) = -i\int_{t,\mathbf{x}} \chi(x)\left(\partial_\mu\partial^\mu\varphi(x) + m^2\varphi(x) + \frac{\lambda}{6}\varphi^3(x)\right)$$

classical EOM

$$-\frac{i\lambda\hbar^2}{24}\int_{t,\mathbf{x}} \chi^3(x)\varphi(x)$$

quantum corrections $O(\hbar^2)$

$$+i\int_{\mathbf{x}} \chi_f(\mathbf{x})\dot\varphi_f(\mathbf{x}) - \chi_0(\mathbf{x})\dot\varphi_0(\mathbf{x})$$

boundary terms

see S. Jeon PRC72 (2005) 014907

Classical aspects of non-equilibrium quantum fields

Non-equilibrium path integral takes the form

$$O(t) = \int [d\varphi_0][d\varphi_f] \int_{\varphi_0}^{\varphi_f} D\varphi \int [d\chi_0] \left\langle \varphi_0 + \frac{\hbar}{2}\chi_0 \middle| \hat{\rho}_0 \middle| \varphi_0 - \frac{\hbar}{2}\chi_0 \right\rangle e^{+i\int_{\mathbf{x}} \chi_0(\mathbf{x})\dot{\varphi}_0(\mathbf{x})}$$

$$\int D\chi \exp\Big[-i \int_{t,\mathbf{x}} \chi(x) \Big(\partial_\mu \partial^\mu \varphi(x) + m^2 \varphi(x) + \frac{\lambda}{6}\varphi^3(x) \Big) - \frac{i\lambda\hbar^2}{24} \int_{t,\mathbf{x}} \chi^3(x)\varphi(x) \Big]$$

$$\int [d\chi_f] \left\langle \varphi_f - \frac{\hbar}{2}\chi_f \middle| \hat{O} \middle| \varphi_f + \frac{\hbar}{2}\chi_f \right\rangle e^{-i\int_{\mathbf{x}} \chi_f(\mathbf{x})\dot{\varphi}_f(\mathbf{x})}$$

boundary terms lead to the appearance of Wigner-function

$$W[\varphi_0, \dot{\varphi}_0] = \int [d\chi_0] \left\langle \varphi_0 + \frac{\hbar}{2}\chi_0 \middle| \hat{\rho}_0 \middle| \varphi_0 - \frac{\hbar}{2}\chi_0 \right\rangle e^{+i\int_{\mathbf{x}} \chi_0(\mathbf{x})\dot{\varphi}_0(\mathbf{x})}$$

$$O_W[\varphi_0, \dot{\varphi}_0] = \int [d\chi_0] \left\langle \varphi_0 - \frac{\hbar}{2}\chi_0 \middle| \hat{O} \middle| \varphi_0 + \frac{\hbar}{2}\chi_0 \right\rangle e^{-i\int_{\mathbf{x}} \chi_0(\mathbf{x})\dot{\varphi}_0(\mathbf{x})}$$

Ignoring $O(\hbar^2)$ effects for the moment one finds

$$O(t) = \int [d\varphi_0][d\varphi_f] \int_{\varphi_0}^{\varphi_f} D\varphi \; W[\varphi_0, \dot{\varphi}_0] \; \delta\Big(\partial_\mu \partial^\mu \varphi(x) + m^2 \varphi(x) + \frac{\lambda}{6}\varphi^3(x) \Big) \; O_W[\varphi_f, \dot{\varphi}_f]$$

where only classical-trajectories contribute

see S. Jeon PRC72 (2005) 014907

Classical-statistical field theory

Evaluation of $$O(t) = \int [d\varphi_0][d\varphi_f] \int_{\varphi_0}^{\varphi_f} D\varphi \, W[\varphi_0, \dot{\varphi}_0] \, \delta\Big(\partial_\mu \partial^\mu \varphi(x) + m^2 \varphi(x) + \frac{\lambda}{6}\varphi^3(x)\Big) \, O_W[\varphi_f, \dot{\varphi}_f]$$

1. Start with initial conditions $\phi(t=0, \mathbf{x}) = \phi_\mathbf{0}$ $\quad \partial_t \phi(t, \mathbf{x})|_{t=0} = \pi_0$

2. Solve classical field equations of motion

   $$\partial_\mu \partial^\mu \phi + m^2 \phi + \frac{\lambda}{6}\phi^3 = 0$$ => classical trajectories $\phi_{\mathrm{cl}}(t, \mathbf{x})$

3. Calculate observables $O_W[\phi_{cl}(t)]$ from classical trajectories

4. Perform statistical average over different initial conditions

Note that quantum aspects may still be encoded in the Wigner functions W / O_W (e.g. W can include quantum fluctuations and may not be positive semi-definite)