Non-equilibrium QCD dynamics on the lattice

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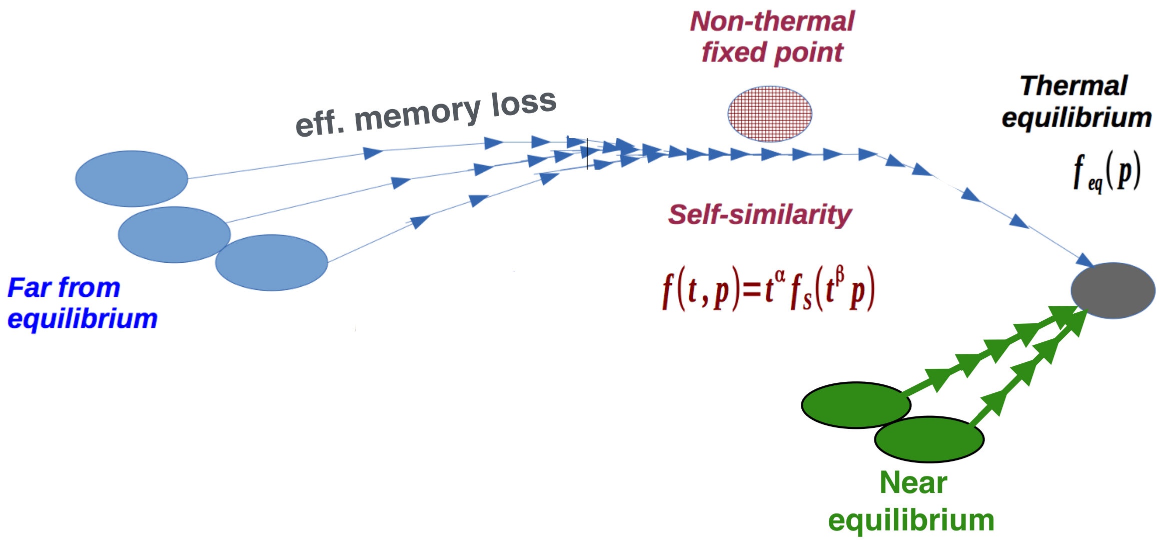
Strong-interaction matter under extreme conditions

Classical-statistical simulations indicate generic features in the non-equilibrium dynamics of far-from-equilibrium systems



Non-thermal fixed points associated with self-similar scaling behavior

Classical-statistical simulations are powerful numerical tool but provide limited physical insight into the underlying processes

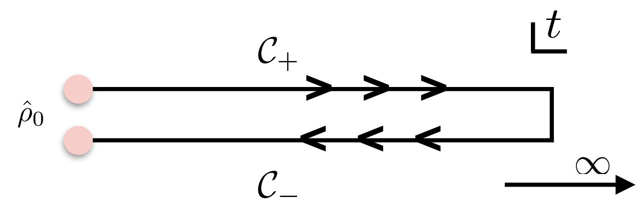
Need different theoretical tools to obtain further insight & extend description to quantum regime

-> Kinetic description of NTFPs

Non-equilibrium path integral

Starting point of our discussion has been the non-equilibrium generating functional

$$Z[J,R,\hat{\rho}_0] = \mathrm{tr}\left(\hat{\rho}_0\ \mathcal{T}_{\mathcal{C}} \exp\left[\frac{i}{\hbar}\Big(\int_{x^0_{\mathcal{C}},\mathbf{x}} J(x)\hat{\phi}_H(x) + \frac{1}{2}\int_{x^0_{\mathcal{C}},y^0_{\mathcal{C}}\mathbf{x},\mathbf{y}} \hat{\phi}_H(x)R(x,y)\hat{\phi}_H(y)\Big)\right]\right)$$



for which we constructed a path integral representation

$$Z[J,R,\hat{\rho}_0] = \int [d\varphi_0^+][d\varphi_0^-]\langle\varphi_0^+|\hat{\rho}_0|\varphi_0^-\rangle \int_{\varphi_0^+}^{\varphi_0^-} D\varphi\ e^{\frac{i}{\hbar}\left[S_{\mathcal{C}}[\varphi]+\int_{x_{\mathcal{C}}} J(x)\varphi(x)+\frac{1}{2}\int_{x_{\mathcal{C}},y_{\mathcal{C}}} \varphi(x)R(x,y)\varphi(y)\right]}$$

and subsequently expanded in $\hbar$ to obtain CSA

Now if we are interested in weakly coupled field theories, should also be able to expand in coupling constant λ to obtain approximate description also valid in the quantum regime

Challenges for perturbative approaches

Naive coupling expansion of generating functional leads to secular divergences*

Decay of a particle:

$$\dot{y}(t) = -\lambda^2 y(t) - \lambda^4 y(t) + \cdots \qquad\qquad y(t) = exp(-\lambda^2 t - \lambda^4 t) y_0$$

perturbative expansion LO: $y(t) = y_0$ NLO: $y(t) = y_0 - \lambda^2 y_0 t$

Physically: State of the system will change as a function of time

e.g. particle decays, scalar field decays, fluctuations grow

perturbative expansion around freely evolved initial state meaningless

Need coupling corrections to free-field EOMs to account for dynamical change of the system

selective resummation of all orders

*see Berges AIP Conf.Proc. 739 (2004) no.1, 3-62 for illustrative example

Effective actions & Quantum EOMs

Quantum effective action provides EOM for <ϕ>

$$W[J,R] = -i\log(Z[J,R]) \qquad \phi(x) = \langle\hat{\phi}(x)\rangle = \frac{\delta W[J,R]}{\delta J(x)}$$

Legendre trafo $$\Gamma[\phi,R] = W[J,R] - \int_{x,\mathcal{C}} J(x)\phi(x)$$

Quantum EOM: $$\frac{\delta\Gamma[\phi,R]}{\delta\phi(x)} = -J(x)$$ Classical EOM: $$\frac{\delta S[\phi]}{\delta\phi(x)} = -J(x)$$

Effects of fluctuations still not treated self-consistently
(diagrammatic contributions to Γ expanded in powers of free propagator G_0)

$$\Gamma[\phi,R=0] = S[\phi] + \frac{i}{2}\mathrm{tr}\log\left[G_0^{-1}[\phi]\right] + \Gamma_1[G_0[\phi],\phi]$$

-> Need to go (at least) one step further to capture evolution of fluctuations self-consistently

Effective actions & Quantum EOMs

Quantum effective action provides EOM for ϕ=<ϕ>, G=<$\mathcal{T}_C\phi\phi$>

$$\frac{\delta W[J,R]}{\delta R(x,y)} = \frac{1}{2}\langle \mathcal{T}_c\hat{\phi}(x)\hat{\phi}(y)\rangle = \frac{1}{2}\Big(G_{\mathcal{C}}(x,y)+\phi(x)\phi(y)\Big)$$

Legendre trafo $$\Gamma[\phi,G] = \Gamma[\phi,R] - \frac{1}{2}\int_{xy,\mathcal{C}} \Big(\phi(x)\phi(y)+G_{\mathcal{C}}(x,y)\Big)R(x,y)$$

$$\frac{\delta\Gamma[\phi,G]}{\delta\phi(x)} = -J(x) - \frac{1}{2}\int_{y,\mathcal{C}}\Big(R(x,y)+R(y,x)\Big)\phi(y) \qquad \frac{\delta\Gamma[\phi,G]}{\delta G(x,y)} = -\frac{1}{2}R(x,y)$$

Challenge in this approach is to calculate contributions to Γ

Somewhat cumbersome calculation shows that general structure is of the form*

$$\Gamma[\phi,G] = S[\phi] + \frac{i}{2}\mathrm{tr}\log G^{-1} + \frac{i}{2}\mathrm{tr}G_0^{-1}[\phi]G + \Gamma_2[\phi,G] + const$$

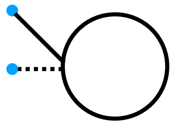
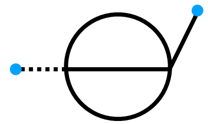
where Γ_2[ϕ,G] contains 2 particle irreducible diagrams in terms of self-consistent propagators and fields (but bare vertices)

*Cornwall,Jackiw,Tomboulis PRD 10, 2428

Effective actions & Quantum EOMs

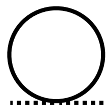
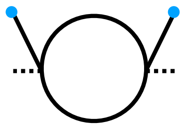
Quantum equations of motion for scalar field theory to 2-loop in Γ_2

EOM for background field

$$\left[\Box + m^2 + \frac{\lambda}{6}\phi(x)^2\right]\phi(x) = -\frac{\lambda}{2}F(x,x)\phi(x) + \frac{\lambda^2}{2}\int_0^{x^0} dy\ \rho(x,y)\Big(F^2(x,y) - \frac{1}{12}\rho^2(x,y)\Big)\phi(y)$$

EOM for propagator most conveniently expressed in the basis of statistical/spectral two-point function

$$\left[\Box_x + m^2 + \frac{\lambda}{2}\phi(x)^2\right]F(x,y) = -\frac{\lambda}{2}F(x,x)F(x,y) + \lambda^2\int_0^{x^0} dz\ \phi(x)F(x,z)\rho(x,z)\phi(z)F(z,y)$$
$$-\frac{\lambda^2}{2}\int_0^{y^0} dz\ \phi(x)\left(F(x,z)+\frac{1}{2}\rho(x,z)\right)\left(F(x,z)-\frac{1}{2}\rho(x,z)\right)\phi(z)\rho(z,y)$$

Berges AIP Conf.Proc. 739 (2004) no.1, 3-62

Effective actions & Quantum EOMs

Quantum evolution equations are EOMs for expectation values of background field & propagator

-> In contrast to CSA which is based on ensemble sample, effects of fluctuations are taken into account in loop expansion

-> Expansion includes both quantum & classical vertices

Quantum evolution equations involve memory integrals

$$\left[\Box + m^2 + \frac{\lambda}{6}\phi(x)^2\right]\phi(x) = -\frac{\lambda}{2}F(x,x)\phi(x) + \frac{\lambda^2}{2}\int_0^{x^0} dy\ \rho(x,y)\Big(F^2(x,y) - \frac{1}{12}\rho^2(x,y)\Big)\phi(y)$$

-> (more) difficult to solve numerically

Beyond loop/coupling expansion there are non-perturbative (1/N) expansion schemes available for scalar field theories

Various problems for gauge theories*
(gauge invariance, soft/collinear regions, …)

***see e.g. Blaizot, Iancu Phys.Rept. 359 (2002) 355-528**
Carrington,Kovalchuk PRD 80 (2009) 085013 and references therein

Off-shell transport & Kinetic theory

Quantum evolution equations feature two very different length/time and scales

Can be seen already at the example of spatially homogenous system free theory:

$$F(t,\mathbf{x},t',\mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \left(n_0(\mathbf{p}) + \frac{1}{2} \right) \cos(\omega_{\mathbf{p}}(t-t'))$$

$$\rho(t,\mathbf{x},t',\mathbf{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \frac{1}{\omega_p} \sin(\omega_{\mathbf{p}}(t-t'))$$

Very rapid oscillations in difference coordinate describing propagation of free particles

$$s = x - y$$

Since system homogenous & properties of free theory do not change no dependence on central coordinate

$$X = \frac{x+y}{2}$$

Off-shell transport & Kinetic theory

Similarly if we are interested of properties of interacting theory on long time/distance scales (e.g. NTFPs) can expect

$$\partial_X F(X+s/2, X-s/2) \ll \partial_s F(X+s/2, X-s/2)$$

Convenient to introduce Wigner transform of spectral/statistical correlation functions

$$\tilde{F}(X,p) = \int d^4s\ e^{ip_\mu s^\mu}\ F(X+s/2, X-s/2)$$
$$\tilde{\rho}(X,p) = -i\int d^4s\ e^{ip_\mu s^\mu}\ \rho(X+s/2, X-s/2)$$

which contain both time/position and frequency/momentum information

Gradient expansion of the evolution equations w.r.t to gradients $\partial_X\ \partial_p$

Off-shell transport & Kinetic theory

Straightforward but lengthy calculation gives
off-shell transport equations

$$2p^\mu \partial_{X^\mu} F(X,p) = \tilde{\Sigma}_\rho(X,p)\tilde{F}(X,p) - \tilde{\Sigma}_F(X,p)\tilde{\rho}(X,p) + \mathcal{O}(\partial_{X^\mu}\partial_{p_\mu})$$

$$2p^\mu \partial_{X^\mu} \rho(X,p) = \qquad 0 \qquad + \mathcal{O}(\partial_{X^\mu}\partial_{p^\mu})$$

with local (in X) interaction terms

$$\tilde{\Sigma}_F(X,p) = -\frac{\lambda^2}{2}\int_{k,q,l}\left(\tilde{F}(X,k)\tilde{F}(X,q) + \frac{1}{4}\tilde{\rho}(X,k)\tilde{\rho}(X,p)\right)\bar{\phi}^2(X,l)\ (2\pi)^4\ \delta^{(4)}(k+q+l-p)$$

$$\tilde{\Sigma}_\rho(X,p) = -\frac{\lambda^2}{2}\int_{k,q,l} 2\tilde{F}(X,k)\tilde{\rho}(X,q)\bar{\phi}^2(X,l)\ (2\pi)^4\ \delta^{(4)}(k+q+l-p)$$

Still includes off-shell processes need further approximation to obtain ordinary kinetic theory

Berges AIP Conf.Proc. 739 (2004) no.1, 3-62
Berges, Borsanyi PRD74 (2006) 045022

Off-shell transport & Kinetic theory

Based on the off-shell transport equations

$$2p^\mu \partial_{X^\mu} F(X,p) = \tilde{\Sigma}_\rho(X,p)\tilde{F}(X,p) - \tilde{\Sigma}_F(X,p)\tilde{\rho}(X,p) + \mathcal{O}(\partial_{X^\mu}\partial_{p_\mu})$$

$$2p^\mu \partial_{X^\mu} \rho(X,p) = \qquad 0 \qquad + \mathcal{O}(\partial_{X^\mu}\partial_{p^\mu})$$

Defining an effective occupation number and making the additional approximation that spectral functions are narrow (consistent with weak coupling expansion)

$$F(X,p) = [n(X,p) + 1/2]\, \rho(X,p)$$

$$\rho(X,p) \approx 2\pi \ \mathrm{sign}(p_0)\ \delta(p_0^2 - \omega_{\mathbf{p}}^2)$$

$$\int_0^\infty \frac{dp_0}{2\pi} 2p^\mu \partial_{X^\mu} F(X,p) = \partial_t n(t,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}^i}{\omega_{\mathbf{p}}} \partial_{\mathbf{x}^i} n(t,\mathbf{x},\mathbf{p}) = C[n](t,\mathbf{x},\mathbf{p})$$

where collision integral obtained from RHS takes the usual form

$$C[n](t,\mathbf{x},\mathbf{p}) = \frac{3\lambda^2 \bar{\phi}^2(t)}{2\omega_0} \Big\{ \int d\Omega_{\mathbf{kl}\leftrightarrow\mathbf{p}\phi} \Big[n_{\mathbf{k}} n_{\mathbf{l}} (1+n_{\mathbf{p}}) - n_{\mathbf{p}}(1+n_{\mathbf{k}})(1+n_{\mathbf{l}}) \Big] \ + \cdots \ \Big\}$$

Off-shell transport & Kinetic theory

Based on the off-shell transport equations

$$2p^\mu \partial_{X^\mu} F(X,p) = \tilde{\Sigma}_\rho(X,p)\tilde{F}(X,p) - \tilde{\Sigma}_F(X,p)\tilde{\rho}(X,p) + \mathcal{O}(\partial_X \ldots$$

$$2p^\mu \partial_{X^\mu} \rho(X,p) = \qquad 0$$

Defining an effective occupati…
and making the additi… $\ldots, p) + 1/2]\,\rho(X,p)$
that spectral fu… $\rho(X,p) \approx 2\pi\ \mathrm{sign}(p_0)\ \delta(p_0^2 - \omega_\mathbf{p}^2)$
(consist… …on)

Note that there are alternative ways to derive kinetic equations from QFT

Can make life simpler by performing approximations in different order

$$\ldots_t n(t,\mathbf{x},\mathbf{p}) + \frac{\mathbf{p}^i}{\omega_\mathbf{p}}\partial_{\mathbf{x}^i} n(t,\mathbf{x},\mathbf{p}) = C[n](t,\mathbf{x},\mathbf{p})$$

w… collision integral obtained from RHS takes the usual form

$$C[n](t,\mathbf{x},\mathbf{p}) = \frac{3\lambda^2\bar{\phi}^2(t)}{2\omega_0}\left\{ \int d\Omega_{\mathbf{kl}\leftrightarrow\mathbf{p}\phi}\left[n_\mathbf{k} n_\mathbf{l}(1+n_\mathbf{p}) - n_\mathbf{p}(1+n_\mathbf{k})(1+n_\mathbf{l})\right] + \cdots \right\}$$

Comparison of range of validity

Kinetic theory:

Expansion of $n(X,p)$ in gradients of X,p

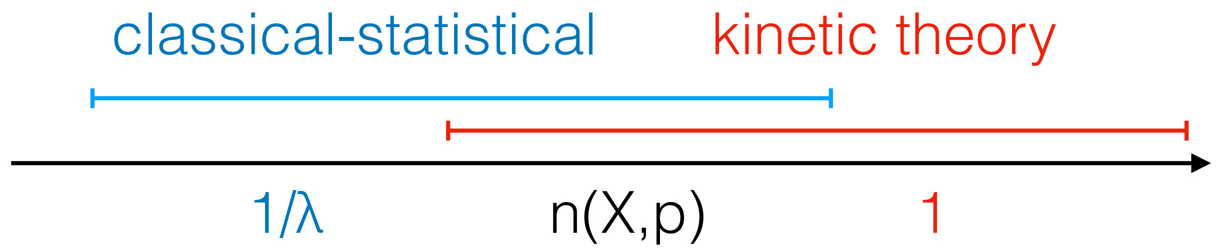
Perturbative expansion in λ and λn

$$\lambda \ll 1 \text{ and } \lambda\, n(X,p) \ll 1$$

Classical-statistical approximation:

Non-perturbative expansion in $n \gg 1$

Kinetic theory and CSA have overlap in range of validity for weakly coupled systems ($\lambda \ll 1$) when occupancies are large but not non-perturbatively large



Kinetic theory for non-abelian plasmas

Effective kinetic description of (isotropic) non-abelian plasmas features two distinct processes

$$(\partial_t + \boldsymbol{v_p} \cdot \partial_{\boldsymbol{x}})\, f(t, \boldsymbol{x}, \boldsymbol{p}) = C_{\rm el}^{2\leftrightarrow 2}[f(\boldsymbol{p})] + C_{\rm inel}^{1\leftrightarrow 2}[f(\boldsymbol{p})]$$

Elastic scattering processes

Separate into small & large momentum transfer

$$C_{\rm el}^{2\leftrightarrow 2}[f(\mathbf{p})] = C_{\rm diff}^{2\leftrightarrow 2}[f(\mathbf{p})] + C_{\rm el}^{2\leftrightarrow 2}[f(\mathbf{p})]$$

Small angle scatterings more frequent, but smaller effect on momenta of the particles

Small angle and large angle scattering equally important for changes in momenta of typical excitation with p~Q

Arnold, Moore, Yaffe, JHEP 0301 (2003) 030
Ghighlieri, Teaney Int.J.Mod.Phys. E24 (2015) no.11, 1530013
Ghiglieri,Moore,Teaney JHEP 1603 (2016) 095

Kinetic theory for non-abelian plasmas

Effective kinetic description of non-abelian plasmas features two distinct processes

$$(\partial_t + \boldsymbol{v_p} \cdot \partial_{\boldsymbol{x}})\, f(t, \boldsymbol{x}, \boldsymbol{p}) = C_{\rm el}^{2\leftrightarrow 2}[f(\boldsymbol{p})] + C_{\rm inel}^{1\leftrightarrow 2}[f(\boldsymbol{p})]$$

In-elastic scattering processes

Elastic scatterings accelerate charged particles
-> possibility to emit/absorp radiation

Since each small scattering can induce radiation it is important to consider coherence effects between multiple scattering

Bethe-Heitler regime vs. Landau Pomeranchuk Migdal regime

Generally relative importance of elastic vs. in-elastic processes depends on momentum scale

Equally important for typical exceptions with p~Q

Arnold, Moore, Yaffe, JHEP 0301 (2003) 030
Ghighlieri, Teaney Int.J.Mod.Phys. E24 (2015) no.11, 1530013
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Kinetic description of NTFPs

Can now study NTFPs directly from Boltzmann equation
by directly searching for self-similar solutions of the form

$$f(t,\mathbf{p}) = t^{\gamma} f_S(t^{-\alpha}\mathbf{p}) \qquad \partial_t f(t,\mathbf{p}) = C[f](t,\mathbf{p})$$

Simply plugging in LHS yields:

$$\partial_t f(t,\mathbf{p}) = \partial_t \left(t^{\gamma} f_S(t^{-\alpha}\mathbf{p}) \right) = t^{\gamma-1} \left[\gamma f_S(\tilde{\mathbf{p}}) - \alpha\, \tilde{\mathbf{p}}\ \nabla_{\tilde{\mathbf{p}}} f_S(\tilde{\mathbf{p}}) \right]_{\tilde{\mathbf{p}}=t^{-\alpha}\mathbf{p}}$$

overall scaling — momentum dependence

Need to consider all processes entering RHS:

Detailed analysis shows that all LO processes exhibit the same scaling properties

Berges, Boguslavski, SS, Venugopalan PRD89 (2014) no.11, 114007
Kurkela, Moore PRD86 (2012) 056008

Kinetic description of NTFPs

Will only elastic scattering (large momentum transfer) as stereotypical example for this analysis

$$C[f](t,\mathbf{p}) = \frac{1}{2}\int_{\mathbf{q},\mathbf{k},\mathbf{l}} \frac{|M(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l})|^2}{2\omega_p\, 2\omega_q\, 2\omega_k\, 2\omega_l} (2\pi)^4\, \delta(\omega_q+\omega_k-\omega_l-\omega_p)\, \delta^{(3)}(\mathbf{q}+\mathbf{k}-\mathbf{l}-\mathbf{p}) \times\Big[(1+f_{\mathbf{p}})(1+f_{\mathbf{l}})f_{\mathbf{q}}f_{\mathbf{k}} - f_{\mathbf{p}}f_{\mathbf{l}}(1+f_{\mathbf{q}})(1+f_{\mathbf{k}})\Big]\,,$$

NTFP occurs in high-occupancy regime $f(t,\mathbf{p}) \gg 1$

$$C[f](t,\mathbf{p}) = \frac{1}{2}\int d\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l})\; f_{\mathbf{p}}f_{\mathbf{l}}f_{\mathbf{q}}f_{\mathbf{k}}\left[f_{\mathbf{p}}^{-1}+f_{\mathbf{l}}^{-1}-f_{\mathbf{q}}^{-1}-f_{\mathbf{k}}^{-1}\right],$$

Goal is to separate time & momentum dependence
-> use scaling ansatz $f(t,\mathbf{p}) = t^{\gamma} f_S(t^{-\alpha}\mathbf{p})$

$$C[f](t,\mathbf{p}) = t^{3\gamma}\,\frac{1}{2}\int d\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l})\; f_S(t^{-\alpha}\mathbf{p})f_S(t^{-\alpha}\mathbf{l})f_S(t^{-\alpha}\mathbf{q})f_S(t^{-\alpha}\mathbf{k}) \times\left[f_S^{-1}(t^{-\alpha}\mathbf{p})+f_S^{-1}(t^{-\alpha}\mathbf{l})-f_S^{-1}(t^{-\alpha}\mathbf{q})-f_S^{-1}(t^{-\alpha}\mathbf{k})\right],$$

Berges, Boguslavski, SS, Venugopalan PRD89 (2014) no.11, 114007

Kinetic description of NTFPS

Next need to consider scaling of phase-space integral

$$\int d\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l}) = \int_{\mathbf{q},\mathbf{k},\mathbf{l}} \frac{|M(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l})|^2}{2\omega_p\ 2\omega_q\ 2\omega_k\ 2\omega_l} (2\pi)^4\ \delta^{(4)}(q+k-l-p)\ ,$$

$$(\quad \text{Substitute: } \tilde{\mathbf{q}} = t^{-\alpha}\mathbf{q},\ \tilde{\mathbf{k}} = t^{-\alpha}\mathbf{k},\ \tilde{\mathbf{l}} = t^{-\alpha}\mathbf{l} \mid \text{Express : } \mathbf{p} = t^{\alpha}(t^{-\alpha}\mathbf{p}) \mid \text{Use: } \omega_{sp} = |s|\omega_p)\ ,$$

$$= \ t^{9\alpha} \int_{\tilde{\mathbf{q}},\tilde{\mathbf{k}},\tilde{\mathbf{l}}} \frac{|M(t^{\alpha}(t^{-\alpha}\mathbf{p}), t^{\alpha}\tilde{\mathbf{q}}, t^{\alpha}\tilde{\mathbf{k}}, t^{\alpha}\tilde{\mathbf{l}})|^2}{2\omega_p\ 2\omega_{t^{\alpha}\tilde{q}}\ 2\omega_{t^{\alpha}\tilde{k}}\ 2\omega_{t^{\alpha}\tilde{l}}} (2\pi)^4\ \delta^{(4)}(t^{\alpha}(\tilde{q}+\tilde{k}-\tilde{l}-t^{-\alpha}p))\ ,$$

$$(\quad \text{Use: } \omega_{sp} = |s|\omega_p,\ \ \delta(sx) = |s|^{-1}\delta(x),\ \ |M(sp,sq,sk,sl)|^2 = |M(p,q,k,l)|^2)$$

$$= \ t^{\alpha} \int_{\tilde{\mathbf{q}},\tilde{\mathbf{k}},\tilde{\mathbf{l}}} \frac{|M(t^{-\alpha}\mathbf{p}, \tilde{\mathbf{q}}, \tilde{\mathbf{k}}, \tilde{\mathbf{l}})|^2}{2\omega_{t^{-\alpha}p}\ 2\omega_{\tilde{q}}\ 2\omega_{\tilde{k}}\ 2\omega_{\tilde{l}}} (2\pi)^4\ \delta^{(4)}(\tilde{q}+\tilde{k}-\tilde{l}-t^{-\alpha}p)\ ,$$

$$(\quad \text{Rename: } \tilde{\mathbf{q}} \to t^{-\alpha}\mathbf{q},\ \tilde{\mathbf{k}} \to t^{-\alpha}\mathbf{k},\ \tilde{\mathbf{l}} \to t^{-\alpha}\mathbf{l} \mid \text{Identify with the first line})$$

$$= \ t^{\alpha} \int d\Omega^{2\leftrightarrow 2}(t^{-\alpha}\mathbf{p}, t^{-\alpha}\mathbf{q}, t^{-\alpha}\mathbf{k}, t^{-\alpha}\mathbf{l})\ .$$

Scaling relation based on scale invariance of matrix element & dispersion

*we neglect logarithmic corrections due to cut-off on momentum transfer

Berges, Boguslavski, SS, Venugopalan PRD89 (2014) no.11, 114007

Kinetic description of NTFPS

Continue analysis of the collision integral

$$
\begin{aligned}
C[f](t,\mathbf{p}) \;&=\; t^{3\gamma}\,\frac{1}{2}\int d\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l})\; f_S(t^{-\alpha}\mathbf{p}) f_S(t^{-\alpha}\mathbf{l}) f_S(t^{-\alpha}\mathbf{q}) f_S(t^{-\alpha}\mathbf{k}) \\
&\quad\times\Big[f_S^{-1}(t^{-\alpha}\mathbf{p}) + f_S^{-1}(t^{-\alpha}\mathbf{l}) - f_S^{-1}(t^{-\alpha}\mathbf{q}) - f_S^{-1}(t^{-\alpha}\mathbf{k})\Big]\,,
\end{aligned}
$$

now using $\int d\Omega^{2\leftrightarrow 2}(\mathbf{p},\mathbf{q},\mathbf{k},\mathbf{l}) \;=\; t^{\alpha}\int d\Omega^{2\leftrightarrow 2}(t^{-\alpha}\mathbf{p},t^{-\alpha}\mathbf{q},t^{-\alpha}\mathbf{k},t^{-\alpha}\mathbf{l})$

$$
\begin{aligned}
&=\; t^{3\gamma+\alpha}\,\frac{1}{2}\int d\Omega^{2\leftrightarrow 2}(t^{-\alpha}\mathbf{p},t^{-\alpha}\mathbf{q},t^{-\alpha}\mathbf{k},t^{-\alpha}\mathbf{l})\; f_S(t^{-\alpha}\mathbf{p}) f_S(t^{-\alpha}\mathbf{l}) f_S(t^{-\alpha}\mathbf{q}) f_S(t^{-\alpha}\mathbf{k}) \\
&\quad\times\Big[f_S^{-1}(t^{-\alpha}\mathbf{p}) + f_S^{-1}(t^{-\alpha}\mathbf{l}) - f_S^{-1}(t^{-\alpha}\mathbf{q}) - f_S^{-1}(t^{-\alpha}\mathbf{k})\Big]\,,
\end{aligned}
$$

up to pre-factor t^X this expression is of same form as the original expression with $f \to f_S \quad p \to t^{-\alpha}p$

$$
C[f](t,\mathbf{p}) = \underbrace{t^{3\gamma+\alpha}}_{\text{overall scaling}}\underbrace{C[f_S](\tilde{\mathbf{p}})\Big|_{\tilde{\mathbf{p}}=t^{-\alpha}\mathbf{p}}}_{\text{momentum dependence}}
$$

overall scaling momentum dependence

Kinetic description of NTFPS

By matching LHS and RHS of Boltzmann equation using their scaling properties

$$\partial_t f(t,\mathbf{p}) = C[f](t,\mathbf{p})$$

$$\partial_t f(t,\mathbf{p}) = t^{\gamma-1}\left[\gamma f_S(\tilde{\mathbf{p}}) - \alpha\,\tilde{\mathbf{p}}\;\nabla_{\tilde{\mathbf{p}}}\,f_S(\tilde{\mathbf{p}})\right]_{\tilde{\mathbf{p}}=t^{-\alpha}\mathbf{p}}$$

$$C[f](t,\mathbf{p}) = t^{3\gamma+\alpha}C[f_S](\tilde{\mathbf{p}})\Big|_{\tilde{\mathbf{p}}=t^{-\alpha}\mathbf{p}}$$

Validity at different times then requires

Scaling relation

$$3\gamma + \alpha = \gamma - 1$$

Spectral shape of fixed point distribution

$$\left[\gamma f_S(\tilde{\mathbf{p}}) - \alpha\,\tilde{\mathbf{p}}\;\nabla_{\tilde{\mathbf{p}}}\,f_S(\tilde{\mathbf{p}})\right] = C[f_S](\tilde{\mathbf{p}})$$

Berges, Boguslavski, SS, Venugopalan PRD89 (2014) no.11, 114007

Kinetic description of NTFPS

Second constraint comes from conserved quantities

Since non-abelian gauge theories feature elastic (2<->2) interactions and inelastic (1<->2) interactions only energy conservation applies

$$\epsilon = \int_{\mathbf{p}} \omega_p \, f(t,\mathbf{p}) = t^{\gamma} \int_{\mathbf{p}} \omega_p \, f_S(t^{-\alpha}\mathbf{p}) = t^{\gamma+3\alpha} \int_{\tilde{\mathbf{p}}} \omega_{t^{\alpha}\tilde{p}} \, f_S(\tilde{\mathbf{p}}) = \boxed{t^{\gamma+4\alpha}} \epsilon_S$$

scaling relation (cons. laws)

$$\boxed{\gamma + 4\alpha = 0}$$

scaling relation (dynamics)

$$\boxed{3\gamma + \alpha = \gamma - 1}$$

$$\gamma = -4/7 \qquad \alpha = 1/7$$

Berges, Boguslavski, SS, Venugopalan PRD89 (2014) no.11, 114007

Kinetic description of NTFPS

Same analysis can be performed for scalar field theory, where the LO process in presence of background field is given by eff. 1<->2 collision kernel

$$C[n](t,\mathbf{x},\mathbf{p}) = \frac{3\lambda^2\bar{\phi}^2(t)}{2\omega_0}\Big\{ \int d\Omega_{\mathbf{kl}\leftrightarrow\mathbf{p}\phi}\Big[n_{\mathbf{k}}n_{\mathbf{l}}(1+n_{\mathbf{p}}) - n_{\mathbf{p}}(1+n_{\mathbf{k}})(1+n_{\mathbf{l}})\Big]$$
$$- \int d\Omega_{\mathbf{pk}\leftrightarrow\mathbf{l}\phi}\Big[n_{\mathbf{p}}n_{\mathbf{k}}(1+n_{\mathbf{l}}) - n_{\mathbf{l}}(1+n_{\mathbf{p}})(1+n_{\mathbf{k}})\Big]$$
$$- \int d\Omega_{\mathbf{lp}\leftrightarrow\mathbf{k}\phi}\Big[n_{\mathbf{l}}n_{\mathbf{p}}(1+n_{\mathbf{k}}) - n_{\mathbf{k}}(1+n_{\mathbf{l}})(1+n_{\mathbf{p}})\Big]\Big\}$$

$$\int d\Omega_{\mathbf{kl}\leftrightarrow\mathbf{p}\phi} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d^3\mathbf{l}}{(2\pi)^3}\frac{(2\pi)^3\delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\ (2\pi)\delta(\omega_k+\omega_l-\omega_p-\omega_0)}{2\omega_k 2\omega_l 2\omega_p}$$

Scaling relation

$$\boxed{2\gamma - \alpha = \gamma - 1}$$

Universality far from equilibrium

Discussion shows that scaling exponents only depend on

order of n-body process
scaling dimension of matrix element
dimension
dispersion relation

but not on microscopic quantities such as

number of color (N_c), coupling strength (g), …

Universality: Different theories characterized by same scaling properties irrespective of mircroscopics

SU(Nc) Yang-Mills (d=3, n>>1)

$$\gamma = -4/7 \qquad \alpha = 1/7$$

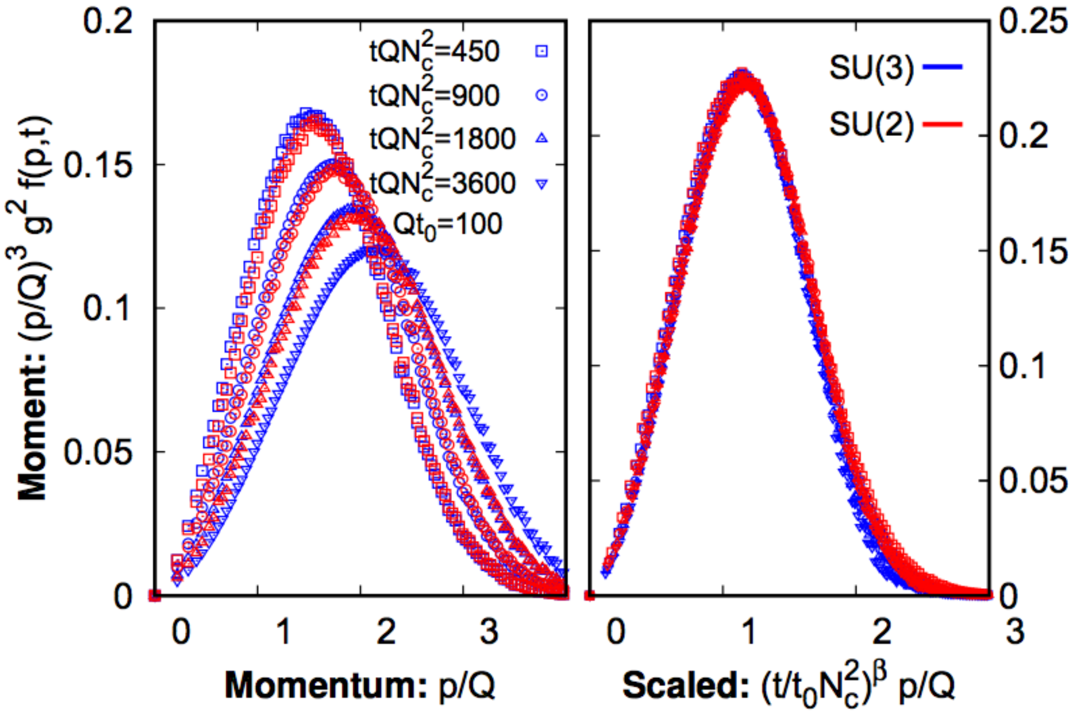
O(N) Scalars (d=3, n>>1)

$$\gamma = -4/5 \qquad \alpha = 1/5$$

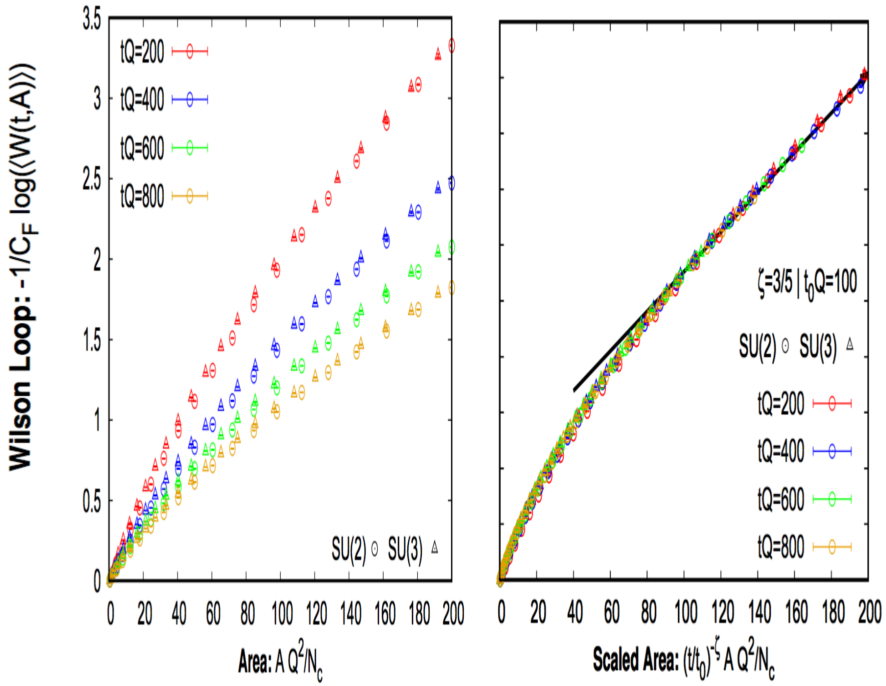
Universality far from equilibrium

Explicit verification for SU(Nc) Yang-Mills theory
from classical-statistical simulations

Hard sector



Soft sector



$SU(2): \quad \beta = -0.145 \pm 0.017\ (\chi^2) \pm 0.002\ (sys.)$

$SU(3): \quad \beta = -0.141 \pm 0.020\ (\chi^2) \pm 0.002\ (sys.)$

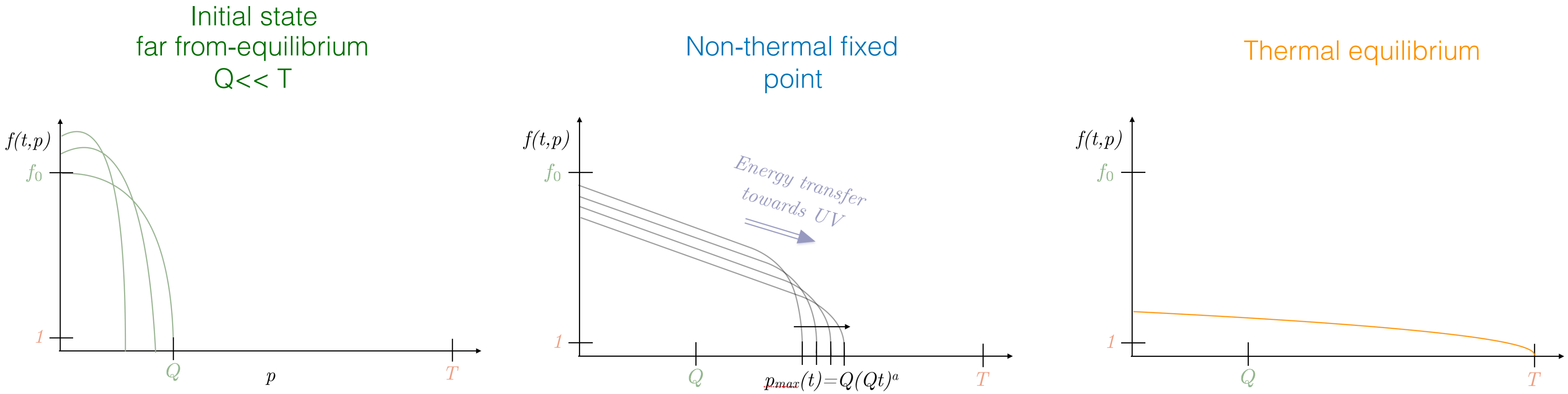
$SU(2): \quad \zeta = 0.603 \pm 0.005\ (\chi^2) \pm 0.004\ (sys.)$

$SU(3): \quad \zeta = 0.604 \pm 0.004\ (\chi^2) \pm 0.005\ (sys.)$

Berges,SS, Mace PRL 118 (2017) no.19, 192005

NTFPs and Thermalization

What does all of this mean in the context of thermalization?



Non-thermal fixed point stable under classical evolution (f>>1)

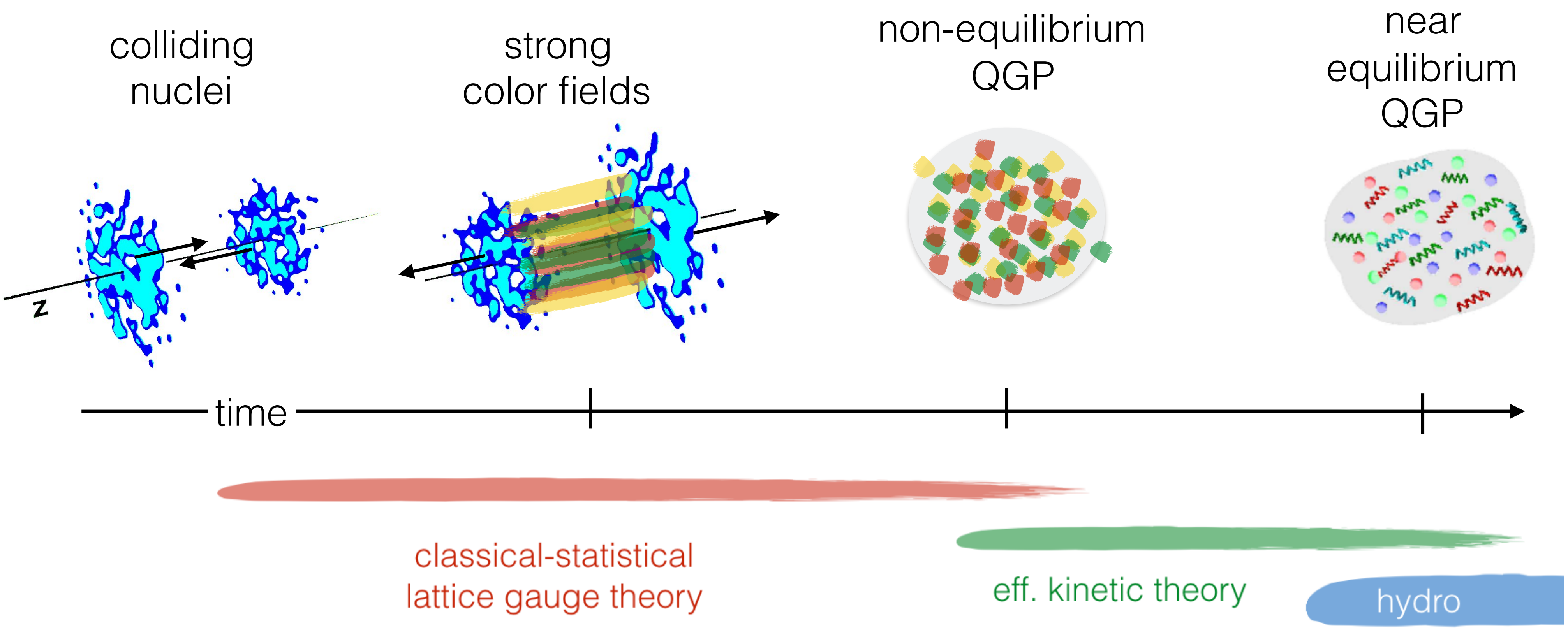
->Breakdown of CSA before system thermalizes f(p~T) ~1

Departure from NTFP and approach to equilibrium when system becomes dilute f(p_{max}(t)) ~1

$$t \sim t_{\rm eq} \sim \alpha_s^{-2} f_0^{-1/4} Q^{-1} \sim \alpha_s^{-2} T^{-1}$$

Early time dynamics of heavy-ion collisions

Different degrees of freedom relevant at different at different stages of the evolution



Need combination of non-equilibrium methods
to describe early time dynamics & approach to equilibrium