Non-equilibrium QCD dynamics on the lattice

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Strong-interaction matter under extreme conditions

Quantum versus classical-statistical field theory

By analyzing structure of non-equilibrium path-integral we derived classicality condition

$$|F(t,\mathbf{x},t',\mathbf{y}) + \phi(t,\mathbf{x})\phi(t',\mathbf{y})| \gg |\rho(t,\mathbf{x},t',\mathbf{y})|$$

Statistical function F -> actual excitations present in the system

Spectral function ρ -> possible excitations of the systems

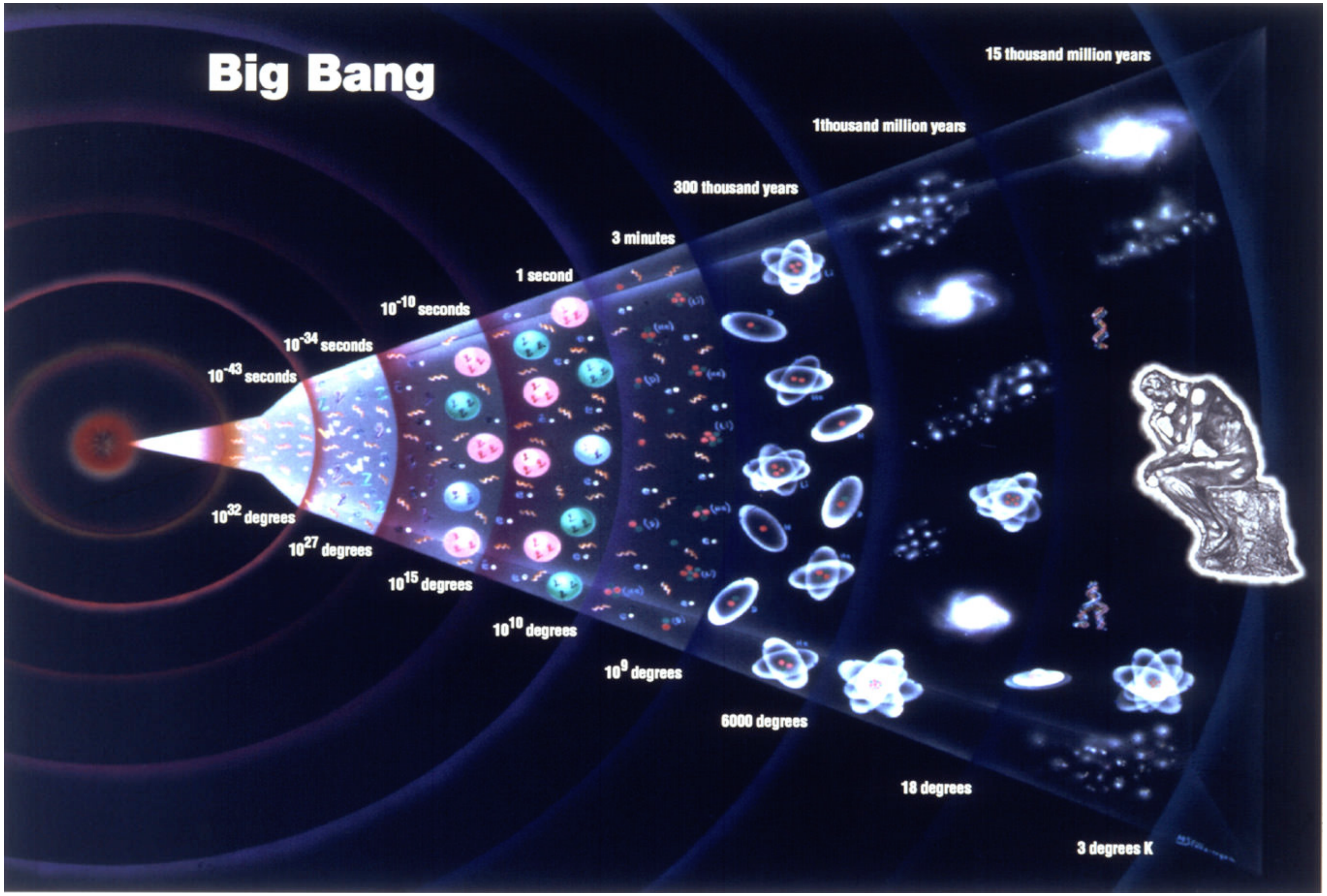
Classical-statistical description accurate in the presence of strong fields/ for highly occupied bosonic quantum fields

=> Correspondence principle

Several interesting problems of this nature in QCD,Cosmology, Cold Atoms

Note that classicality criterion is time dependent; classical-statistical description typically breaks down after some time

Example: Scalar field reheating



Negative pressure to create accelerated expansion during inflation usually realized through scalar field(s)

see e.g. https://wwwmpa.mpa-garching.mpg.de/~komatsu/lecturenotes/Kaloian_Lozanov_on_Reheating.pdf

Example: Scalar field reheating

We will consider simplest possible example of single component scalar field

$$S[\phi] = \int d^4x \sqrt{-g(x)} \Big\{ \frac{1}{2} g^{\mu\nu}(x)(\partial_\mu \phi(x))(\partial_\nu \phi(x)) - \frac{m^2}{2}\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \Big\}$$

in spatially homogenous/isotropic (FRW) metric

$$g_{\mu\nu}(x) = \mathrm{diag}(1, -a(t), -a(t), -a(t))$$

such that EOMs features Hubble damping

$$a^3(t)\left[\partial_t^2 \phi + 3\frac{\partial_t a(t)}{a(t)}\partial_t \phi - a^{-1}(t)\partial_i^2 \phi + m^2\phi + \frac{\lambda}{6}\phi^3\right] = 0$$

Based on scale transformation introducing conformal time variable EOMs can be recast

$$dt = a(t)d\eta \qquad \mathbf{x} = \mathbf{x}_{\mathrm{phys}}/a(t)$$

$$\phi(t) \to a(t)\, \phi_{\mathrm{phys}}(t)$$

$$\partial_\eta^2 \phi - \partial_i^2 \phi + \left(m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}\right)\phi + \frac{\lambda}{6}\phi^3 = 0$$

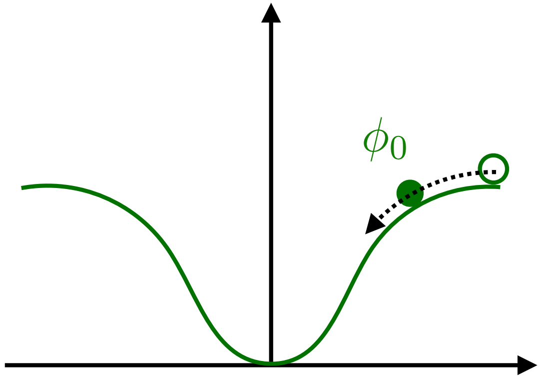
such that for massless fields in radiation dominated universe a''(η)≈0 problem is mapped to scalar field dynamics in static space-time

Example: Scalar field reheating

Initial conditions at the end of inflation (η=0):

Split field into homogenous background field and spatial fluctuations

$$\phi(\eta, \mathbf{x}) = \bar{\phi}(\eta) + \delta\phi(\eta, \mathbf{x})$$



Energy is initially contained in homogenous background field

$$\langle \phi(\eta = 0) \rangle = \bar{\phi}_0 \qquad \langle \partial_\eta \phi(\eta = 0) \rangle \approx 0$$

Vacuum fluctuations around homogenous background field described by Wigner function

$$\langle \delta\phi(\eta = 0, \mathbf{p}) \delta\phi(\eta = 0, \mathbf{q}) \rangle = \frac{1}{2\omega_{\mathbf{p}}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{q})$$

λ<<1: Gaussian fluctuations

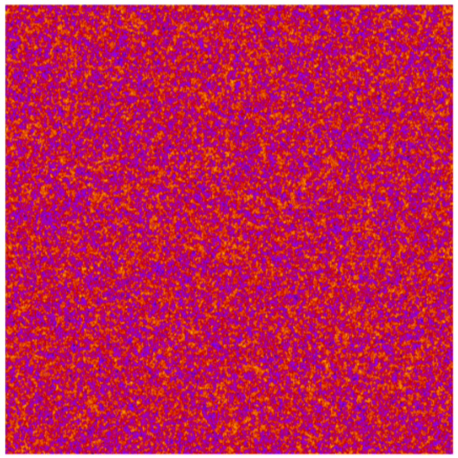
$$\langle \partial_\eta \delta\phi(\eta = 0, \mathbf{p}) \partial_\eta \delta\phi(\eta = 0, \mathbf{q}) \rangle = \frac{\omega_{\mathbf{p}}}{2} (2\pi)^3 \delta(\mathbf{p} - \mathbf{q})$$

with quasi-particle energy $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + \lambda \bar{\phi}_0^2 / 2}$

Example: Scalar field reheating

Clearly initial state is far-from-equilibrium

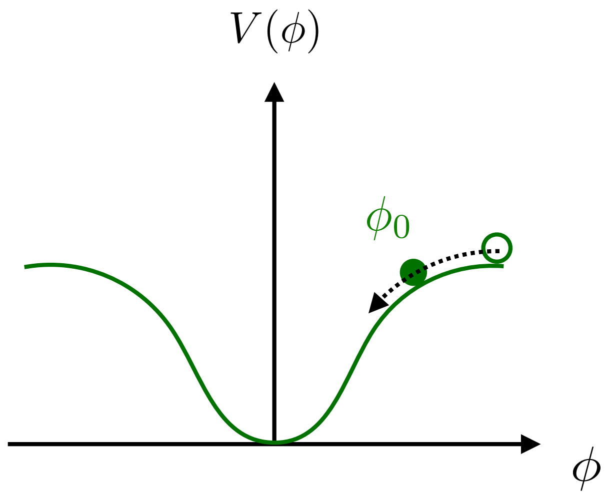
initial state	thermal equilibrium	
$\epsilon \sim \lambda \bar{\phi}_0^4$	$\epsilon \sim T^4$	$T \sim \lambda^{1/4} \bar{\phi}_0$
$l_{\mathrm{coh}} \sim L$	$l_{\mathrm{coh}} \sim 1/T$	$l_{\mathrm{coh}} \sim 1/\lambda^{1/4} \phi_0 \ll L$



Need to accomplish energy transfer from long-wavelength to short wave-length excitations

Example: Scalar field reheating

Split EOM into EOM for homogenous background field and spatial fluctuations



$$\phi(\eta, \mathbf{x}) = \bar{\phi}(\eta) + \delta\phi(\eta, \mathbf{x})$$

EOM for background field

$$\partial_\eta^2 \bar{\phi}(\eta) + \frac{\lambda}{6}\bar{\phi}^3(\eta) = 0$$ (+ non-linear effects of fluctuations)

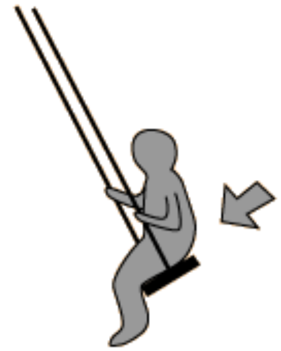
shows anharmonic oscillations

$$\bar{\phi}(\eta) = \bar{\phi}_0 \mathrm{cn}\Big(\sqrt{\frac{\lambda \bar{\phi}_0^2}{6}}\eta, 1/2\Big)$$

Boyanovski, de Vega, Holman, Salgado PRD 54 (1996) 7570-7598

Example: Scalar field reheating

Split EOM into EOM for homogenous background field and spatial fluctuations



$$\phi(\eta, \mathbf{x}) = \bar{\phi}(\eta) + \delta\phi(\eta, \mathbf{x})$$

EOM for spatial fluctuations

$$\left[\partial_\eta^2 + \mathbf{p}^2 + 3\frac{\lambda\bar{\phi}_0^2}{6}\mathrm{cn}^2\left(\sqrt{\frac{\lambda\bar{\phi}_0^2}{6}}\eta, 1/2\right)\right]\delta\phi(\eta, \mathbf{p}) = 0$$ (+ non-linear effects of fluctuations)

Instability (parametric resonance), leads to exponential growth of fluctuations in small resonance band

Non-linear effects of fluctuations quickly become important

Classical-statistical simulations can address dynamics beyond linear instability regime

Boyanovski, de Vega, Holman, Salgado PRD 54 (1996) 7570-7598

Example: Scalar field reheating

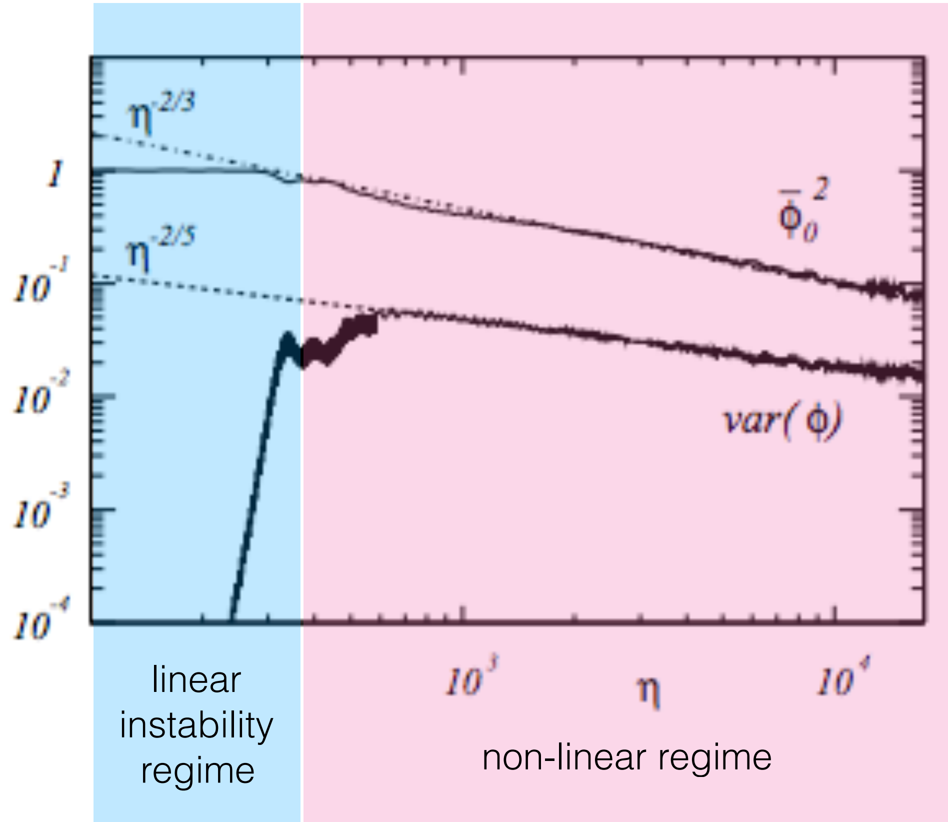
Numerics analogous to equilibrium study of scalar field dynamics

Different initial conditions to represent non-equilibrium initial conditions

$$\phi(\eta=0,\mathbf{x}) = \phi_0 + \sum_{\mathbf{p}} \frac{1}{\sqrt{2\omega_p}}\left[\xi_{\mathbf{p}} e^{-i\mathbf{p}\mathbf{x}} + \xi^*_{\mathbf{p}} e^{+i\mathbf{p}\mathbf{x}}\right]$$

$$\pi(\eta=0,\mathbf{x}) = \sum_{\mathbf{p}} i\sqrt{\frac{\omega_p}{2}}\left[\xi_{\mathbf{p}} e^{-i\mathbf{p}\mathbf{x}} - \xi^*_{\mathbf{p}} e^{+i\mathbf{p}\mathbf{x}}\right]$$

Now statistical averaging over different microscopic realizations of vacuum fluctuations



fast early time dynamics   slow evolution towards equilibrium

Micha, Tkachev Phys.Rev. D70 (2004) 043538

Example: Scalar field reheating

Quantify non-equilibrium evolution further based on correlation functions of fields

Equilibrium: $F(\omega,p) = -i\,(n_{BE}(\omega)+1/2)\,\rho(\omega,p)$

Non-equilibrium: No unique concept of occupation number for interacting fields

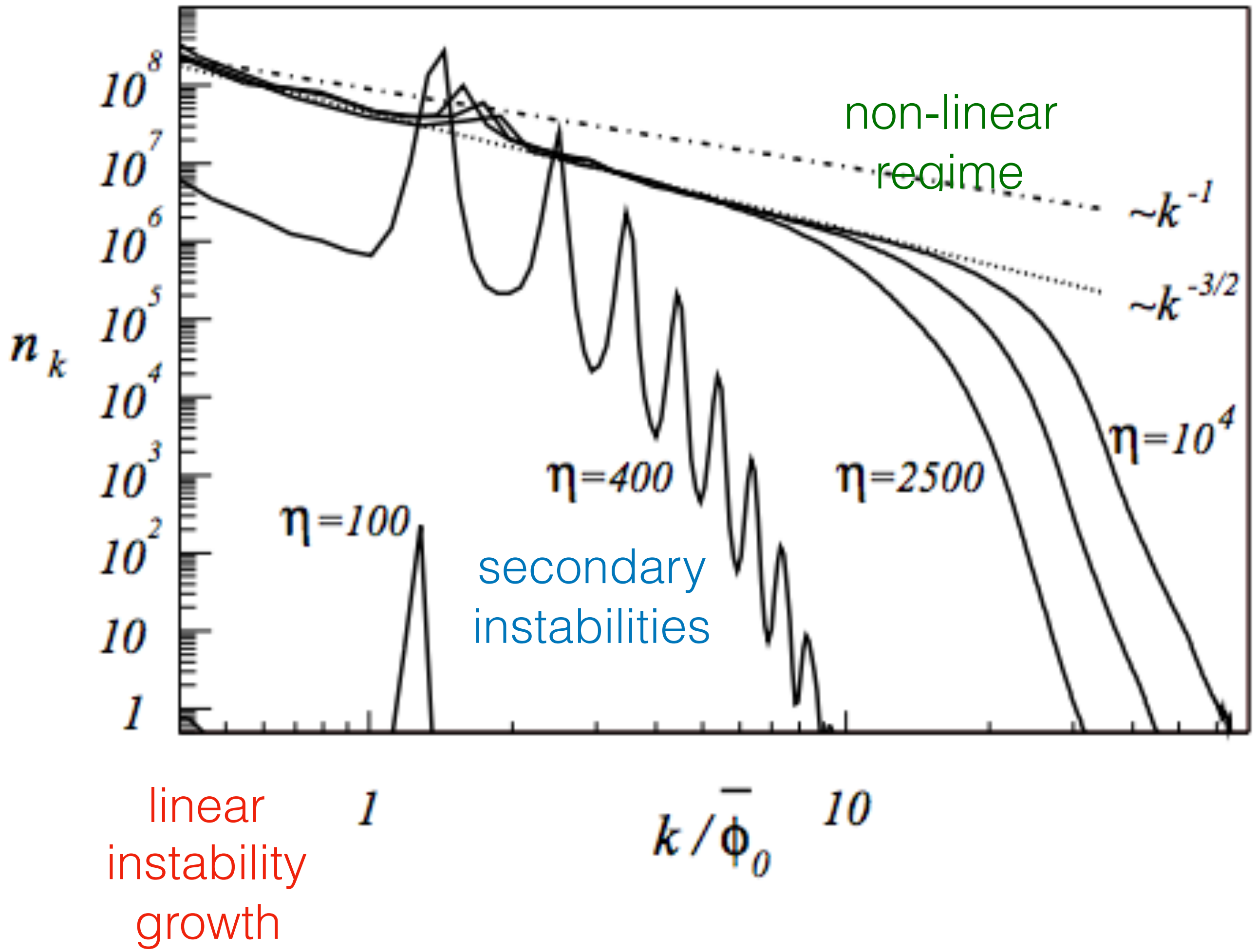
Definition of effective occupation number, by assuming that after the time of interest no further interactions take place

$$\phi(\eta,\mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2\omega_p}}\left[a_{\mathbf{p}} e^{+i(\omega_{\mathbf{p}}\eta - \mathbf{p}\mathbf{x})} + a^*_{\mathbf{p}} e^{-i(\omega_{\mathbf{p}}\eta - \mathbf{p}\mathbf{x})}\right] \qquad \pi(\eta,\mathbf{x}) = \sum_{\mathbf{p}} i\sqrt{\frac{\omega_p}{2}}\left[a_{\mathbf{p}} e^{+i(\omega_{\mathbf{p}}\eta - \mathbf{p}\mathbf{x})} - a^*_{\mathbf{p}} e^{-i(\omega_{\mathbf{p}}\eta - \mathbf{p}\mathbf{x})}\right]$$

$$n(\eta,\mathbf{p}) = \langle a^*_{\mathbf{p}} a_{\mathbf{p}} \rangle = \frac{\omega_{\mathbf{p}}}{2}\left\langle \left| \tilde{\phi}(\eta,\mathbf{p}) - i\frac{\tilde{\pi}(\eta,\mathbf{p})}{\omega_{\mathbf{p}}} \right|^2 \right\rangle$$

Example: Scalar field reheating

Characteristic behavior of unstable systems



Micha, Tkachev Phys.Rev. D70 (2004) 043538

Example: Scalar field reheating

Non-linear dynamics beyond early times dynamics becomes surprisingly simple

Energy transfer towards UV achieved via a self-similar cascade

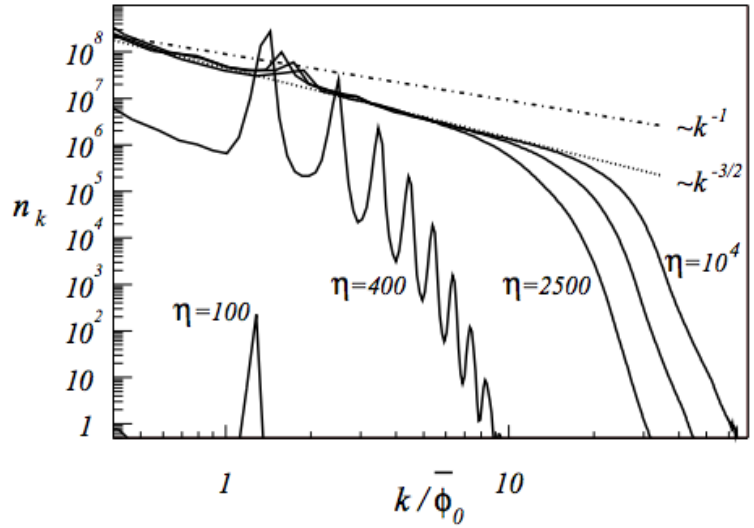
$$n(t,\mathbf{p}) = t^{\gamma} n_S(t^{-\alpha}\mathbf{p})$$

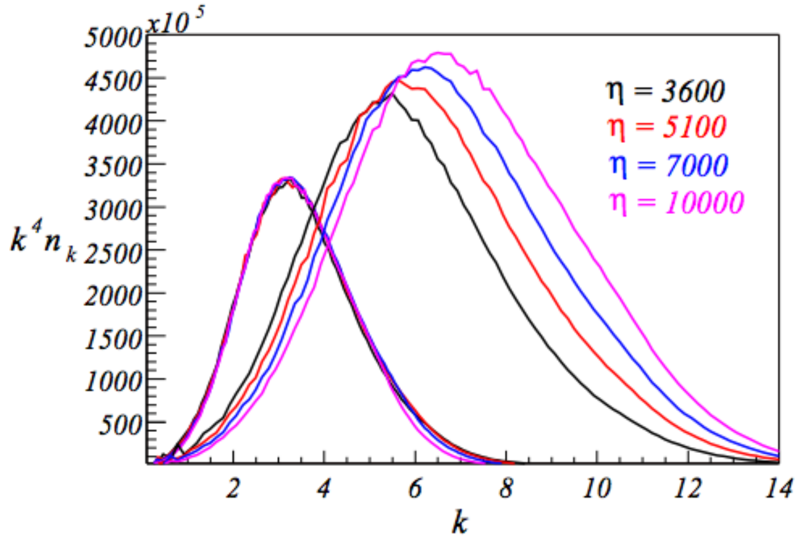
stationary fixed point distribution

$$n_S(x) \simeq \begin{cases} x^{-3/2} & \text{for} \quad x \lesssim 1 \\ \ll 1 & \text{for} \quad x \gtrsim 1 \end{cases}$$

scaling exponents

$$\gamma \approx -4/5 \qquad \alpha \approx 1/5$$



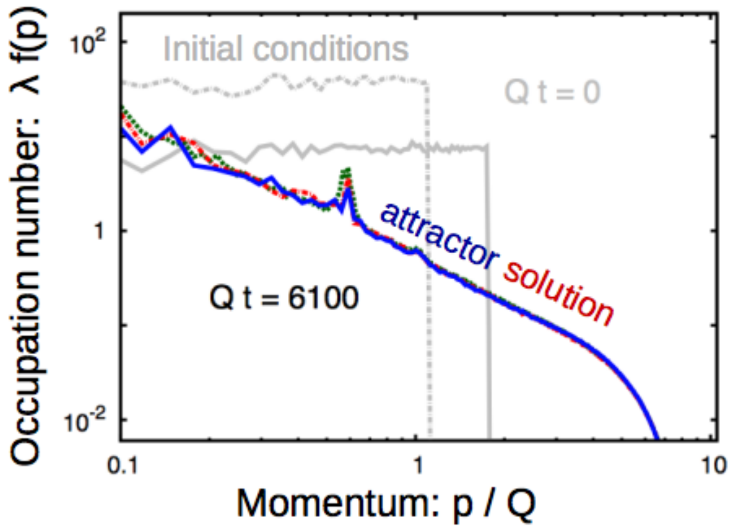


Micha, Tkachev Phys.Rev. D70 (2004) 043538

Example: Scalar field reheating

Effective memory loss occurs at early times; different initial conditions lead to the same attractor solution

-> Non-thermal fixed point (NTFP) associated with UV cascade

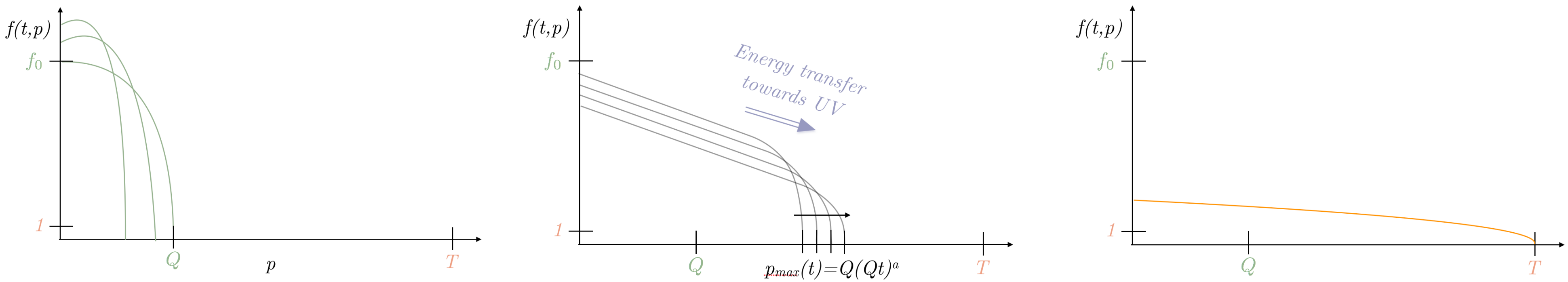


Basic problem is the same classical problem of turbulence

Initial state far from-equilibrium Q<< T

Non-thermal fixed point

Thermal equilibrium

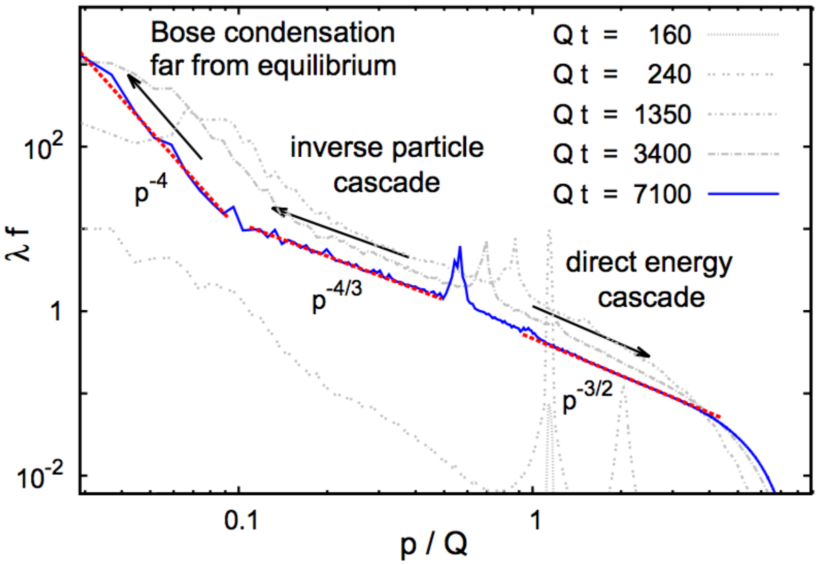


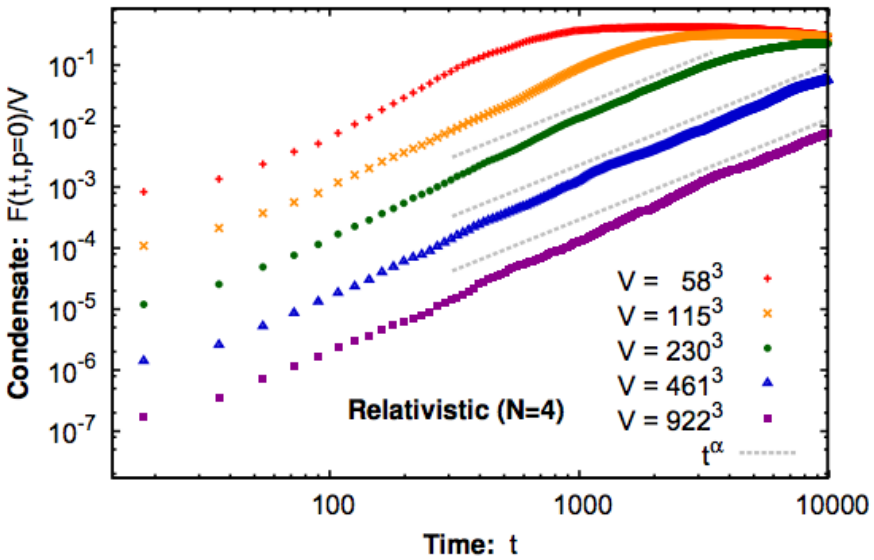
Example: Scalar field reheating

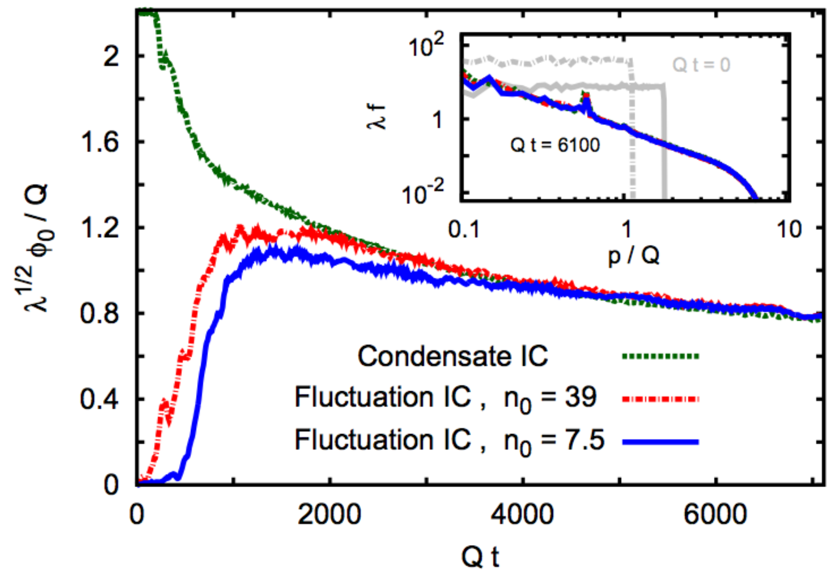
Non-equilibrium Bose condensation of relativistic scalar field

-> Non-thermal fixed point (NTFP) associated with IR cascade

Divergence of condensation time in the infinite volume limit







Berges, Boguslavski, SS, Venugopalan JHEP 1405 (2014) 054

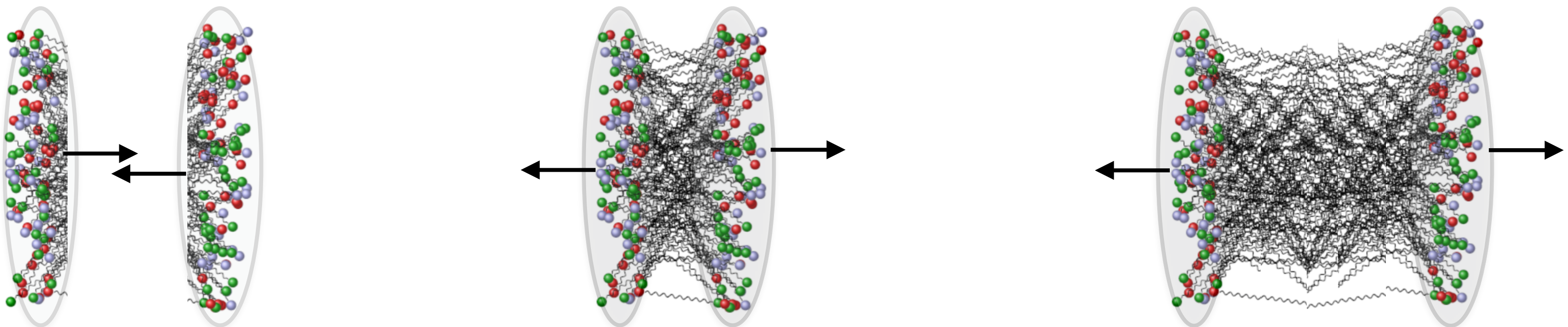
Berges, Sexty PRL 108 (2012) 161601

Orioli, Boguslavski, Berges PRD92 (2015) no.2, 025041

Example: Non-abelian gauge theories

Heavy-Ion collisions (g<<1)

Description within Color-Glass Condensate EFT



Strong color fields mediate the interaction of quarks & gluons inside nuclei

Due to large phase-space occupancy of gluons can describe dynamics early time dynamics semi-classically

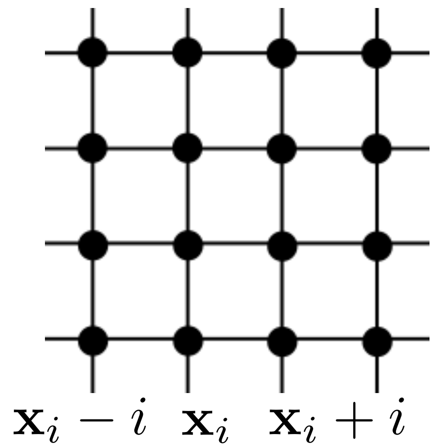
Even though longitudinal expansion extremely important in Heavy-Ion collisions we will first consider simpler problem of non-expanding $SU(N_c)$ Yang-Mills plasma

Discretization of non-abelian gauge theories

So far lattice discretization of scalar fields trivial as it only involves global symmetries (Z_2,O(N),..)

Challenge with regards to discretization of gauge fields is to preserve as much as possible of the local gauge symmetry

$$A_\mu^G(x) = G(x)A_\mu(x)G^\dagger(x) + \frac{i}{g}G(x)\partial_\mu G^\dagger(x)$$



Solution in lattice QCD is to discretize Wilson lines instead of gauge fields

$$U_\mu(x_i) = \mathcal{P}\exp\left(ig\int_{x_i}^{x_i+i} dx^\mu A_\mu(x)\right) \qquad U_\mu^G(x_i) = G(x_i)U_\mu(x_i)G^\dagger(x_i+i)$$

Example: Non-abelian gauge theories

Discretization of non-abelian gauge theories

Sufficiently close to the continuum limit can recover the gauge fields

$$U_\mu(x_i) = \mathcal{P}\exp\left(ig\int_{x_i}^{x_i+i} dx^\mu A_\mu(x)\right) \approx \exp\left(igA_i(x_i+i/2)a_i\right) \approx 1 + igA_i(x_i+i/2)a_i$$

Discretization should preserve invariance under discrete set of gauge transformations $G(t,x_i)$ at each lattice point

Euclidean lattice QCD

action formalism

$a_t \sim a_s$

Classical-statistical real-time QCD

Hamiltonian formalism in $A_0=0$ gauge

$a_t << a_s$

Need to build Hamiltonian which is invariant under residual gauge symmetry (time independent gauge transformations $G(x_i)$ at each lattice point)

Example: Non-abelian gauge theories

Discretization of non-abelian gauge theories

Electric fields described in terms of dimensionless variables

$$E^i_{\mathbf{x}} \equiv \frac{g_c a^3}{a_i} E^{i,cont}_{\mathbf{x}} = \frac{g_c a^3}{a_i} \partial_{x^0} A_{\mathbf{x},i} \qquad E^i_{\mathbf{x}} \to G_{\mathbf{x}} E^i_{\mathbf{x}} G^\dagger_{\mathbf{x}}$$

Magnetic fields described in terms of plaquettes

$$U^{\square}_{i,j}(\mathbf{x}) = U_{i,\mathbf{x}} U_{j,\mathbf{x}+i} U^\dagger_{i,\mathbf{x}+j} U^\dagger_{j,\mathbf{x}} \qquad U^{\square}_{ij}(\mathbf{x}) \to G_{\mathbf{x}} U^{\square}_{ij}(\mathbf{x}) G^\dagger(\mathbf{x})$$

Gauge invariant lattice Hamiltonian

$$H_{YM} = \sum_{\mathbf{x},i} \frac{a_i^2}{g^2 a^3} \mathrm{Tr}\left[E^i_{\mathbf{x}}\right] + \sum_{\mathbf{x},i,j} \frac{a^3}{g_c^2 a_i^2 a_j^2} \mathrm{ReTr}\left[\mathbf{1} - U^{\square}_{ij}(\mathbf{x})\right] \ .$$

Defining variation of lattice gauge links w.r.t gauge fields

$$\frac{\delta U_{\mathbf{x},i}}{\delta A^a_{\mathbf{y},j}} = (+i g a_i t^a)\ U_{\mathbf{x},i}\ \delta_{ij}\ \frac{\delta_{\mathbf{x},\mathbf{y}}}{a^3}$$ one obtains Hamiltons EOM in the usual way

Example: Non-abelian gauge theories

Initial conditions

Details of initial conditions irrelevant so can choose simple ones

typically incoherent superposition of free field modes

$$A_i^a(t=0,x) = \sum_{p,\lambda} \frac{1}{\sqrt{2\omega_p}} \sqrt{n(t=0,p)} \left[\eta_{p\lambda a}\ \xi_i^{(\lambda)}(p) e^{-ipx} + c.c.\right]$$

$$E_i^a(t=0,x) = \sum_{p,\lambda} i\sqrt{\frac{\omega_p}{2}} \sqrt{n(t=0,p)} \left[\eta_{p\lambda a}\ \xi_i^{(\lambda)}(p) e^{-ipx} - c.c.\right]$$

$n(t=0,p)$ initial phase space density

$\xi_i^{(\lambda)}(p)$ polarization vectors (Coulomb gauge)

$\eta_{p\lambda a}$ complex Gaussian random numbers

Special attention required to satisfy Gauss law constraint

$$G(x) = \sum_i D_i E_x^i = \sum_i \frac{E_x^i - U_{x-i} E_x^i U_{x-i}^\dagger}{a_i} = 0$$

EOMs guarantee $\partial_t G(x) = 0$

still need to make sure that initial conditions satisfy constraint

typically achieved by minimizing $\sum_x \mathrm{tr}[G(x)G(x)]$ via gradient descent

Example: Non-abelian gauge theories

Observables

Gauge invariant observables straightforward to implement

Energy momentum tensor, Wilson loops, …

Definition of occupation number based on equal time correlation functions of gauge fields

gauge invariant correlation functions	$E^i_x U_{x\to y} E^j_y$
gauge fixed correlation functions	$\left(E^i_x E^j_y\right)_{\text{Coulomb gauge}}$

Exploit residual gauge freedom to fix physical gauge condition at each time when occupation number is calculated

Example: Non-abelian gauge theories

Coulomb gauge fixing & occupation numbers

Exploit the fact that Coulomb gauge is minimal gauge

$$F[G] = \sum_i \int d^3x \ \mathrm{tr}[A_i^2(x)] \qquad\qquad F[G] = \sum_i \sum_x \frac{2a^3}{g^2 a_i^2} \mathrm{tr}[1 - U_i^G(x)]$$

and optimize by gradient descent $\qquad U_i^G(x_i) = G(x_i) U_i(x_i) G^\dagger(x_i + i)$

$$\frac{\partial F[G]}{\partial G} = -\sum_i \frac{2a^3}{g^2 a_i^2} \mathrm{tr}\Big[it^a \Big(G(x) U_i(x) G^\dagger(x+i) - G(x-i) U_i(x-i) G^\dagger(x)\Big)\Big]$$

based on expansion of the gauge links $\qquad G(x) U_i(x) G^\dagger(x+i) \approx 1 + i g a_i A_{i,b}^G(x) t^b$

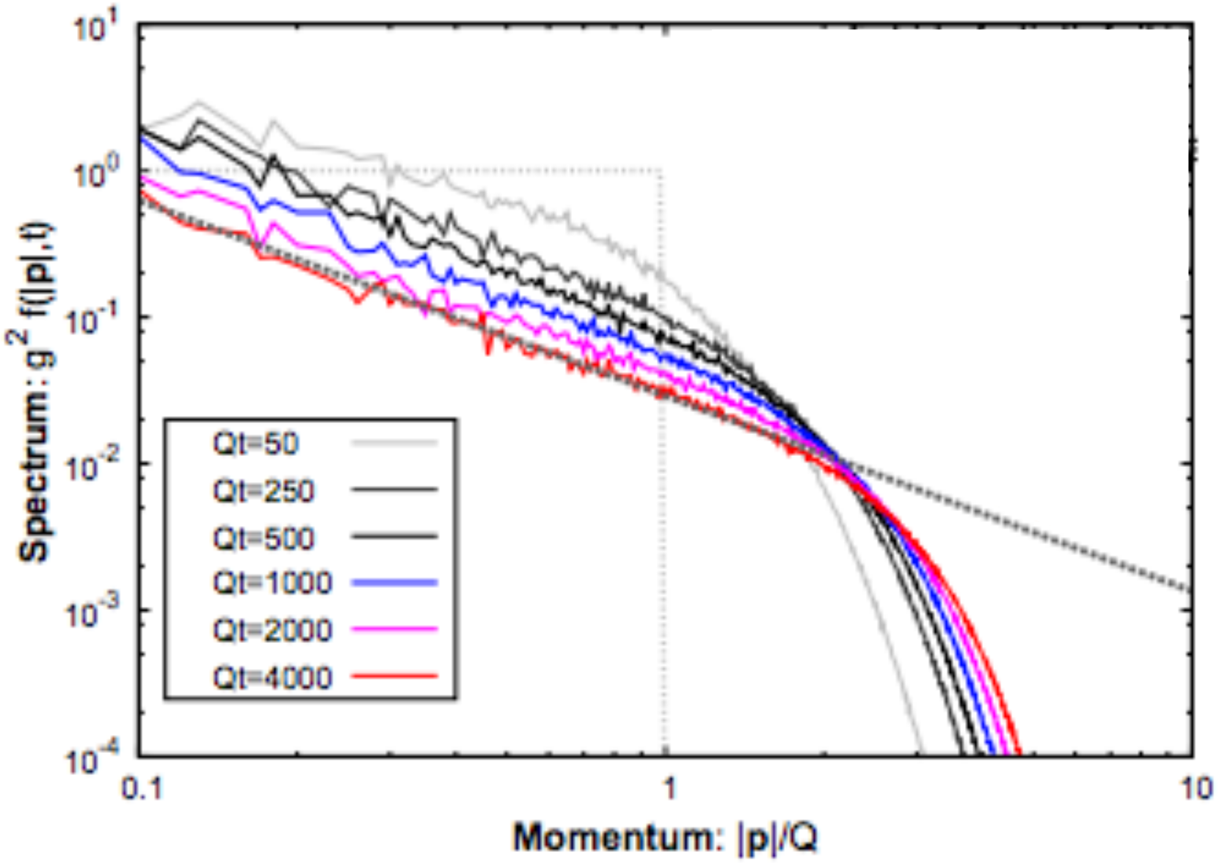
$$\frac{\partial F[G]}{\partial G} = 0 \qquad \text{<=>} \qquad \frac{A_i^G(x) - A_i^G(x-i)}{a_i} = 0$$

(Coulomb gauge)

Note existence of different local minima due to Gribov ambiguity

Can strongly affect IR modes, should only trust UV modes for gauge fixed observables

Example: Non-abelian gauge theories



Early time dynamics strongly dependent on initial conditions

Beyond early times approach to NTFP

Berges, Boguslavski, SS, Venugopalan PRD89 (2014) no.11, 114007

Example: Non-abelian gauge theories

Energy transfer towards UV achieved via a self-similar cascade

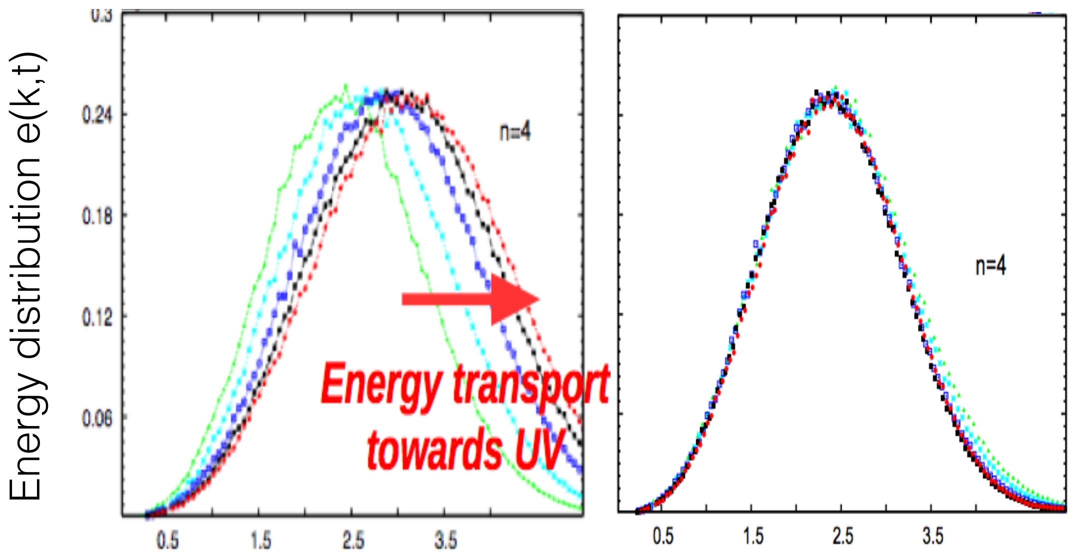
$$n(t,\mathbf{k}) = t^{\alpha}\, n_S(t^{\beta}\mathbf{k})$$

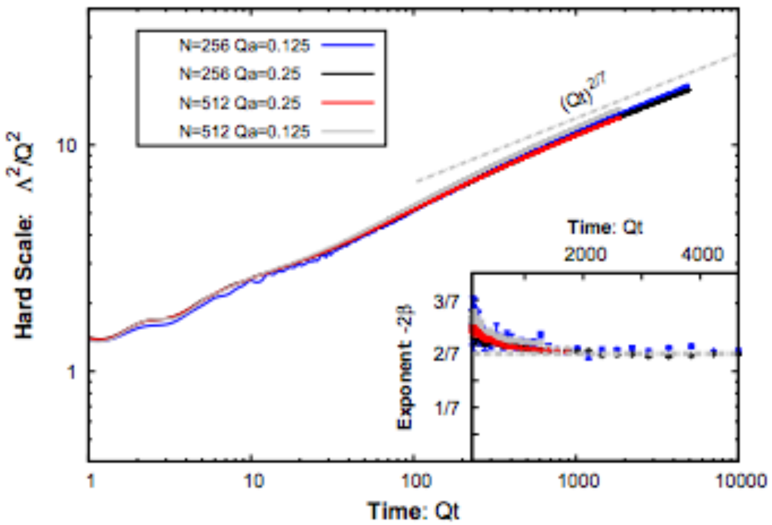
scaling exponents

$$\gamma \approx -4/7 \qquad \alpha \approx 1/7$$

Scaling properties can also be established on the basis of gauge invariant local operator definitions

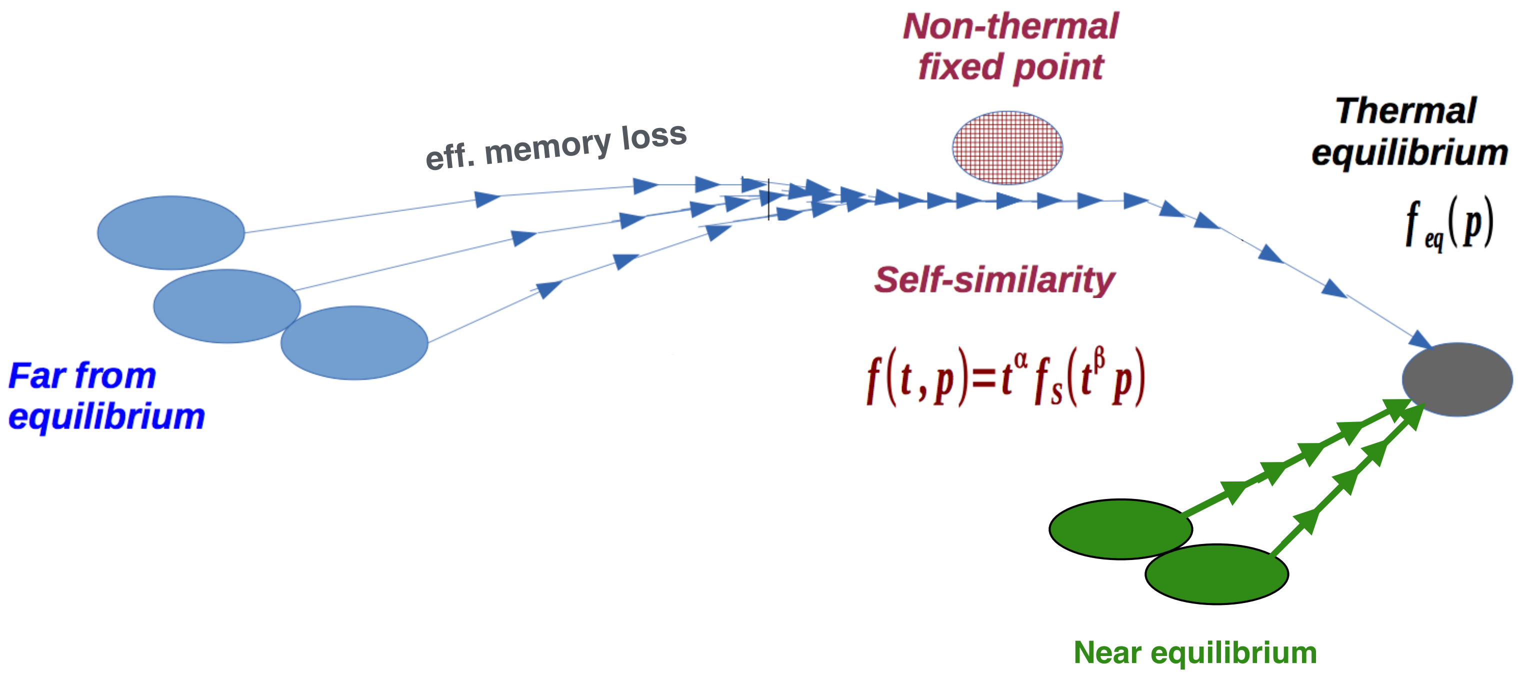
$$\Lambda^2(t) = \frac{\langle \mathrm{tr}[D_i F^{ij}(t,x) D_i F^{ij}(t,x)]\rangle}{\langle \mathrm{tr}[F_{ij}(x,t)F^{ij}(t,x)]\rangle} \sim t^{2\alpha}$$





Example: Non-abelian gauge theories

<u>Qualitvative picture of the thermalization process</u>



Questions: What is the origin of this phenomenon & what is its role in the thermalization process?