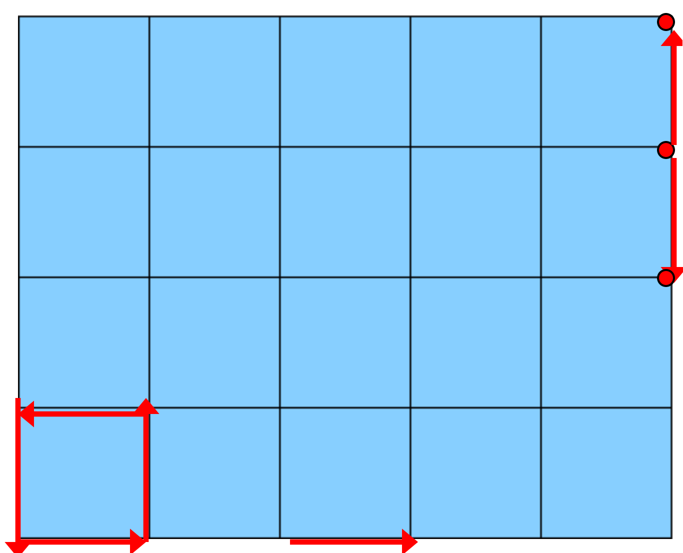
Basics of lattice QCD

- Lattice regularization and gauge symmetry : Wilson gauge action, fermion doubling
- Different fermion formulations
- Meson correlation function and Wilson loops
- Scale setting, continuum limit and lines of constant physics (LCP)
- Lattice calculations and Monte-Carlo methods
- Improved actions in lattice QCD

Quarks and gluon fields on a lattice

Wilson, PRD10 (1974) 2445



$\bar{\psi}(x+\nu)$

$\psi(x)$

$\bar{\psi}(x-\nu)$

$$\Rightarrow \quad \bar{\psi}\ \gamma_\nu D_\nu\ \psi$$

$g = 0$:

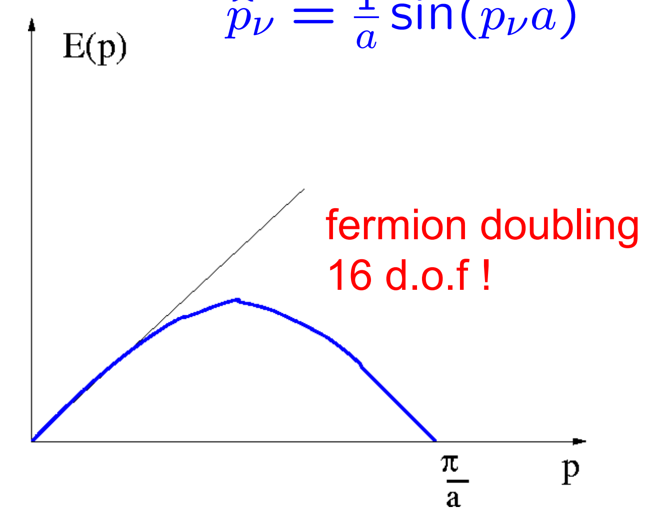
$$\hat{\partial}_\nu \psi(x) = \frac{1}{2}[\psi(x+\nu) - \psi(x-\nu)]$$

$$\langle \psi(x)\bar{\psi}(x)\rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \frac{-i\sum_\nu \gamma_\nu \hat{p}_\nu + m}{\sum_\nu \hat{p}_\nu^2 + m^2}$$

$$U_\mu(x) = \exp(igaA_\mu(x))$$

$$U_\mu(x) \simeq 1 + igaA_\mu(x)$$

$$\hat{p}_\nu = \frac{1}{a}\sin(p_\nu a)$$



$$U_P(x) = U_{\mu\nu}(x) = S^{1\times 1}_{\mu\nu}(x) =$$
$$U_\mu(x)U_\nu(x+\mu)U^\dagger_\mu(x+\mu+\nu)U^\dagger_\nu(x)$$

Gauge transformation: $U_\mu(x) \to G^\dagger(x)U_\mu(x)G(x+\mu)$

$$S_{Wilson} = \beta\sum_x(1 - \frac{1}{3}Re\ trU_P(x)),\ \beta = \frac{6}{g^2}$$

$$S_{Wilson}|_{a\to 0} = \int d^4x\ trF^2_{\mu\nu}$$

Wilson fermions

$$S_f^W = \int_x \left[\bar{\psi}\ \gamma_\nu D_\nu\ \psi - a\frac{r}{2}\bar{\psi}\ \Box\ \psi - m\bar{\psi}\psi \right]$$ Wilson (1975)

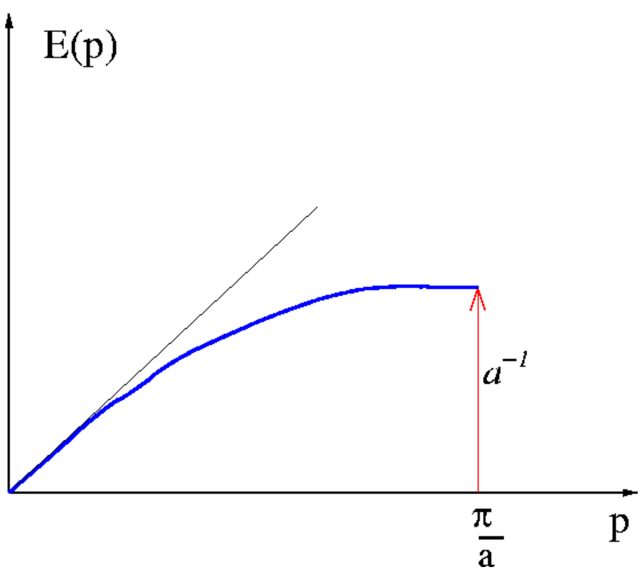
$$S_f^W = \int_x \bar{\psi}\ D^W\ \psi, \quad \int_x = \sum_x a^4$$

$$D^w(x,y) = \delta_{x,y}(4+m) + \sum_\mu (1+\gamma_\mu)\delta_{x+\mu,y} - (1-\gamma_\mu)\delta_{x-\mu,y})$$

$U_\mu(x)$ $\qquad$ $U_\mu^\dagger(x)$

$$\langle \psi(x)\bar{\psi}(x)\rangle = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\frac{-i\sum_\nu \gamma_\nu \hat{p}_\nu + m'(p)}{\sum_\nu \hat{p}_\nu^2 + m'^2(p)}$$

$$m'(p) = m + \frac{2r}{a}\sum_\mu \sin^2(\frac{p_\mu a}{2})$$



$$SU(3)_V \times SU(3)_A \to SU(3)_V$$

chiral symmetry is broken even in the massless case !

⇓

additive mass renormalization

⇓

Wilson Dirac operator is not bounded from below

⇓

difficulties in numerical simulations

Discretization errors ~ $a\ g^2$, used for study of hadron properties, spectral functions

Staggered fermions

$$\psi(x) = T(x)\chi(x), \quad \bar{\psi}(x) = \bar{\chi}T^\dagger(x)$$ Kogut, Susskid, PRD 11 (1975) 395

$$T(x)\gamma_\mu T^\dagger(x+\mu) = 1 \cdot \eta_\mu(x)$$

$$\eta_\mu(x) = (-1)^{x_1+\ldots+x_{\mu-1}}, \eta_1(x) = 1$$
$$T(x) = \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$$

$$S_f^{stagg} = \sum_x \sum_\alpha [\eta_\mu(x)\bar{\chi}_\alpha(x)\hat{\partial}_\mu \chi_\alpha(x) + m\bar{\chi}_\alpha(x)\chi_\alpha(x)]$$

$\Downarrow$ omit index α and the sum ($16 \rightarrow 4$)

$$\sum_{x,y}[\bar{\chi}(x)D_{stagg}(x,y)\chi(y)], \quad D_{stag}(x,y) = \delta_{x,y}m + \sum_\mu \eta_\mu(x)(\delta_{x+\mu,y} - \delta_{x-\mu,y})$$

different flavors, spin componets sit in different corners of the Brillouin zone or in 2^4 hypercube

$\sum_{x,y}[\bar{\chi}(x)D(x,y)\chi(y)] \rightarrow$ 4-flavor theory $\qquad U(4)_V \times U(4)_A \rightarrow U_{e-o}(1) \subset SU_A(4)$

$$\chi(x) \rightarrow U_o\chi(x), \quad \bar{\chi}(x) \rightarrow \bar{\chi}(x)U_e^\dagger, \quad \sum_{i=1}^{4} x_i \text{ even}$$

$$\chi(x) \rightarrow U_e\chi(x), \quad \bar{\chi}(x) \rightarrow \bar{\chi}(x)U_o^\dagger, \quad \sum_{i=1}^{4} x_i \text{ odd}$$

$$\bar{\chi}(x)(\chi(x+\mu) - \chi(x-\mu)) \rightarrow \bar{\chi}(x)U_e^\dagger U_e(\chi(x+\mu) - \chi(x-\mu))$$

$$m\bar{\chi}(x)(\chi(x) \rightarrow m\bar{\chi}(x)U_e^\dagger U_o\chi(x)$$

$||D_{stagg}|| > m$ useful in numerical simulations !

Chiral fermions on the lattice ?

$$S_F = a^4 \sum_{x,y} \bar{\psi}(x) D(x-y) \psi(y)$$

We would like the following properties for the lattice Dirac operator:

1. $D(x)$ should be local, i.e. $||D(x)|| \leq C\,\text{exp}(-\gamma x)$

2. $D(p) = i \sum_\mu \gamma_\mu p_\mu + O((ap)^2)$ (cubic symmmetry)

3. no doubler exist, i.e. $D(p)$ is invertible for $p \neq 0$

4. $\gamma_5 D + D \gamma_5 = 0$ (chiral symmetry)

Nielsen-Ninomiya no-go theorem :
conditions one 1-4 cannot be satisfied simultaneously

Nielsen, Ninomiya, Phys. Lett. 105B (1981) 219

Wilson fermion formulation gives up 4)
Staggered fermion formulation gives up 3)

Ginsparg-Wilson fermions

$$\gamma_5 D + D\gamma_5 = 0$$

$$\Downarrow$$

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = a2R\gamma_5$$

$$\gamma_5 D + D\gamma_5 = aD2R\gamma_5 D$$

R local operator ($R = 1/2$ in most of applications)

Ginsparg, Wilson, PRD 25 (1982) 2649

- anti-commutation properties are recovered in the continuum limit (a->0)
- the r.h.s. of the Ginsparg-Wilson relation is zero for the solutions

$\downarrow$

mildest way to break the chiral symmetry on the lattice : physical consequences of the chiral symmetry are maintained (e.g. chiral perturbation theory)

Generalized chiral symmetry and topology

$$\psi(x) \to \psi(x) + \delta\psi(x), \bar{\psi}(x) \to \bar{\psi}(x) + \delta\bar{\psi}(x)$$

$$\delta\psi_i = iT^a_{ij}\phi^a\gamma_5(1 - a\tfrac{1}{2}D)\psi_j(x) \;\; flavor\ non-singlet$$

$$\delta\psi_i = \gamma_5(1 - a\tfrac{1}{2}D)\psi_i(x) \;\; flavor\ singlet$$

$$\delta\bar{\psi}_i = \bar{\psi}_j\gamma_5(1 - a\tfrac{1}{2}D)i\phi^a T^a_{ij} \;\; flavor\ non-singlet$$

$$\delta\bar{\psi}_i = \bar{\psi}_i\gamma_5(1 - a\tfrac{1}{2}D) \;\; flavor\ singlet$$

GW relation ⇨ $\delta(\bar{\psi}D\psi) = 0$ Luescher, PLB 428 (1998) 34

flavor singlet transformation :

$$\delta[d\bar{\psi}d\psi] = \mathrm{Tr}(a\gamma_5 D)[d\bar{\psi}d\psi] = 2N_f(n_- - n_+)[d\bar{\psi}d\psi]$$

$$index(D) = n_- - n_+$$

$q(x) = \frac{1}{2}\mathrm{tr}\gamma_5 D(x,x)$ -topological charge density

Hasenfratz, Laliena, Niedermeyer PLB 427 (1998) 125

for flavor non-singlet transformation $\mathrm{Tr}(a\gamma_5 D T^a) = 0$ no anomaly !

Constructing chiral fermion action I

Overlap fermions :

$$A = 1 - aD_w$$
$$D = \frac{1}{a}(1 - \frac{A}{\sqrt{AA^\dagger}})$$

Neuberger, PLB 417 (1998) 141

using $\gamma_5 D_w \gamma_5 = D_w^\dagger$ it can be shown that

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

GW relation with *R=1/2*

no $\mathcal{O}(a^2)$ -discretization errors

large numerical cost due to evaluation of $(AA^\dagger)^{-1/2}$ (polynomial approximation), especially for gauge field configurations which change topology

costs $> 100\times$ costs of Wilson formulations

Constructing chiral fermion action II

Domain wall fermions : introduce the fictitious 5th dimension of extent N_s :

$$S_{dwf} = -\sum_{x,y,s,s'} \bar{\psi}^5_{y,s'}(D_{x,y}\delta_{s,s'} + D_{s,s'}\delta_{x,y})\psi^5_{x,s}$$

$$D_{x,y} = \frac{1}{2}\sum_\mu((1+\gamma_\mu)U_{x,\mu}\delta_{x+\mu,y} + (1-\gamma_\mu)U^\dagger_{x,\mu}\delta_{x-\mu,y} + (M-4)\delta_{x,y})$$

$$D_{s,s'} = (P_R\delta_{s+1,s'} + P_L\delta_{s-1,s'} - \delta_{s,s'}) + (P_R\delta_{1,s'} + P_L\delta_{N_s-2,s'}) - (mP_L\delta_{N_s-1,s'} + mP_R\delta_{0,s'} + \delta_{0,s'} + \delta_{N_s-1,s'})$$

$$P_{R,L} = (1 \pm \gamma_5)/2$$

$$\psi_x = \frac{1+\gamma_5}{2}\psi^5_{x,0} + \frac{1-\gamma_5}{2}\psi^5_{x,N_s-1}$$

Kaplan PLB 288 (1992) 342
Shamir NPB 406 (1993) 90

$N_s \to \infty$ two chiral fermions bounded to the 4d walls

$$m_q = mM(M-2)$$

costs $= N_s \times$ costs of Wilson formulations $\quad N_s = 16 - 32$

Extensively used in mumerical simulations

(see Boyle, PoS LATTICE 2007 (2007) 005 for review)

QCD at finite baryon density

The naive continuum prescription of introducing chemical potential by adding a term $\mu \int d^4 x_E \bar{\psi}\gamma_0\psi$ does not work !

$$S = a^3 \sum_x \left[ma\bar{\psi}_x\psi_x + \mu a\bar{\psi}_x\gamma_0\psi_x + \frac{1}{2}\sum_\mu (\bar{\psi}_x\gamma_\mu\psi_{x+\mu} - \bar{\psi}_{x-\mu}\gamma_\mu\psi_x) \right]$$

$$\Downarrow$$

$\epsilon(\mu) \sim \mu^2/a^2$ instead of $\epsilon(\mu) \sim \mu^4$

The correct prescription is

$$U_0(x) \to e^{\mu a} U_0(x), \quad U_0^\dagger(x) \to e^{-\mu a} U_0^\dagger(x)$$

Hasenfratz, Karsch, PLB 125 (83) 308

$$S = (\overline{\psi}_x e^{\mu a} U_0(x)\psi_{x+0} - \overline{\psi}_x e^{-\mu a} U_0^+(x)\psi_{x-0}) + \sum_{x,i} \eta_i(x)\, (\overline{\psi}_x U_i(x)\psi_{x+i} - \overline{\psi}_x U_i^+(x)\psi_{x-i}) + am\sum_x \overline{\psi}_x\psi_x$$

$$\Downarrow$$

det M is complex => sign problem *det M exp(-S)* cannot be a probability

see also talks by Rajiv Gavai

Meson correlators and Wilson loops

Meson states are created by quark bilenear operators:

$$J(x,y;\tau) = \psi(\tau,x)\Gamma\mathcal{U}(x,y)\psi(y,\tau)$$

Fixes the quantum number of of mesons, Γ is one
Of the Dirac matrices

Most often one considers point operators *x*=*y* and their correlation function:

$$G(\tau) = \langle J(\tau)J^\dagger(0)\rangle = \sum_{n=1}^{\infty} f_n^2 e^{-M_n\tau},\ f_n = |<0|J|n>|$$

decay constant

$$\tau \to \infty : G(\tau) \simeq f_1 e^{-M_1\tau}$$

Consider static quarks :

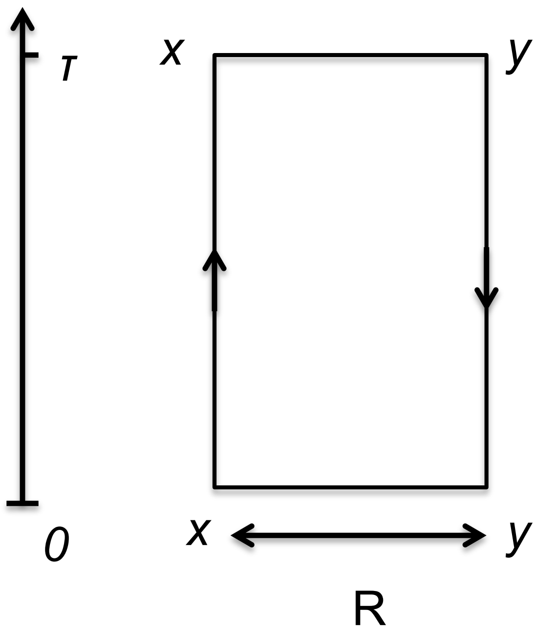
$$(-i\partial_\tau - gA_0(\tau,x))\,\psi(\tau,x) = 0$$

formal solution $\psi(\tau,x) = \mathcal{P}\exp\left(ig\int_0^\tau d\tau' A_0(\tau',x)\right)\psi(0,x) = L(\tau;x)\psi(0,x)$

lattice : $L(\tau;x) = \prod_{x_0=0}^{\tau} U_0(x,x_0)$

Static meson correlation function functions after integrating out the static quark fields:

$$G(\tau) = \langle J(\tau) J^\dagger(0) \rangle = \mathrm{Tr}[L(\tau; x)\mathcal{U}(x,y)L^\dagger(\tau; y)\mathcal{U}^\dagger(x,y)] = W_C(\tau, R)$$



$$W_C(\tau, R) = \sum_{n=1}^{\infty} c_n e^{-E_n(R)\tau},$$

Static quark anti-quark potential

$\tau \to \infty$ ground state $E_1 = V(R)$ dominates

$$V(R) = -\alpha/R + \sigma R$$

confinement

String tension

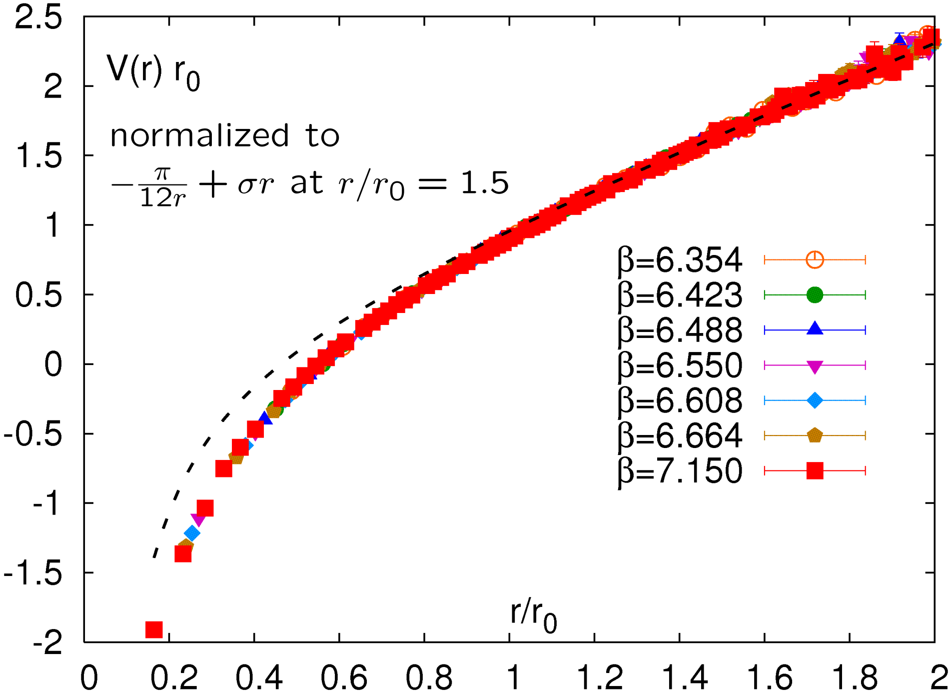
$$W_C(\tau, R) = \exp(-\sigma R \tau),$$

area law for large R and τ

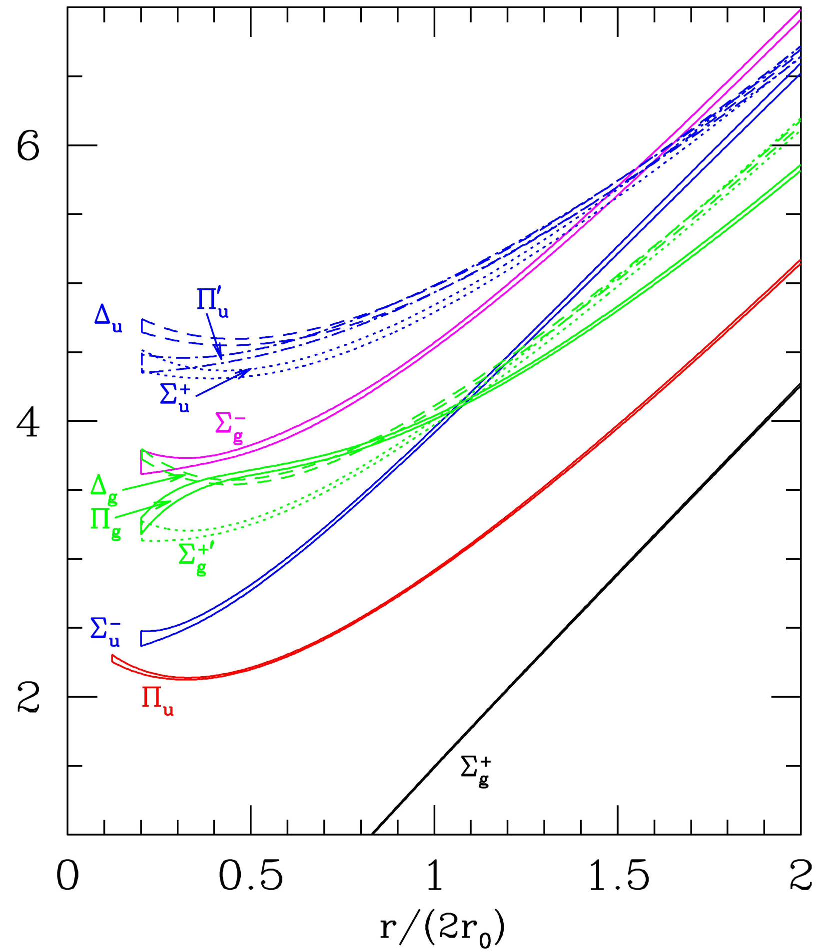
n=2 and larger : hybrid potentials

Numerical results on the potentials

Static quark anti-quark potential



Hybrid potentials



$$\sqrt{\sigma} \simeq 470 \text{ MeV}$$

$$\left(r^2 \frac{dV_{q\bar{q}}(r)}{dr}\right)_{r=r_0} = 1.65, \quad r_0 = 0.468(4) fm$$

Sommer scale

Scale setting in lattice QCD and continuum limit

Hadron masses in lattice QCD are dimensionless: $m = m_{phys}\, a$

Continuum limit: $a \to 0$

$m \to 0 \quad \Rightarrow \xi = 1/m \to \infty$ divergent correlation length $\Rightarrow$ second order phase transition (universality)

Physics does not depend on the details on the regularization, e.g dimensionless ratios :

$$r_0 m, \quad m/\sqrt{\sigma}$$

Should be independent of the lattice spacing

The gauge coupling constant depends on the lattice spacing:

$$a \to 0 \quad \Rightarrow g \to 0 \quad (\beta = 6/g^2 \to \infty) \quad a\Lambda = R(g) = (\beta_0 g)^{-\beta_1/(2\beta_0))} e^{-\frac{1}{2\beta_0 g^2}}$$

There are corrections to this asymptotic expression $\sim g^{2n} a^{2n}, \; n = 1, 2...$

$a\Lambda = c_0 R(g)(1 + c_2 g^2 R^2(g) + c_4 g^4 R^4(g) + ...)$

Allton Ansatz

We need to tune the bare quark mass m_{bare} on lattice to keep the physical ratio $m_{meson}/\sqrt{\sigma}$ fixed for each g:

$\Rightarrow m_{bare} = m_{bare}^{LCP}(g)$ defining the line of constant physics (LCP).

Lattice QCD calculations

$$\langle O \rangle = \int \mathcal{D}A_\mu \mathcal{D}[\psi\bar{\psi}] O e^{-\int_0^\beta d\tau d^3x \mathcal{L}_{QCD}} \quad A_\mu(0,\mathbf{x}) = A_\mu(\beta,\mathbf{x}) \;\; \psi(0,\mathbf{x}) = -\psi(\beta,\mathbf{x})$$

$$O = O^G[U_\mu](\bar{\psi}_{x_1}\psi_{x_2}...)$$

$$\langle O \rangle = \int \prod_{x,\mu} dU_\mu(x) D[\bar{\psi}_x\psi_x] O^G[U_\mu](\bar{\psi}_{x_1}\psi_{x_2}...) e^{-\sum_x S_G[U_\mu(x)] - \sum_{x,y,f} \bar{\psi}^f(x) D_{x,y}[U_\mu]\psi^f(y)}$$

$$\langle O \rangle = \int \prod_{x,\mu} dU_\mu(x) O^G[U_\mu](D^{-1}_{x_1,x_2}[U_\mu]...)(det D[U_\mu, m, \mu])^{n_f} e^{-\sum_x S_G[U_\mu(x)]}$$

Staggered fermions : we get $4n_f$ flavors to get 1-flavor replace n_f by ¼ (rooting trick)

$\mu = 0$ ➡ Monte-Carlo Methods, importance sampling: $\{U^i_\mu(x)\}, i = 1, .., N$

$$\langle O \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} O^G(U^i_\mu(x))(D^{-1}_{x_1,x_2}[U^i_\mu]..)$$

$\mu \neq 0$: $det D_q(U, m, \mu)$ complex ➡ sign problem

Quenched QCD: $det D_q(U, m, \mu) = 1$

continuum limit $a \to 0$

$T = 1/(aN_\tau)$, $N_\tau \to \infty$, N_σ/N_τ fixed

Costs : $\sim a^{-7} \sim N_\tau^7, \;\; m^{-1}$

improved discretization schemes are needed : p4, asqtad, stout, HISQ

Simulating lattice QCD

$$\langle O\rangle = \int \prod_x dU_\mu(x) O(det D[U,m]) e^{-\sum_x S_G[U(x)]}$$

because of the determinant no local updates are possible (e.g. Metropolis)

$det D = 1$ is real ($\mu = 0$) but not positive: $det D \rightarrow det(DD^\dagger)$ (2-flavor theory)

for 1-flavor use rational approximation of $det D \rightarrow det(DD^\dagger)^{1/2}$

pseudo-fermion fields : $$det(DD^\dagger) = \int [d\phi d\phi^+] \exp(-\phi^+ [DD^\dagger]^{-1} \phi\ )$$

$$S_G \rightarrow S_{eff} = S_G + \phi^+ [DD^\dagger]^{-1} \phi$$

introduce (Gaussian) conjugate momenta

$$H[P,\pi,U,\phi] = \frac{1}{2} \sum_{x\mu,a} P^2_{x,\mu a} + \sum_x \pi_x^+ \pi_x + S_{eff}[U,\phi]$$

$$\langle O\rangle \sim \int [dPdUd\pi d\pi^+ d\phi d\phi^+] \exp(-H[P,\pi,U,\phi])$$

Partition function of classical system with Hamiltonian *H* => if the lattice is large enough the partition function cab be calculated using micro-canonical ensemble => molecular dynamic (MD) (assuming ergodicity)

Introduce simulation time *t* and solve classical Hamilton's equation of motion

$$\dot\phi_x(t) = \pi_x(t),$$

$$\dot\pi_x(t) = -\sum_y D^{-1}_{x,y}[U]\phi_y(t),$$

$$\dot U_{\mu,x}(t) = P_{\mu,x}(t)U_{\mu,x}(t),$$

$$\dot P_{\mu,x} = -\frac{\partial S_G}{\partial U_{\mu,x}(t)} - \sum_{x,y}\phi^+_x(t), \frac{\partial D^{-1}_{x,y}[U]}{\partial U_{\mu,x}}\phi_y(t).$$

Problems with MD: exact only for infinite lattice volume and zero step-size *dt=0* limit, egrodicity is not implicitly implemented

Hybrid algorithm:

Use the fact that distribution in of momenta is Gaussian and generate the momenta randomly (from Gaussian distribution): $\tilde\pi_x$ and the use MD with leapfrog

$$\pi_x(t) = \tilde\pi_x - \frac{dt}{2}\sum_y D^{-1}_{x,y}[U]\phi_x(t),$$

$$\phi_x(t+1) = \phi_x(t) + dt\pi_x(t), \quad \pi_x(t+1) = \pi_x(t) - dt\sum_y D^{-1}_{x,y}[U]\phi_x(t+1) \leftarrow$$ repeat *N* times

N dt =MD length

$$\Rightarrow \{P', \pi', U', \phi'\}$$

Ergodic but has step-size errors of order dt^2 extrapolation to zero stepsize is needed (for N=1 is equivalent to Langevin algorithm)

Hybrid Monte-Carlo (HMC) algorithm = Hybrid algorithm + global Metropolis step

$$P = min(1, \quad \exp(-H[P', \pi', U', \phi'] + H[P, \pi, U, \phi]))$$

if step-size is small energy is almost conserved and new configuration is likely accepted

Ergodic and exact !

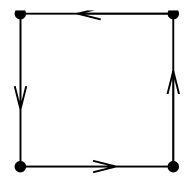
Improved gauge action

$$
\begin{aligned}
S^{(1,1)}_{\mu,\nu}(x) &= 1 - \frac{1}{N}\mathrm{Re}\ \mathrm{Tr} U_{x,\mu} U_{x+\hat{\mu},\nu} U^{+}_{x+\hat{\nu},\mu} U^{+}_{x,\nu} \\
&= -\frac{1}{2N} g^2 a^4 \Big(F_{\mu,\nu} F_{\mu,\nu} + \frac{1}{12} a^2 F_{\mu,\nu} (\partial^2_\mu + \partial^2_\nu) F_{\mu,\nu} \\
&\qquad + O(a^4) \Big) \quad .
\end{aligned}
$$

can be eliminated by adding larger loops

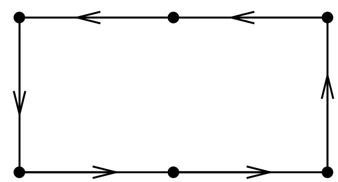
Luescher and Weisz,
Commun. Math. Phys. 97 (1985) 59

$$
S^{\mathrm{imp}}_{\mathrm{gluon}} = \beta \Bigg[c_0(g^2) \sum_{x,\mu<\nu} \Big(1 - \frac{1}{N}\mathrm{ReTr}
$$



$$
{}_{\mu,\nu}(x)\Big)
$$

$$
+ \ c_1(g^2) \sum_{x,\mu<\nu} \Big(1 - \frac{1}{N}\mathrm{ReTr}
$$



$$
{}_{\mu,\nu}(x)\Big)\Bigg]
$$

$$\mathcal{O}(g^0 a^2)$$

$$c_0^{(0)} = 5/3 \quad c_1^{(0)} = -1/6$$

Improved staggered fermion actions

Heller, Karsch, Sturm, PRD60 (1999) 114502

Standard staggered action has discretization errors ~ a^2

Eliminate those using higher order difference scheme

$$S_F = m \sum_x \bar{\chi}(x)\chi(x) + \sum_x \sum_\mu \eta_\mu(x)\bar{\chi}(x)[\;\; c_{1,0}\,(\chi(x+\mu)-\chi(x-\mu)) +$$

$$+c_{3,0}\,(\chi(x+3\mu)-\chi(x-3\mu))$$

$$\sum_{\nu\neq\mu} c_{1,2}\,(\chi(x+\mu+2\nu)-\chi(x-\mu+2\nu)) + \chi(x+\mu-2\mu)-\chi(x-\mu-2\mu) \;\;]$$

Free quark propagator: $\dfrac{-i\sum_\mu \gamma_\mu h_\mu(p)+m}{D^{(0)}(p)+m^2}$

$$\begin{array}{lcl} s_\mu(p) & \equiv & \sin(p_\mu) \\ c_\mu(p) & \equiv & \cos(p_\mu) \end{array}$$

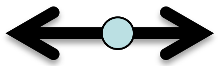
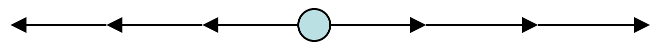
$$h_\mu(p) \quad = \quad 2\; s_\mu(p)\left[c_{1,0}^{(0)} + 2\; c_{1,2}^{(0)} \sum_{\nu\neq\mu} c_\nu(2p)\right] + 2\; c_{3,0}^{(0)} s_\mu(3p)$$

$$D^{(0)}(p) = \sum_\mu h_\mu(p)h_\mu(p) = \sum_\mu A p_\mu^2 A \left(A + 2B_1 p_\mu^2 + 2B_2 \sum_{\nu\neq\mu} p_\nu^2\right) + \mathcal{O}(p^6)$$

$$A \quad = \quad 2c_{1,0}^{(0)} + 12c_{1,2}^{(0)} + 6c_{3,0}^{(0)} \qquad B_1 \quad = \quad -\frac{1}{3}c_{1,0}^{(0)} - 2c_{1,2}^{(0)} - 9c_{3,0}^{(0)} \qquad B_2 \quad = \quad -8c_{1,2}^{(0)}$$

The different staggered which flavors sit in different corners of the Brillouin zone
Are completely equivalent in the free theory => flavor symmetry
Not the case in the interacting theory:
exchange with gluons with momenta ~ *π/a* can change the quark flavor (taste) as it brings it to another corner of the Brillouin zone

Rotational symmetry at order p^4 : $B_1 = B_2$ ⇨ $c^{(0)}_{1,0} + 27\ c^{(0)}_{3,0} + 6\ c^{(0)}_{1,2} = 24\ c^{(0)}_{1,2}$

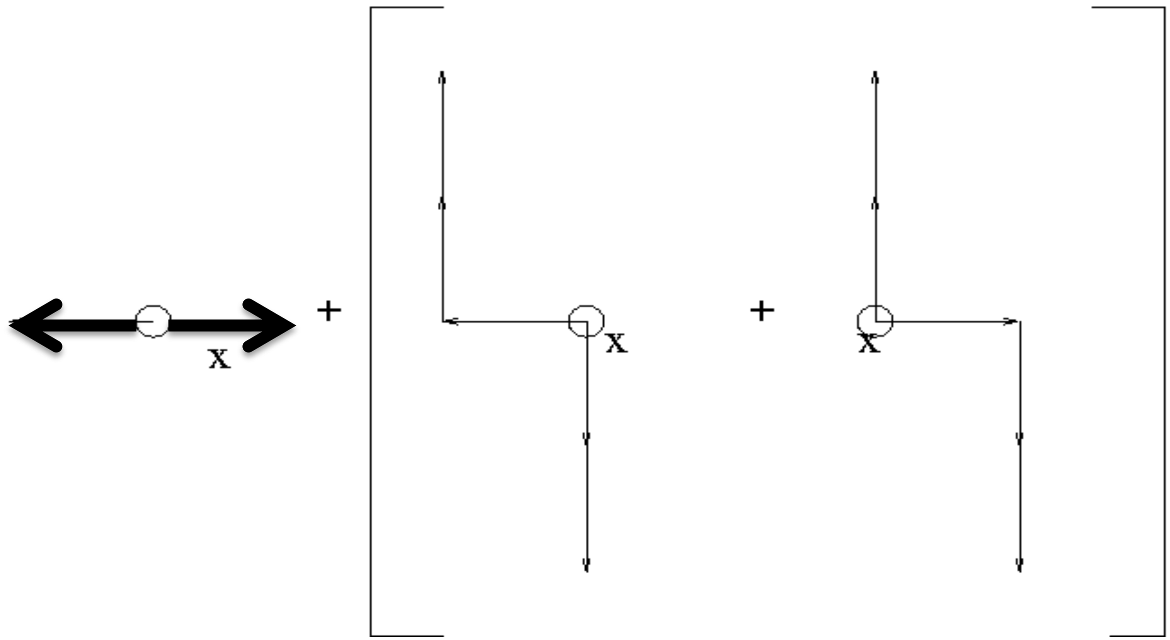
$C_{1.0}$  + $C_{3,0}$ 

Naik action:

$$\boxed{c^{(0)}_{1,0} = \frac{9}{16}\ ,\quad c^{(0)}_{3,0} = -\frac{1}{48}}$$

Normalization: $c^{(0)}_{1,0} + 3\ c^{(0)}_{3,0} + 6\ c^{(0)}_{1,2} \quad = \quad 1/2$

$C_{1,0}$



$c_{1,2}$

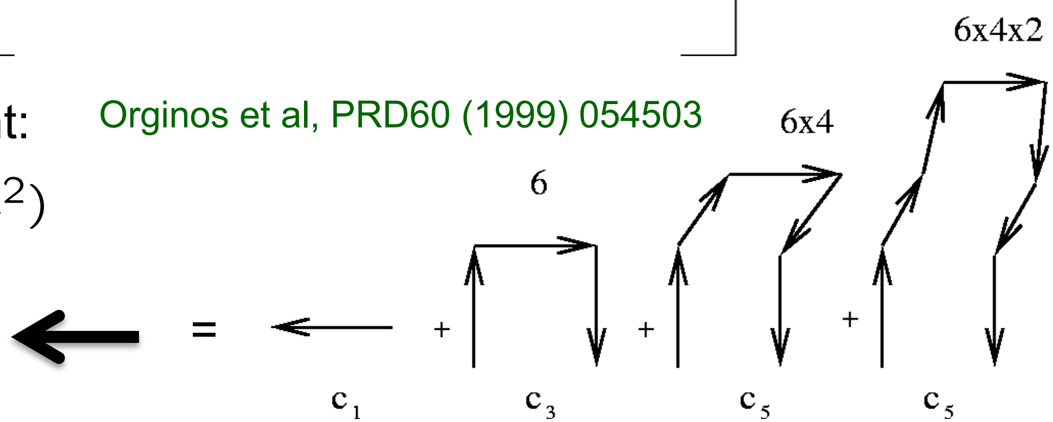
p4 action:

$$\boxed{c^{(0)}_{1,0} = \frac{3}{8}\ ,\quad c^{(0)}_{1,2} = \frac{1}{48}}$$

Taste symmetry improvement:

Orginos et al, PRD60 (1999) 054503

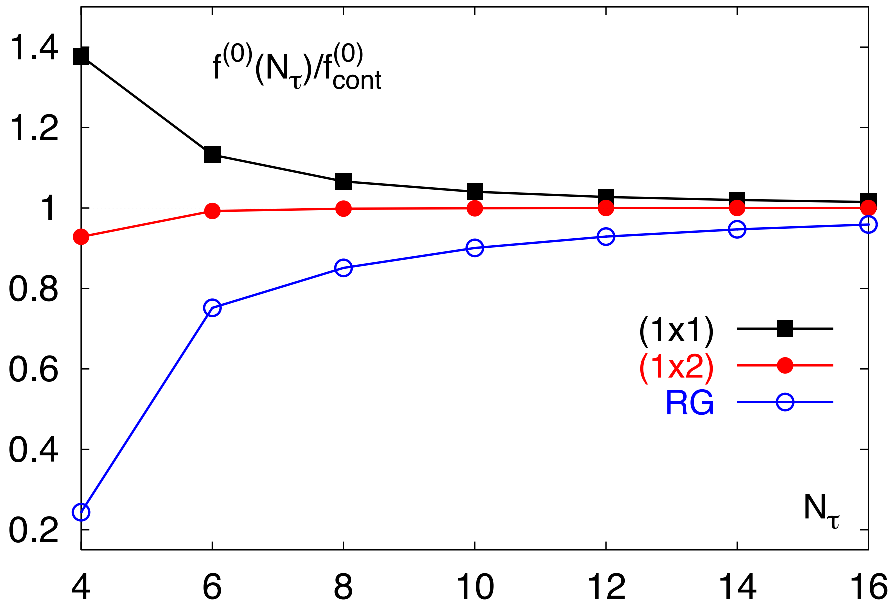
no taste breaking at $\mathcal{O}(g^2 a^2)$

Fat (smeared) link: 

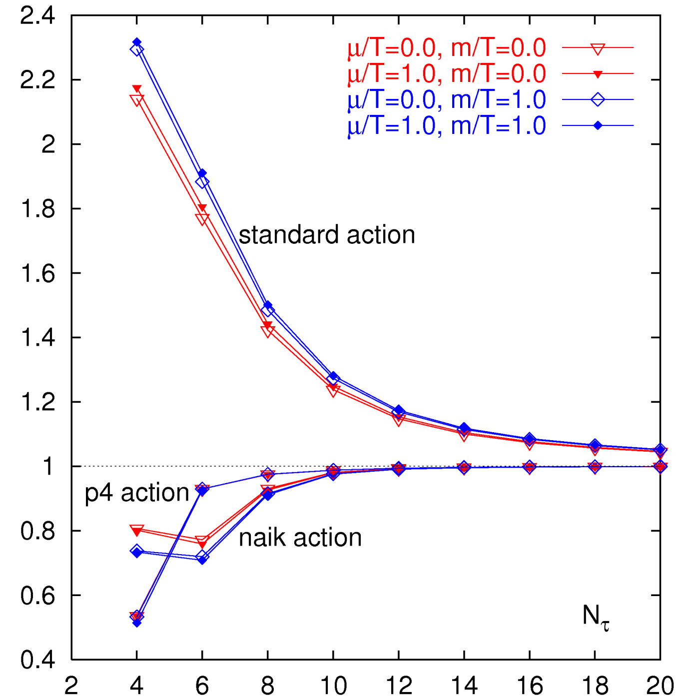
Projection to U(3) => HISQ action

Why improved actions ?

Pressure of the ideal gluon gas



Pressure of the ideal quark gas



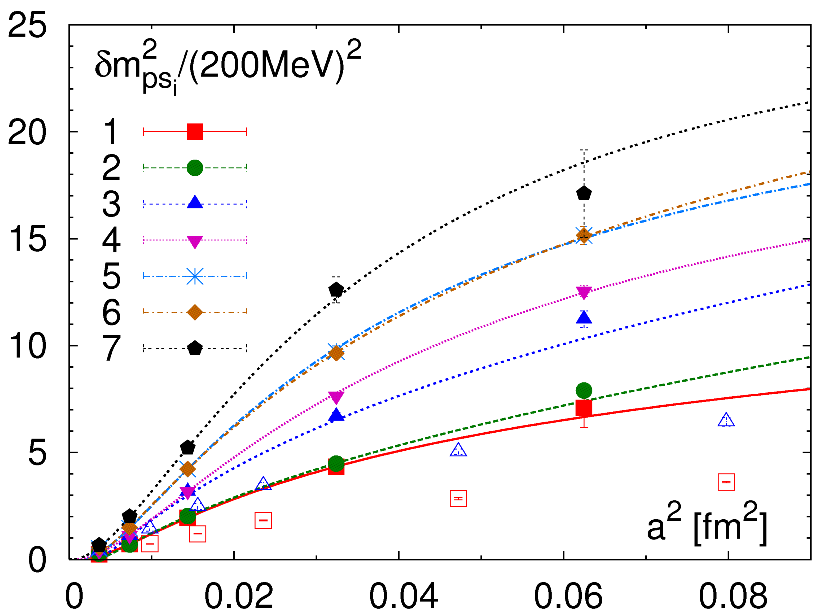
$$\frac{p(N_\tau)}{p_{cont}} = 1 - \frac{1143}{980}\left(\frac{\pi}{N_\tau}\right)^4 + \frac{73}{2079}(1+6528c_{30})\left(\frac{\pi}{N_\tau}\right)^6 + \mathcal{O}(N_\tau^{-8})$$

$c_{30} = 0$ for p4, $c_{30} = -1/48$ for Naik

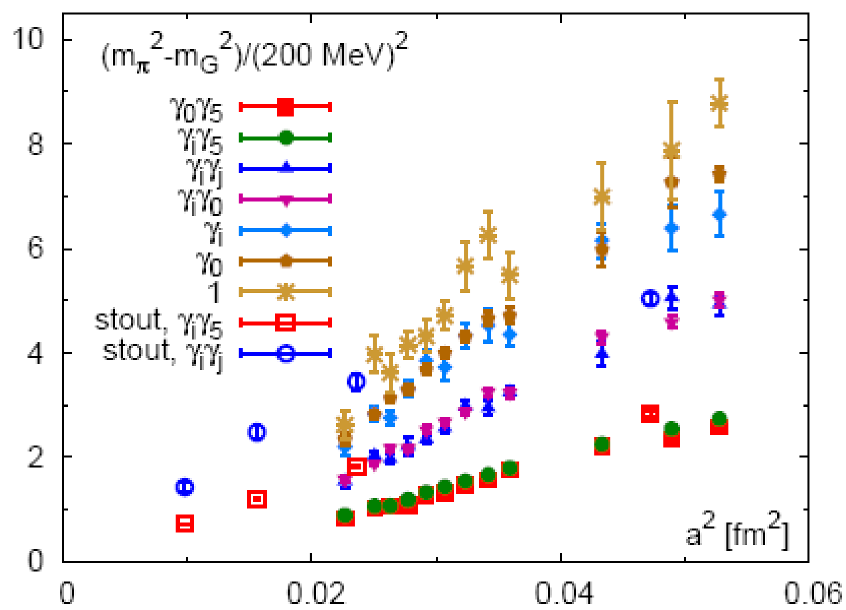
Mass splitting of pseudo-scalar mesons

Only one out of 16 PS mesons has zero mass in the chiral limit, the quadratic mass splitting is the measure of flavor symmetry breaking

asqtad



HISQ



PS meson splittings in HISQ calculations are reduced by factor ~ 2.5 compared to asqtad at the same lattice spacing and are even smaller than for stout action => discretizations effects for $N_\tau=8$ HISQ calculations are similar to those in $N_\tau=12$ asqtad calculations

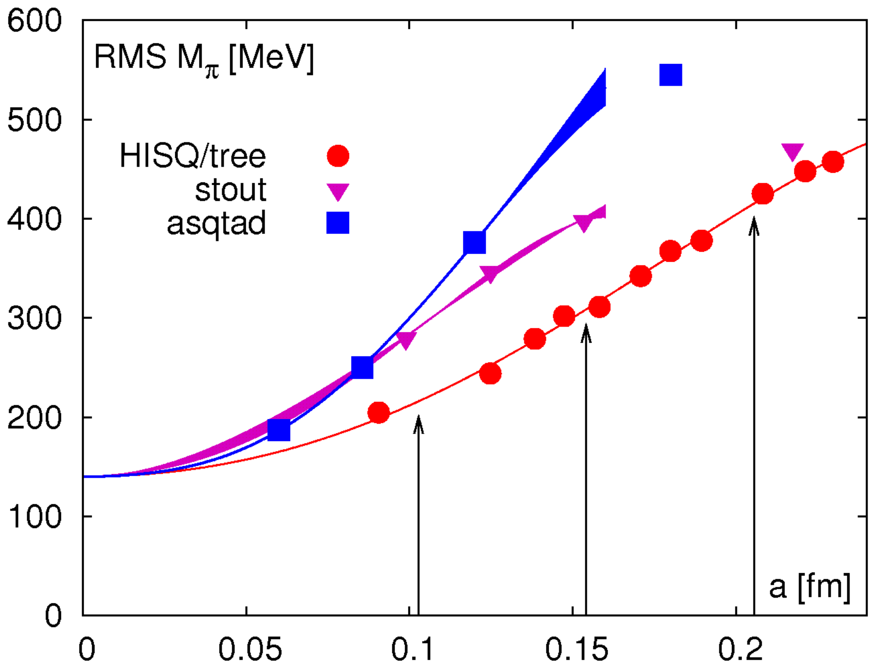
Glossary of improved staggered actions

p4 = std. staggered Dslash with 3-step (fat3) link +p4 term

asqtad = std. staggered Dslash with 7-step (fat7) link + Naik term

HISQ = std. staggered Dslash with re-unitarized doubly smeared 7-step (fat7) link

stout = std. staggered Dslash with re-unitarized doubly smeared 3-step (fat7) link



p4, asqtad, HISQ, stout