Homoclinic classes and equilibrium states

Lecture 1 – Homoclinic classes: general properties

Sylvain Crovisier

CNRS / Univ. Paris-Sud

Smooth and homogeneous dynamics

Bangalore, september 23-27th 2019

Dynamics of diffeomorphisms

M: a compact boundaryless connected manifold,
f: a C^r-diffeomorphism, $r > 1$.

Goal: *Describe the orbits* $\{f^n(x)\}_{n\in\mathbb{Z}}$.

Steps:
- decompose the system (identify attractors, invariant pieces,...),
- analyze each piece (eg. build a coding),
- study limit behaviors (invariant measures, speed of convergence,...),
- ...

This is well understood for uniformly hyperbolic diffeomorphisms.

What about general diffeomorphisms?

Program:
- *Lecture 1:* decomposition of the dynamics,
- *Lecture 2:* the case of surfaces,
- *Lecture 3:* coding,
- *Lecture 4:* limit measures.

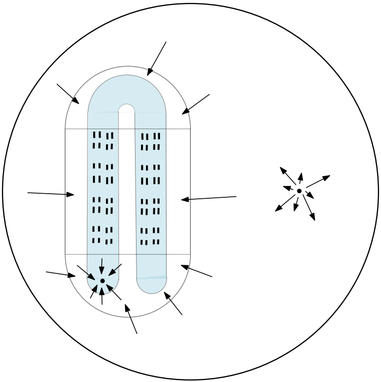
Uniformly hyperbolic diffeomorphisms

Hyperbolic diffeomorphisms (1): definition

f is *hyperbolic* if for any $x \in M$, one of the following holds:

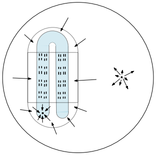
- *no recurrence*: there is U open s.t. $f(\overline{U}) \subset U$ and $x \in U \setminus f(U)$,
- *hyperbolicity*: there is $N \geq 1$ and $T_xM = E^s \oplus E^u$ s.t. for $\ell \in \mathbb{Z}$, $k \geq 1$,

$$\|Df^{kN}|_{Df^{\ell}(E^s)}\| \leq 2^{-k}, \quad \|Df^{-kN}|_{Df^{\ell}(E^u)}\| \leq 2^{-k}.$$



Hyperbolic diffeomorphisms (2): decomposition

f: a hyperbolic diffeomorphism



Smale's spectral decomposition. *The set $\Omega(f)$ of points which are not trapped is a finite disjoint union*

$$\Omega(f) = K_1 \cup \cdots \cup K_\ell$$

of sets K_i which are compact, invariant, and transitive:

for any balls U, V of K, $\quad f^k(U) \cap V \neq \emptyset$, *for some $k \geq 1$.*

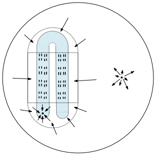
The sets K_i are called *basic sets* of f.

Remark. There is a finer decomposition $K_i = A \cup f(A) \cup \ldots \cup f^{m-1}(A)$ s.t. A is preserved by f^m and *topologically mixing*:

for any balls U, V of K, $\quad f^{km}(U) \cap V \neq \emptyset$ for **all** large k.

Hyperbolic diffeomorphisms (3): coding

f: a hyperbolic diffeomorphism



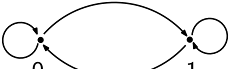
Markov partition (Adler-Weiss, Sinaï, Bowen). *For each basic set* K*, there are a symbolic system* (Σ, σ) *and* $\pi : \Sigma \to K$ *continuous s.t.*

– Σ*: space of itineraries* (a_n) *on a finite oriented connected graph,*

– $\sigma : \Sigma \to \Sigma$ *is the shift map* $(a_n) \mapsto (a_{n+1})$*,*

– $\pi : \Sigma \to K$ *is surjective and semiconjugates:* $f \circ \pi = \pi \circ \sigma$*,*

– *each preimage* $f^{-1}(x)$ *is finite.*



Hyperbolic diffeomorphisms (4): invariant measures

f: a hyperbolic diffeomorphism

$\mathcal{M}$: space of probabilities μ that are invariant, $f_*(\mu) = \mu$.

μ is ergodic if for any A invariant measurable set, $\mu(A) = 0$ or 1.

Physical measures. (Sinaï, Ruelle, Bowen).
There exists finitely many ergodic probabilities $\nu_1, \dots, \nu_J$, s.t. for Lebesgue almost every $x \in M$ and every continuous $\varphi\colon M \to \mathbb{R}$,
$$\frac{1}{n}(\varphi(x)+\varphi\circ f(x)+\ldots+\varphi\circ f^{n-1}(x)) \text{ converges to one } \int \varphi d\nu_j.$$

Periodic equidistribution. (Bowen). *For each basic set K, let $Per_n(K)$: set of periodic points $x \in K$ with period $\leq n$. Then,*
$$\frac{1}{Card(Per_n(K))} \sum_{x \in Per_n(K)} \delta_x \text{ converges to some } \mu_K \in \mathcal{M} \text{ as } n \to +\infty.$$

Hyperbolic diffeomorphisms (4): properties of the measures

f: a hyperbolic diffeomorphism

The physical measures ν_j and the periodic limit μ_K:

– are solutions of variational problems (called *equilibrium states*),

– $\simeq$ Bernoulli measures on Markov chains (as measured transformations),

– in particular, they are *mixing*: for any measurable sets A, B,

$$\mu(A \cap f^{-n}(B)) \underset{n\to\infty}{\longrightarrow} \mu(A)\mu(B).$$

Exponential mixing. *Assume that f is topologically mixing on Supp(μ). For any Hölder maps $\varphi, \psi\colon M \to \mathbb{R}$ there exists $C > 0$, $\theta \in (0, 1)$ s.t.*

$$\left|\int \varphi \circ f^n.\psi d\mu - \int \varphi d\mu \int \psi d\mu\right| \leq C.\theta^n.$$

Central limit theorem. *For any Hölder map $\varphi\colon M \to \mathbb{R}$ s.t. $\int \varphi d\mu = 0$, the variable $\frac{1}{\sqrt{n}}(\varphi(x) + \varphi(f(x) + \cdots + \varphi(f^{n-1}(x)))$ converges in law as $n \to +\infty$ towards some Gaussian law $\frac{1}{\sigma\sqrt{2\pi}} \int e^{-t^2/2\sigma^2} dt$.*

Remark. $\sigma = 0$ iff $\varphi = \psi \circ f - \psi$ for some $\psi \in L^2(\mu)$. In this case, it converges to δ_0.

How does this extends to general diffeomorphisms?

Homoclinic classes (1): definition

An orbit O with period k is *hyperbolic* if
for $p \in O$, $Df^k(p)$ has no eigenvalue on the unit circle.

The *stable* and *unstable* sets of $p \in O$ are immersed submanifolds:

$$W^s(p) := \{x \in M, d(f^n(x), f^n(p)) \to 0 \text{ as } n \to +\infty\},$$

$$W^u(p) := \{x \in M, d(f^{-n}(x), f^{-n}(p)) \to 0 \text{ as } n \to +\infty\}.$$

$$W^{s/u}(O) := \cup_{p \in O} W^{s/u}(p).$$

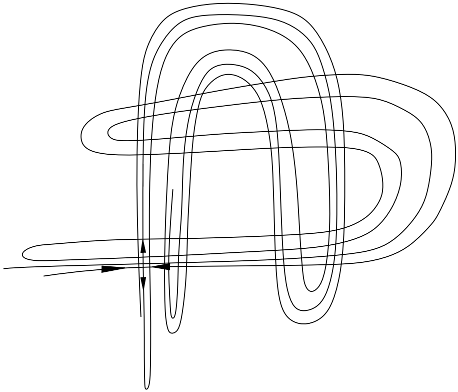
Definition (Newhouse). The *homoclinic class* of O is

$$H(O) := Closure(W^s(O) \pitchfork W^u(O)),$$

where $\pitchfork$ denotes the set of transverse intersections.

Homoclinic classes (1): definition

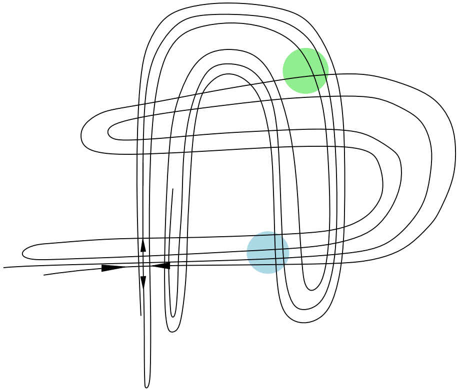
$$H(O) := Closure(W^s(O) \pitchfork W^u(O)).$$



Remark. For hyperbolic diffeomorphisms, homoclinic classes and basic sets coincide. (Consequence of the *shadowing lemma*.)

Homoclinic classes (2): transitivity

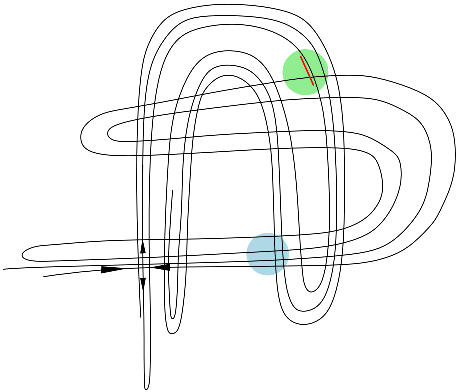
$H(O) = Closure(W^s(O) \pitchfork W^u(O))$ is transitive, invariant, compact.



(Consequence of the *inclination lemma*.)

Homoclinic classes (2): transitivity

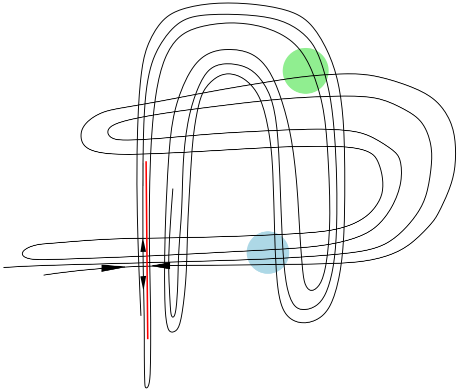
$H(O) = Closure(W^s(O) \pitchfork W^u(O))$ is transitive, invariant, compact.



(Consequence of the *inclination lemma*.)

Homoclinic classes (2): transitivity

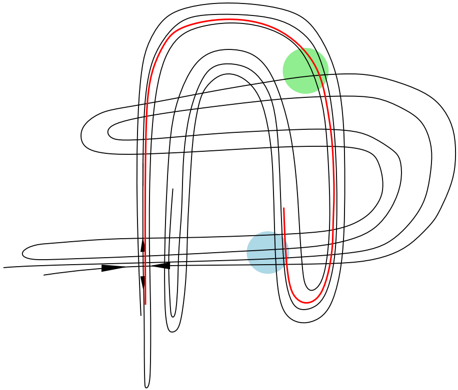
$H(O) = Closure(W^s(O) \pitchfork W^u(O))$ is transitive, invariant, compact.



(Consequence of the *inclination lemma*.)

Homoclinic classes (2): transitivity

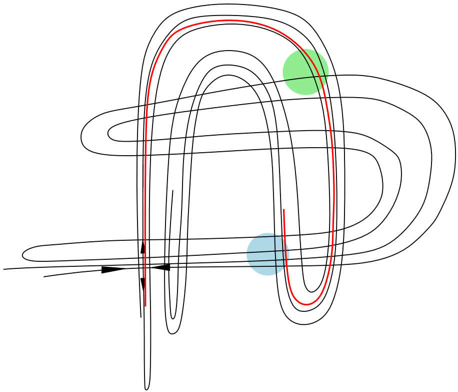
$H(O) = Closure(W^s(O) \pitchfork W^u(O))$ is transitive, invariant, compact.



(Consequence of the *inclination lemma*.)

Homoclinic classes (2): transitivity

$H(O) = Closure(W^s(O) \pitchfork W^u(O))$ is transitive, invariant, compact.



(Consequence of the *inclination lemma*.)

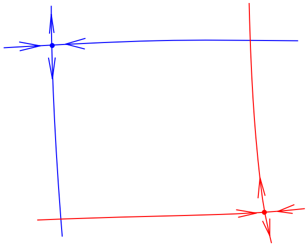
! The number of homoclinic class may be infinite.
!! In general, two distinct homoclinic classes may not be disjoint.

Homoclinic classes (3): density of periodic points

$H(O) := Closure(W^s(O) \pitchfork W^u(O))$.

q: a hyperbolic periodic point.

Definition. The point q is *homoclinically related* to O if $W^s(q) \pitchfork W^u(O) \neq \emptyset$ and $W^u(q) \pitchfork W^s(O) \neq \emptyset$. (One notes $q \sim O$.)



Properties. (1) The relation $\sim$ is an equivalence relation.

(2) $H(O) = Closure\{q \sim O\}$.

(Another consequence of the inclination lemma!)

Homoclinic classes (4): period of the class

Consider $p \in O$ and $h_p := Closure(W^s(p) \pitchfork W^u(p))$.

Property. *$H(0) = h_p \cup f(h_p) \cup \dots f^{k-1}(h_p)$ and $f^k(h_p) = h_p$, where k is the gcd of the periods of the $q \sim O$. Moreover h_p is a topologically mixing set of f^k.*

k is called *period* of the homoclinic class $H(O)$.

(Still uses the inclination lemma!!)

Homoclinic classes (5): hyperbolic measures

Consider $\mu \in \mathcal{M}$ ergodic.

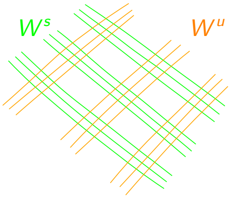
Theorem (Oseledets) *There is an inv. measurable decomposition $T_xM = E_1 \oplus \ldots \oplus E_k$ on a full measure set and $\lambda_1 < \ldots < \lambda_k$ s.t.*

$$\frac{1}{n} \log \| Df^n_{E_i}.v\| \underset{n\to\pm\infty}{\longrightarrow} \lambda_i \text{ for any } v \in E_i \setminus \{0\}.$$

μ is *hyperbolic* if its *Lyapunov exponents* λ_i are all different from 0.

$\Rightarrow$ there exists a (non-uniform) splitting $T_xM = E^s \oplus E^u$ μ-a.e.

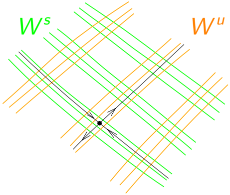
Theorem (Pesin) *If μ is hyperbolic, for μ-ae x, the stable and unstable sets $W^{s/u}(x)$ are immersed submanifolds tangent to $E^{s/u}_x$.*



Homoclinic classes (5): hyperbolic measures

Definition. μ is *homoclinically related* to O if for μ-ae x,
$W^s(x) \pitchfork W^u(O) \neq \emptyset$ and $W^u(x) \pitchfork W^s(O) \neq \emptyset$.
(We note $\mu \sim O$.)

Theorem (Katok's theorem revisited) *Each hyperbolic measure μ is homoclinically related to a hyperbolic periodic orbit O.*



(Uses a non-uniform shadowing lemma.)

Dynamics on surfaces

The (topological) entropy

Definition. A conjugacy invariant:

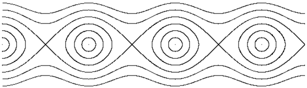
$$h_{top}(f) = \lim_{\varepsilon \to 0} h_{top}(f, \varepsilon)$$

where $h_{top}(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \#\{\text{orbits of length } n \text{ distinct at scale } \varepsilon\}$.

Surface dynamics with zero entropy:

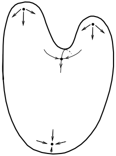
– Conservative examples

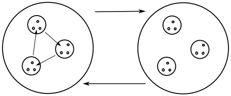
(eg. translations on $\mathbb{T}^2$, hamiltonian systems).



– Dissipative examples

(eg. Morse-Smale systems, odometers).





Some classification results: Franks-Handel, LeCalvez-Tal, C.-Pujals.

Surface dynamics with positive entropy : statement

A generalized spectral decomposition theorem.

Theorem. (Buzzi-C-Sarig) f: a C^∞ diffeomorphism of a surface.

(a) Covering. $\forall$ *inv. compact* A, $h_{top}\left(A \cap \left(\cup_O H(O)\right)\right) = h_{top}(A)$.

(b) Disjointness. $\forall O, O'$, *either* $O \sim O'$ *or* $h_{top}(H(O) \cap H(O')) = 0$.

(c) Uniqueness. *f transitive $\Rightarrow$ at most one non-triv. homoclinic class.*

(d) Finiteness. $\forall \delta > 0$, *the set* $\{H(O)\colon h_{top}(H(O)) > \delta\}$ *is finite.*

(e) Properties of homoclinic classes (coding, equilibrium states).

Properties a: consequence of Katok's theorem.
Properties b, c, d: lecture 2.
Property d: lectures 3,4.

Some references

R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes in Mathematics **470**.

J. Buzzi, S. Crovisier, O. Sarig. *Measures of maximal entropy for surface diffeomorphisms.* ArXiv:1811.02240.

Homoclinic classes and equilibrium states

Lecture 2 – Homoclinic classes on surfaces: disjointness and finiteness

Sylvain Crovisier

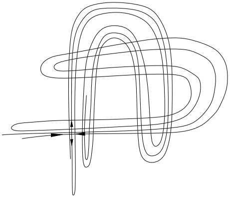
CNRS / Univ. Paris-Sud

Smooth and homogeneous dynamics

Bangalore, september 23-27th 2019

Homoclinic classes

M: a compact boundaryless connected manifold,
f: a C^r-diffeomorphism, $r > 1$,



Definition. The *homoclinic class* of a hyperbolic periodic orbit O is

$$H(O) := Closure(W^s(O) \pitchfork W^u(O)).$$

Properties. – If $O \sim O'$, then $H(O) = H(O')$.

– $H(O)$ is an invariant compact and transitive set.

– Any (ergodic) hyperbolic measure is supported on a homoclinic class.

Entropy

Entropy of invariant compact sets: $h_{top}(K) = h_{top}(f|_K)$.

Goal. *Study the dynamics up to invariant sets with zero entropy.*

One also defines the entropy $h(f, \mu)$ of an invariant probability μ.

Variational principle. $h_{top}(K) = \sup\{h(f, \mu),\ supp(\mu) \subset K\}$.

Key property on surfaces.

Measures with positive entropy are hyperbolic.

Surface dynamics with positive entropy

Goal. Obtain a generalized spectral decomposition theorem for arbitrary surface diffeomorphisms with positive entropy.

(Joint work with Jérôme Buzzi and Omri Sarig.)

Theorem. f: a C^∞ diffeomorphism of a surface.

(a) Covering. μ *ergodic with positive entropy* $\Rightarrow \mu(\cup_O H(O)) = 1$.

(b) Disjointness. $\forall O, O'$, *either* $O \sim O'$ *or* $h_{top}(H(O) \cap H(O')) = 0$.

(c) Uniqueness. f *transitive* $\Rightarrow$ *at most one non-triv. homoclinic class.*

(d) Finiteness. $\forall \delta > 0$, *the set* $\{H(O) \colon h_{top}(H(O)) > \delta\}$ *is finite.*

(e) Properties of homoclinic classes (coding, equilibrium states).

Disjointness

Disjointness

Theorem. f: C^∞ surface diffeomorphism, O, O': periodic saddles.
Then, either $O \sim O'$ or the entropy of $H(O) \cap H(O')$ vanishes.

Definition. $$Bilip(f) = \lim_{n\to+\infty} \frac{1}{n} \log \max(\|Df^n\|, \|Df^{-n}\|).$$

Remark. $Bilip(f) \geq h_{top}(f)$ (Ruelle's inequality).

Theorem. f: C^r surface diffeomorphism, O, O': periodic saddles such that $h_{top}(f|_{H(O)}),\ h_{top}(f|_{H(O')}) > Bilip(f)/r$.
Then, either $O \sim O'$ or the entropy of $H(O) \cap H(O')$ vanishes.

Problem. Does there exists a C^r diffeomorphism and O, O' not homoclinically related such that $h_{top}(H(O) \cap H(O')) > 0$?

Disjointness (2): proof

Assume that $h_{top}(H(O) \cap H(O')) > 0$.

(1) There exists μ ergodic and hyperbolic on $H(O) \cap H(O')$.

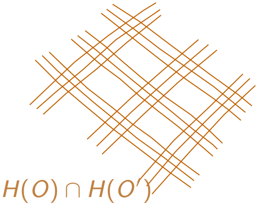
- *If one gets* **transverse** *intersections,*
  one concludes that $O \sim O'$, hence $H(O) = H(O')$. $\square$

Disjointness (2): proof

Assume that $h_{top}(H(O) \cap H(O')) > 0$.

(1) There exists μ ergodic and hyperbolic on $H(O) \cap H(O')$.

(2) The Pesin foliations $\mathcal{W}^s(\mu)$ and $\mathcal{W}^u(\mu)$ define small rectangles.



- *If one gets* **transverse** *intersections,*
  one concludes that $O \sim O'$, hence $H(O) = H(O')$. □

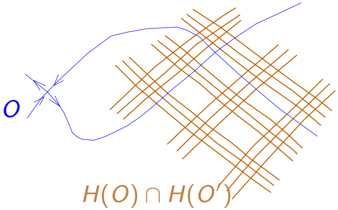
Disjointness (2): proof

Assume that $h_{top}(H(O) \cap H(O')) > 0$.

(1) There exists μ ergodic and hyperbolic on $H(O) \cap H(O')$.

(2) The Pesin foliations $\mathcal{W}^s(\mu)$ and $\mathcal{W}^u(\mu)$ define small rectangles.

(3) $W^s(O), W^u(O)$ intersect the rectangles $\Rightarrow$ "cross" $\mathcal{W}^s(\mu), \mathcal{W}^u(\mu)$.



- *If one gets* **transverse** *intersections,*
  one conclude*s* that $O \sim O'$, hence $H(O) = H(O')$. $\square$

Disjointness (2): proof

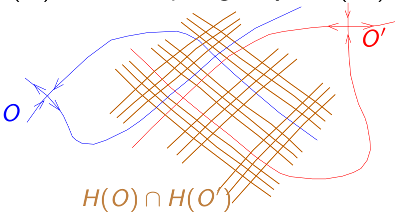
Assume that $h_{top}(H(O) \cap H(O')) > 0$.

(1) There exists μ ergodic and hyperbolic on $H(O) \cap H(O')$.

(2) The Pesin foliations $\mathcal{W}^s(\mu)$ and $\mathcal{W}^u(\mu)$ define small rectangles.

(3) $W^s(O), W^u(O)$ intersect the rectangles $\Rightarrow$ "cross" $\mathcal{W}^s(\mu), \mathcal{W}^u(\mu)$.

(4) This holds for O and O' $\Rightarrow$ $W^s(O)$ crosses topologically $W^u(O')$.



- *If one gets* **transverse** *intersections,*
  one conclude s that $O \sim O'$, hence $H(O) = H(O')$. $\square$

Disjointness (2): proof

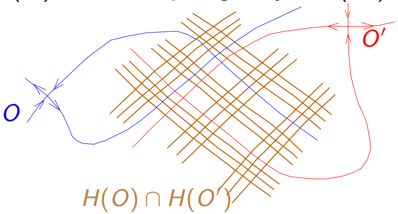
Assume that $h_{top}(H(O) \cap H(O')) > 0$.

(1) There exists μ ergodic and hyperbolic on $H(O) \cap H(O')$.

(2) The Pesin foliations $\mathcal{W}^s(\mu)$ and $\mathcal{W}^u(\mu)$ define small rectangles.

(3) $W^s(O), W^u(O)$ intersect the rectangles $\Rightarrow$ "cross" $\mathcal{W}^s(\mu), \mathcal{W}^u(\mu)$.

(4) This holds for O and O' $\Rightarrow$ $W^s(O)$ crosses topologically $W^u(O')$.

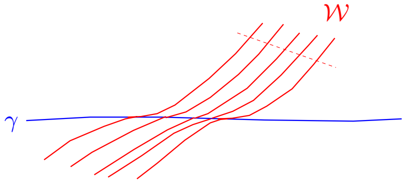


- *If one gets* **transverse** *intersections,*
  one conclude that $O \sim O'$, hence $H(O) = H(O')$. □

A "dynamical" Sard theorem

Theorem.
– γ: a C^r-curve,
– $\mathcal{W}$: a lamination by C^r-leaves, continuous in C^r-topology, with Lipschitz holonomies ($\Rightarrow$ transverse dimension well defined).

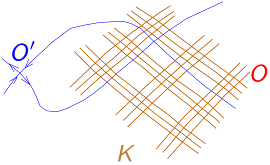
Then $\mathcal{T} := \{$leaves of $\mathcal{W}$ tangent to $\gamma\}$ has transverse dimension $\leq 1/r$.



Remark. When $\mathcal{W}$ is a C^r-foliation, one recovers the usual C^r-Sard lemma.

A “dynamical” Sard theorem (2): a consequence

f: a C^r-diffeomorphism
K: a transitive hyperbolic set
with entropy larger than $Bilip(f)/r$
O, O': two saddles such that $O \in K$.



Corollary. *If $W^s(O)$ crosses topologically $W^u(O')$, then $W^s(O) \pitchfork W^u(O') \neq \emptyset$.*

Proof. $W^u(O')$ is a C^r-curve γ.
The stable lamination $\mathcal{W}^s$ of K has C^r-leaves, continuous in C^r-topology.

Lemma. *$\mathcal{W}^s$ has Lipschitz holonomies.* (since 1-codim.)

Lemma (Manning). *The dimension of K inside its unstable leaves is $\geq h_{top}(f|_K)/\log \|Df\|$.*

$\Rightarrow$ The transverse dimension of $\mathcal{W}^s$ is $\geq h_{top}(f|_K)/Bilip(f) > 1/r$.

By Sard, one leaf of $\mathcal{W}^s$ intersects $W^u(O')$ transversally.

Since $W^s(O)$ is C^1 dense in $\mathcal{W}^s$, it intersects $W^u(O')$ transversally. $\square$

Uniqueness

Corollary. *f a C^r-diffeomorphism of a surface and K a transitive compact set such that $h_{top}(f|_K) > Bilip(f)/r$.*
Then K contains at most one non-trivial homoclinic class.

Proof.
If $H(O)$ and $H(O')$ are non-trivial,
the transitivity forces $W^s(O)$ and $W^u(O')$ to cross topologically.

Dynamical Sard Lemma gives the transverse intersection. □

Finiteness

Finiteness

Notation. $Bilip(f) := \lim_{n\to+\infty} \frac{1}{n} \log \max(\|Df^n\|, \|Df^{-n}\|)$.

Theorem. Let f be a C^r diffeomorphism of a surface.
For any $\chi > Bilip(f)/r$, the number of homoclinic classes such that $h_{top}(f|_{H(O)}) > \chi$ is finite.

Remark. The bound $Bilip(f)/r$ is optimal.

The proof uses:

– the tail entropy,

– Yomdin theory,

– 2-dim arguments.

Entropy at small scales: tail entropy (1)

Topological entropy: $h_{top}(f) = \lim_{\varepsilon \to 0} h_{top}(f, \varepsilon)$

where $h_{top}(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \#\{\text{orbits of length } n \text{ distinct at scale } \varepsilon\}$.

Entropy of an ergodic measure: $h(\mu) = \lim_{\varepsilon \to 0} h_{top}(f, \mu, \varepsilon)$ (Katok)

$$h(f, \mu, \varepsilon) = \limsup_{n \to \infty} \inf_{\mu(X)=1/2} \frac{1}{n} \log \#\{\text{orbits of length } n \text{ distinct at scale } \varepsilon \text{ meeting } X\}.$$

Local contribution: $h^*(f, \varepsilon) = \sup_{x \in M} h_{top}(Dyn.Ball(f, x, \varepsilon))$,

$$h^*(f, \mu, \varepsilon) = \inf_{\mu(X)=1/2} \sup_{x \in X} h_{top}(Dyn.Ball(f, x, \varepsilon)),$$

where $Dyn.Ball(f, x, \varepsilon) = \{y : \ \forall n, \ d(f^n(x), f^n(y)) \leq \varepsilon\}$.

Entropy at small scales: tail entropy (2)

Definition. *Tail entropy.* $$h^*(f) = \lim_{\varepsilon \to 0} h^*(f, \varepsilon).$$

Proposition. (Misiurewicz, Newhouse)

$$h(f, \mu) \leq h(f, \mu, \varepsilon) + h^*(f, \varepsilon).$$

$$\limsup_n h(f, \mu_n) \leq h(f, \mu) + h^*(f) \qquad \text{if } \mu_n \to \mu.$$

Yomdin theory.

Theorem. (Yomdin, Newhouse, Buzzi, Downarowicz, Burguet,...)
f: C^r-diffeomorphism of surface.

$$h^*(f) \leq \frac{Bilip(f)}{r}.$$

Entropy at small scales: summary

Corollary. For a C^r-diffeomorphism of surface,

$$h(f,\mu) \leq h(f,\mu,\varepsilon) + \frac{Bilip(f)}{r}.$$

$$\limsup_n h(f,\mu_n) \leq h(f,\mu) + \frac{Bilip(f)}{r} \qquad \text{if } \mu_n \to \mu.$$

Proof of the finiteness (1)

Theorem. Let f: C^r diffeomorphism on a surface and any $\delta > 0$

$$\left\{ H(O) : \ h_{top}(H(O)) > \frac{Bilip(f)}{r} + \delta \right\} \text{ is finite.}$$

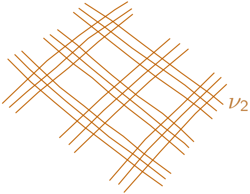
Consider a family of $H(O_n)$ supporting μ_n with $h(f, \mu_n) > \frac{Bilip(f)}{r} + \delta$.
▷ We have to show that there are $n \neq m$ such that $O_n \sim O_m$.

Assume $\mu_n \to \nu$. Then $h(f, \nu) > \limsup h(f, \mu_n) - Bilip(f)/r > 0$.

Decompose $\nu = \alpha\nu_1 + (1 - \alpha)\nu_2$ such that $\alpha > 0$ and:
$h(f, \nu_1) = 0$ and all components of ν_2 have positive entropy.

(1) Fix a $\varepsilon > 0$ small and N_0 large: there is a large ν_1-measure set X st.
$\frac{1}{n} \log \#\{\text{orbits of length } N_0 \text{ distinct at scale } \varepsilon \text{ meeting X}\} \ll 1$.

(2) ν_2 is approximated by a hyperbolic set $K \sim \nu_2$:
there exist squares $R_1, \dots, R_n$
bounded by $W^s(K)$ and $W^u(K)$
with diameter smaller than ε
and large total ν_2-measure.

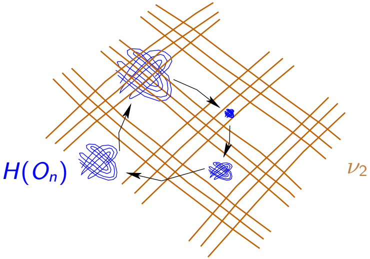


Proof of the finiteness (2)

Each $H(O_n)$ decomposes as $A_1 \cup \cdots \cup A_\ell$, cyclically permuted by f.

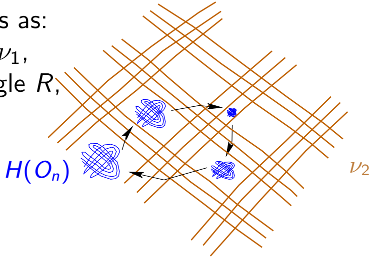
First case. For n large,
some A_i meets a rectangle R and R^c.

The $W^s(A_i)$ and $W^u(A_i)$ are connected.
Consequently $O_n \sim \nu_2$.
If this occurs for distinct n and m:
▷ We get $O_n \sim O_m$.



Second case. μ_n-typical orbits decomposes as:
– segments of orbits of length N_0 near ν_1,
– iterates in a A_i contained in a rectangle R,
– other iterates (small proportion).

Conclusion: the entropy $h(\mu_n, \varepsilon)$ is small.
but $h(\mu_n) \le h(\mu_n, \varepsilon) + Bilip(f)/r$.
▷ A contradiction.



Proof of the “dynamical” Sard theorem

Proof of the "dynamical" Sard theorem

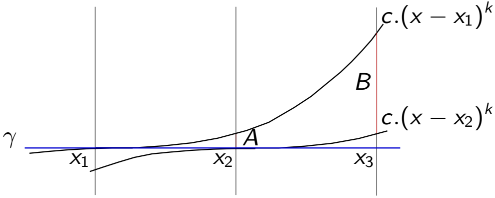
Theorem.
– γ: a C^r-curve,
– $\mathcal{W}$: a lamination by C^r-leaves, continuous in C^r-topology, with Lipschitz holonomies ($\Rightarrow$ transverse dimension well defined).

Then $\mathcal{T} := \{$leaves of $\mathcal{W}$ tangent to $\gamma\}$ has transverse dimension $\leq 1/r$.

$\mathcal{T}_k := \{$leaves of $\mathcal{W}$ with contact of order k with $\gamma\}$.

Lemma. *$\mathcal{T}_k$ is at most countable for $k < r$.*

Proof. Two close leaves at x_1, x_2.
$x_2 - x_1 = d$
$x_3 - x_2 = K.d$
$A = c.d^k$
$B = c.((K+1)^k - K^k).d^k$
Contradicts the Lipschitz holonomy
(if K is large)



Proof of the “dynamical” Sard theorem

Theorem.
– *γ: a C^r-curve,*
– *$\mathcal{W}$: a lamination by C^r-leaves, continuous in C^r-topology, with Lipschitz holonomies ($\Rightarrow$ transverse dimension well defined).*

Then $\mathcal{T}:=\{$leaves of $\mathcal{W}$ tangent to $\gamma\}$ has transverse dimension $\leq 1/r$.

Lemma. *$\mathcal{T}_r = \{$leaves with contact of order $\geq r\}$ has dimension $\leq 1/r$.*

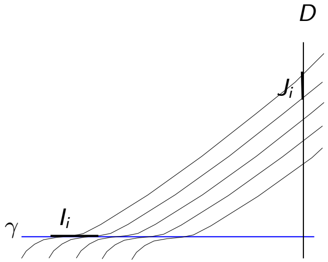
Proof.

Cover γ by small intervals I_i

$$\sum |I_i| < 1.$$

Project by holonomy as interval J_i in a transversal D.

$$\sum |J_i|^{1/r} < 1.$$



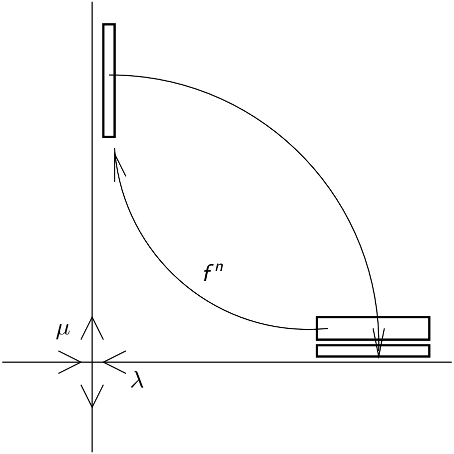
Examples

Measures of large entropy: examples

Newhouse construction:

$\lambda < 1 < \mu < \lambda^{-1}$

$\mu \simeq \lambda^{-1}$

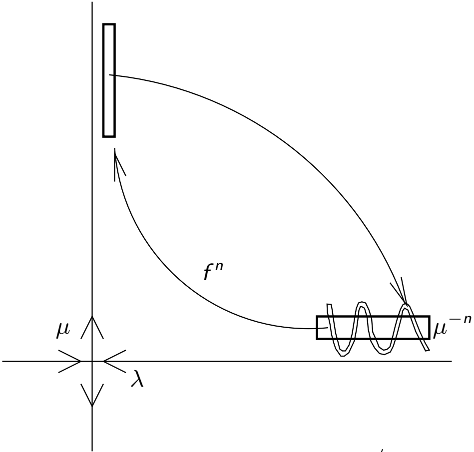


Measures of large entropy: examples

Newhouse construction:

$\lambda < 1 < \mu < \lambda^{-1}$

$\mu \simeq \lambda^{-1}$

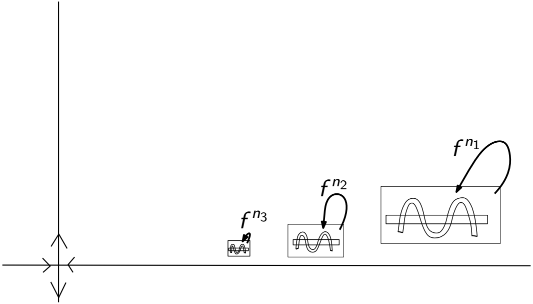


Number of waves after a C^r-perturbation: $\mu^{n/r}$.

$\Rightarrow h_{top}(f) \geq \frac{\log \mu}{r} \sim Bilip(f)/r.$

Measures of large entropy: examples

Proposition. *For any $r > 1$ and $\eta > 0$, there exists a C^r-diffeomorphism with infinitely many disjoint homoclinic classes with entropy larger than $(1-\eta).Bilip(f)/r$.*



Homoclinic classes and equilibrium states

Lecture 3 – Homoclinic classes: coding

Sylvain Crovisier

CNRS / Univ. Paris-Sud

Smooth and homogeneous dynamics

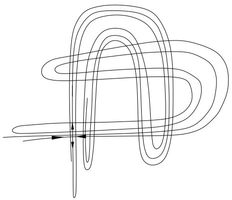
Bangalore, september 23-27th 2019

Homoclinic classes

M: a compact boundaryless connected manifold,
f: a C^r-diffeomorphism, $r > 1$.

The *homoclinic class*
of a hyperbolic periodic orbit O is
$H(O) := Closure(W^s(O) \pitchfork W^u(O))$.

It is an invariant compact and transitive set.



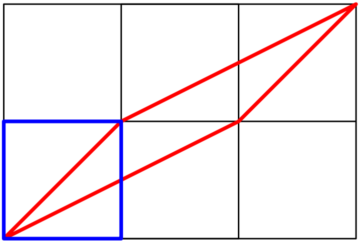
Program:

- *Lecture 1:* decomposition of the dynamics,
- *Lecture 2:* the case of surfaces,
- *Lecture 3:* coding,
- *Lecture 4:* limit measures.

Coding of uniformly hyperbolic systems

(Adler-Weiss, Sinaï, Bowen)

Anosov "cat map":
$f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$.

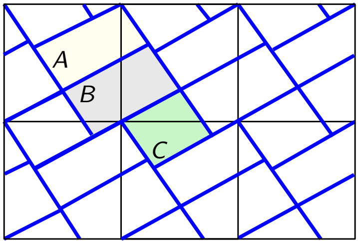


Coding of uniformly hyperbolic systems

(Adler-Weiss, Sinaï, Bowen)

Anosov “cat map”:
$f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$.

Partition:
rectangles A, B, C parallel to E^s, E^u

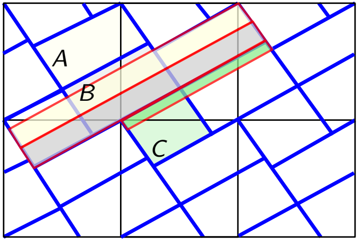


Coding of uniformly hyperbolic systems

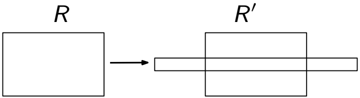
(Adler-Weiss, Sinaï, Bowen)

Anosov "cat map":
$f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2.$

Partition:
rectangles A, B, C parallel to E^s, E^u

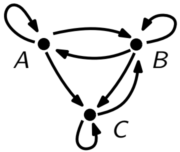


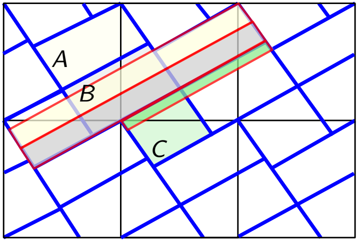
Markov property:
If $f(R)$ intersects *interior*(R'), it crosses.



Coding of uniformly hyperbolic systems

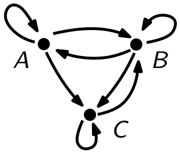
Transitions: a finite oriented graph

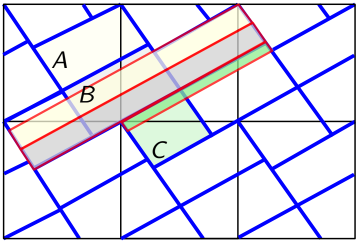




Coding of uniformly hyperbolic systems

Transitions: a finite oriented graph





Symbolic space Σ: space of admissible sequences.

Projection $\pi : \Sigma \to M$ defined by

$$\pi(\dots R_{-1}, R_0, R_1, \dots) = \bigcap f^{-i}(\overline{R_i}).$$

- π is continuous, surjective, generally not injective but finite-to-one.

Coding the non-uniformly hyperbolic set (global version)

Def. μ is *χ-hyperbolic* if its Lyapunov exponents belong to $\mathbb{R}\backslash[-\chi,\chi]$.

Theorem (Sarig, Ben Ovadia). *For $\chi > 0$, there exist:*
– a locally compact Markov shift $(\widehat{\Sigma}, \sigma)$ on a countable alphabet,
– a Hölder map $\pi : \widehat{\Sigma} \to M$ satisfying $\pi \circ \sigma = f \circ \pi$,
such that:

(a) $\pi(\widehat{\Sigma}^{\#})$ has full measure for every χ-hyperbolic measure,

(b) $\pi^{-1}(x) \cap \widehat{\Sigma}^{\#}$ is finite for every $x \in M$.

Here $\widehat{\Sigma}^{\#}=$ set of orbits with arb. large forward & backward iterates in a compact set.

Remarks. – Compact sets in $\widehat{\Sigma}$ project to unif. hyperbolic sets in M.
– Invariant probabilities in a transitive component of $\widehat{\Sigma}$ project to measures that are homoclinically related.

Coding of a homoclinic class

$H(O)$: a homoclinic class of a C^r diffeomorphism f, $r > 1$.

Theorem (Buzzi, C-, Sarig). For any $\chi > 0$, there exists:

- a locally compact Markov shift on a countable alphabet (Σ, σ),
- a Hölder map $\pi\colon \Sigma \to H(O)$ satisfying $\pi \circ \sigma = f \circ \pi$,

such that

(a) $\mu(\pi(\Sigma^\#)) = 1$ for any χ-hyperbolic measure $\mu \sim O$,
(b) $\pi^{-1}(y) \cap \Sigma^\#$ is finite for all $y \in \pi(H(O))$,
(c) (Σ, σ) is **transitive** (irreductible).

Here $\Sigma^\# =$ set of orbits with arb. large forward & backward iterates in a compact set.

Remark. Any χ-hyperbolic $\mu \sim O$ lifts as an inv. probability $\widehat{\mu}$ on $\Sigma^\#$.
From (b), the entropy of $\widehat{\mu}$ and μ coincide.

Coding of a homoclinic class

Three steps:

I. Construction of a highly redundant coding Σ_0.

II. Refinement to a finite-to-one global coding $\widehat{\Sigma}$.

III. Extraction of an irreducible component $\Sigma \subset \widehat{\Sigma}$.

Quality of hyperbolicity

Fix $\varepsilon, \beta > 0$ small. To simplify, M is a surface.

The *Non-Uniformly Hyperbolic set* NUH_χ:

set of $x \in M$ with directions e^s, e^u and angle $\alpha(x)$ s.t.

- $s(x) := \left(\sum_{n \geq 0} e^{2\chi n} \| Df^n(x).e^s \|^2 \right)^{1/2} < \infty$,
- $u(x) := \left(\sum_{n \geq 0} e^{2\chi n} \| Df^{-n}(x).e^u \|^2 \right)^{1/2} < \infty$,
- $\frac{1}{n} \log Q(f^n(x)) \longrightarrow 0$ as $n \to \pm\infty$,

where $\quad Q(x) := \max(\alpha(x), 1/s(x), 1/u(x)))^{1/\beta}$.

Points $x \in \mathrm{NUH}_\chi$ have a *Pesin chart* of size: $\quad Q(x)$.

Size of stable manifold: $\quad q^s(x) = \min\{e^{\varepsilon n} Q(f^n(x)),\ n \geq 0\}$,

Size of unstable manifold: $q^u(x) = \min\{e^{\varepsilon n} Q(f^{-n}(x)),\ n \geq 0\}$.

I- Markov covering: construction

$\mathcal{A}$: collection of Pesin charts $\Psi_x^{p^s,p^u}$ for $x \in NUH_\chi$, $p^s, p^u < Q(x)$,
with p^s = stable size and p^u = unstable size, such that:

– it "covers" NUH_χ,

– it is discrete: the set of charts $\Psi_x^{p^s,p^u}$ with $p^s, p^u > t$ is finite.

Transitions. $\Psi_x^{p_1^s,p_1^u} \to \Psi_y^{p_2^s,p_2^u}$ if $f(x) \sim y$ and $p_1^{s/u} \sim p_2^{s/u}$.

Σ_0: space of sequences in $\mathcal{A}$ compatible with the transitions.

Non-uniform shadowing $\pi_0(\underline{\Psi})$: For any $\underline{\Psi} = (\Psi_n)_{\mathbb{Z}} \in \Sigma_0$, there is a unique point $\pi_0(\underline{\Psi}) = x \in M$ such that $f^n(x) \in Im(\Psi_n)$ for all $n \in \mathbb{Z}$.

$\Rightarrow$ Any orbit in NUH_χ lifts by $\pi_0 : \Sigma_0 \to M$.

I- Markov covering: local finiteness

We have built
$$\pi_0 : \Sigma_0 \to M$$
$$\underline{\Psi} = (\Psi_n) \mapsto \pi_0(\underline{\Psi}).$$

Inverse theorem.

There exists $c > 0$ st. for any $\underline{\Psi} \in \Sigma_0^{\#}$ and $n \in \mathbb{Z}$,
$$c^{-1} \leq \frac{p_n^s}{q^s(f^n(x))} \leq c \quad \text{and} \quad c^{-1} \leq \frac{p_n^u}{q^u(f^n(x))} \leq c.$$

Summary. Let $Z_\Psi = \pi_0(\{(\Psi_n), \Psi_0 = \Psi\} \cap \Sigma_0^{\#})$ (projected cylinder), $\Psi \in \mathcal{A}$.

One gets a locally finite covering $\mathcal{Z}$ of $\pi_0(\Sigma_0^{\#})$ by "Markov rectangles":

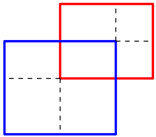
$x \in \pi_0(\Sigma_0^{\#})$ belongs to finitely many Z_Ψ, but has maybe infinitely many lifts in $\Sigma_0^{\#}$.

II- The global coding: construction

$\mathcal{Z}$: collection of projected rectangles Z_Ψ with transition relation $\Psi \to \Psi'$.

Bowen-Sinaï refinement.

Any $Z_\Psi, Z_{\Psi'}$ which intersect generate seven rectangles.



$\mathcal{R}$: partition of $\pi_0(\Sigma_0^\#)$ induced by all rectangles generated by pairs $Z_\Psi, Z_{\Psi'} \in \mathcal{Z}$.

Transitions: $R \to R'$ if $f(R) \cap R' \neq \emptyset$.

$\widehat{\Sigma}$: space of sequences on $\mathcal{R}$ compatible with transitions.

Projection: $\pi(\underline{R}) = \cap_{n\in\mathbb{Z}} f^{-n}(Closure(R_n))$.

$\Rightarrow$ A well-defined continuous map $\pi : \widehat{\Sigma} \to M$.

II- The global coding: properties

Bowen property. There exists a relation $\sim$ on $\mathcal{R}$ (affiliation) st:

- *For any* $\underline{R}, \underline{R}' \in \widehat{\Sigma}^{\#}, \quad \pi(\underline{R}) = \pi(\underline{R}') \Leftrightarrow (\forall n, R_n \sim R'_n)$.
- *For any* R*, the set* $\{R' \sim R\}$ *is finite.*

$\sim$ is defined by: $\quad R \sim R'$ iff $R \subset Z_\Psi$, $R' \subset Z_{\Psi'}$ and $Z_\Psi \cap Z_{\Psi'} \neq \emptyset$.

Corollary. $\pi : \widehat{\Sigma}^{\#} \to M$ *is finite-to-one.*

Consider $\underline{R} = (R_n)$ and R_-, R_+ such that $R_n = R_-$ for infinitely many $n < 0$,
$R_n = R_+$ for infinitely many $n > 0$.

Then $\#\{\underline{R}' \in \widehat{\Sigma}^{\#},\ \pi(\underline{R}') = \pi(\underline{R})\}$ is bounded by a $Cte(R_-, R_+)$.

Characterization of the uniform hyperbolicity.

Uniformly χ*-hyperbolic sets in* M *can be lifted as compact sets in* $\widehat{\Sigma}$*.*
The transitivity is preserved.

III- Selection of a transitive component Σ of $\widehat{\Sigma}$

$H(O)$: a homoclinic class.

Proposition.
There exists a transitive component $\Sigma \subset \widehat{\Sigma}$ containing periodic orbits that lift all the periodic orbits $O' \sim O$ that are χ-hyperbolic.

Proof.
Consider these periodic orbits $O_1, O_2, \dots$.

There exists transitive hyperbolic set K_n which contains $O_1, \dots, O_n$.

Lift $K_n \subset M$ as a transitive compact set $\widehat{K}_n \subset \widehat{\Sigma}$.

Finiteness-to-one property on $\Sigma^\#$
$\Rightarrow$ there is a transitive component $\Sigma \subset \widehat{\Sigma}$ containing infinitely many $\widehat{K}_n$.

□

III- Lifting the measures in Σ

$\mu \sim O$ a χ-hyperbolic measure

Proposition.
There exists a measure ν on Σ such that $\pi_*(\nu) = \mu$.

Proof.
- Lift $\widehat{\nu}$ on a transitive component Σ' of $\widehat{\Sigma}$ (maybe not Σ).
- Approximate $\widehat{\nu}$ by periodic orbits $\widehat{O}_1, \widehat{O}_2, \dots$ in $\widehat{\Sigma}$:
there are $\widehat{p}_i \in \widehat{O}_i$ such that $\widehat{p}_i \to \widehat{x}$ typical for $\widehat{\nu}$.
- Project in M as periodic orbits $O_1, O_2, \dots$ s.t. χ-hyperbolic and $\sim O$.
- Lift them as periodic orbits $\widetilde{O}_1, \widetilde{O}_2, \dots$ in Σ.
There is $\widetilde{p}_i \in \widetilde{O}_i$ such that $\pi(\widetilde{p}_i) = \pi(\widehat{p}_i)$.
- $(\widetilde{p}_i)$ is precompact (Bowen property) $\Rightarrow$ converges to some $\widetilde{x} \in \Sigma^\#$.
- This lift to $\Sigma^\#$ all points of a set of full μ-measure and average
    $\Rightarrow$ defines a measure ν on $\Sigma^\#$ which lifts μ.

□

Summary

$H(O)$: a homoclinic class of a C^r diffeomorphism of surface
NUH_χ: set of points in $H(O)$ that are 'χ-hyperbolic'.

Theorem (Local coding). For any $\chi > 0$, there exists:

- a loc. compact Markov shift on a countable alphabet (Σ, σ),
- a Hölder map $\pi \colon \Sigma \to H(O)$ satisfying $\pi \circ \sigma = f \circ \pi$,

such that
(a) $\pi(\Sigma^{\#}) \supset NUH_\chi$,
(b) π is finite-to-one on $\Sigma^{\#}$,
(c) (Σ, σ) is transitive.

Questions.

- How does Σ behave at infinity?
- Is it possible to code the whole $\bigcup_\chi NUH_\chi$ (when f is C^∞)?
- How to address $H(O) \setminus \pi(\Sigma^{\#})$?

Homoclinic classes and equilibrium states

Lecture 4 – Equilibrium states: existence and uniqueness

Sylvain Crovisier

CNRS / Univ. Paris-Sud

Smooth and homogeneous dynamics

Bangalore, september 23-27th 2019

Measures of maximal entropy

μ *maximizes the entropy if it realizes the supremum*

$$h_{top}(f) = h(f, \mu).$$

Motivation (Burguet). *For a C^∞-diffeomorphism on surface, the periodic orbits with Lyapunov exponents δ-far from 0 equidistribute towards the measures maximizing the entropy.*

Existence?

- *Yes* under expansivity (hyperbolic diffeomorphisms,...).
- *No* in general for C^r-diffeomorphisms. (Buzzi)

Theorem. (Newhouse) *If f is a C^∞ diffeomorphism on a compact manifold, it has a measure of maximal entropy.*

Measures of maximal entropy: finiteness

Finiteness of the set of ergodic measures maximizing the entropy?

was known for: – hyperbolic diffeomorphisms,...

– some non-uniformly hyperbolic Hénon maps. (Berger)

Theorem. f: a C^∞-diffeo of surface with $h_{top}(f) > 0$.
The number of ergodic measures maximizing the entropy is finite. Moreover, if f is transitive, it is equal to 1 (uniqueness).

Remark. When f is C^r:

– the same holds if $h_{top}(f) > Bilip(f)/r$,

– this may fail when $h_{top}(f) < Bilip(f)/r$.

Physical / SRB measures

μ: a hyperbolic measure of a C^2-diffeomorphism.

Definition. μ is SRB if its desintegrations along W^u are abs continuous.

Equivalent definitions (Ledrappier, Young, Tsujii,...):

(1) μ is "strongly" physical: for x in a set of positive Lebesgue measure, $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)} \to \mu$ and the forward orbit of x has the same exponents as μ.

(2) $h(f,\mu)$ equals the sum of the positive Lyapunov exponents of μ.

Restatement. On introduces the *geometrical potential* $\phi_{geom} : M \to \mathbb{R}$.

$$\phi_{geom}(x) = \begin{cases} -\log|\det(Df|_{E^u(x)})| & \text{if } x \text{ has an unstable space,} \\ -\infty & \text{otherwise.} \end{cases}$$

If μ is an SRB it maximizes $h(f,\mu) + \int \phi_{geom} d\mu$.

Physical / SRB measures: finiteness

Theorem (Hertz-Hertz-Tahzibi-Ures). f: a C^2 diffeo of surface.
Each homoclinic class supports at most one SRB measure.

Corollary. *On a transitive attractor, there is at most one SRB measure.*

Theorem (BCS). f: a C^∞ diffeo of surface. Fix $\delta > 0$.
If Leb. a.e. point thas an upper Lyapunov exponent $> \delta$, then there exist at most finitely many ergodic SRB measures.

Remark. When f is C^r, the same holds if $\delta > Bilip(f)/r$.

Equilibrium measures

$\phi \colon H(O) \to \mathbb{R} \cup \{-\infty\}$: a measurable potential.

Definition. μ is an *equilibrium state* for ϕ if it realizes the supremum:

$$P_f(\phi) := \sup_\nu \left(h(f,\nu) + \int \phi d\nu \right).$$

Remark (*small potential condition.*) For surface diffeomorphisms, the equilibrium states are hyperbolic provided that:

$$\sup \varphi - \inf \varphi < h_{top}(f).$$

Uniqueness

Equilibrium measures: uniqueness

Theorem. *Consider f, a C^2 diffeomorphism of a compact manifold, O a hyperbolic periodic orbit and φ either Hölder or $= \varphi_{geom}$.*

Then there is at most one hyperbolic equilibrium state $\mu \sim O$.

Its support coincides with $H(O)$; if period$(H(O))=1$, μ is Bernoulli.

Other approaches under various hyperbolic settings, using:

– the specification for the original system

(for instance the recent works by Climenhaga, Thompson, Burns, Fisher,...),

– the geometrical properties of measures

(for instance Hopf argument),

– ...

Equilibrium measures: uniqueness

Fix $\chi > 0$ small.

(1) There exists a coding by transitive Markov shift $\pi : \Sigma \to M$ st:
– π is Hölder continuous,
– any χ-hyperbolic measure $\mu \sim O$ lifts as a measure $\widehat{\mu}$ on Σ,
– $h(f, \mu) = h(\sigma, \widehat{\mu})$.

(2) χ-hyperbolic equilibrium states $\mu \sim O$ lift as eq. states on Σ.
Hölder or geometrical potentials lift as Hölder bounded potentials on Σ.

(3) The Bernoulli property is preserved by factor maps. (Ornstein)

Conclusion. We are reduced to a problem on Markov shifts.

Properties of Markov shifts

(Σ, σ): a transitive locally compact Markov shift on a countable alphabet and with finite entropy.

Theorem (Gurevich, Buzzi-Sarig). $\phi\colon \Sigma \to \mathbb{R}$ Hölder and bounded. *Then ϕ admits at most one equilibrium measure.*
When it exists, it has full support and
it is isomorphic to Bernoulli $\times$ finite permutation.

Proof. In the case $\phi = 0$.
Denote by $[i]$ the 0-cylinders of Σ.

Any measure μ has a Markov approximation $\bar{\mu}$:
$\bar{\mu}[i] := \mu[i]$ with transitions $P_{i,j} := \mu[i,j]/\mu[i]$.

Then $h(\mu) \leq h(\bar{\mu}) = -\sum_{i,j} \bar{\mu}[i]\ P_{i,j}\ \log P_{i,j}$. □

Existence

Equilibrium measures: existence

Theorem. *Take f C^∞ on a compact manifold and φ continuous. Then there exists an equilibrium state.*

Proof. Yomdin theory for a C^r-diffeomorphism gives:

$$\limsup_n h(f,\mu_n) \leq h(f,\mu) + h^*(f) \qquad \text{if } \mu_n \to \mu,$$

and

$$h^*(f) \leq \frac{Bilip(f)}{r}.$$

Hence $h \mapsto h(f,\mu)$ is semi-continuous for C^∞ diffeomorphisms.

Thus one considers any limit of measures approaching the supremum. □

Yomdin theory

Notation. $B_f(x, n, \varepsilon) := \{z, \;\; d(f^i(x), f^i(z)) \leq \varepsilon, \; 0 \leq i \leq n\}.$

$B_f(x, \infty, \varepsilon) := \{z, \;\; d(f^i(x), f^i(z)) \leq \varepsilon, \; 0 \leq i\}.$

Local contribution to entropy at scale ε

$$h^*(f, \varepsilon) = \sup_{x \in M} h_{top}(B_f(x, \infty, \varepsilon)),$$

$$h^*(f, \mu, \varepsilon) = \inf_{\mu(X)=1/2} \sup_{x \in X} h_{top}(B_f(x, \infty, \varepsilon)),$$

Theorem. f: C^r-diffeomorphism of surface.

$$h^*(f) := \lim_{\varepsilon \to 0} h^*(f, \varepsilon) \leq \frac{Bilip(f)}{r}.$$

Yomdin theory: steps of the proof

– A variational principle: $h^*(f) \leq \sup_\mu h^*(f,\mu,\varepsilon)$
(Downarowicz-Newhouse)

– Newhouse's bound: $h^*(f,\mu,\varepsilon) \leq L_r^*(f,2\varepsilon)$

where

$$L_r^*(f,\varepsilon) = \sup_{C^r-curve\ \gamma} \left(\limsup_n \sup_x \frac{1}{n} \log^+ \mathrm{Length}(f^n(\gamma \cap B_f(x,n,\varepsilon))) \right).$$

– Yomdin's bound: $\lim_{\varepsilon\to 0} L_r^*(f,\varepsilon) \leq \frac{Bilip(f)}{r}$

Entropy at small scales: Yomdin's reparametrization lemma

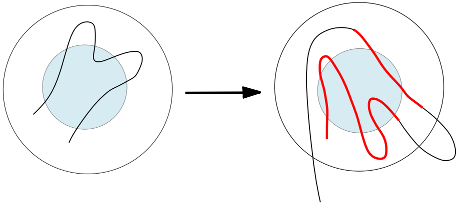
Given $\gamma\colon [0,1] \xrightarrow{C^r} M$, how to bound $\text{Length}(f^n(\gamma \cap B_f(x,n,\varepsilon)))$?

Consider $I_1, \ldots, I_{\ell(n)} \subset [0,1]$ and parametrizations $\psi_i\colon [0,1] \to I_i$ st:

(a) $\|f^n \circ \gamma \circ \psi_i\|_{C^r} \leq 1$

(b) $Im(\gamma) \cap B_f(x,n,\varepsilon) \subset \cup_i Im(\gamma \circ \psi_i)$.

The growth of $\ell(n)$ is estimated by induction: $\ell(n) \lesssim Lip(f)^{n/r}$.



Entropy at small scales: Yomdin's reparametrization lemma

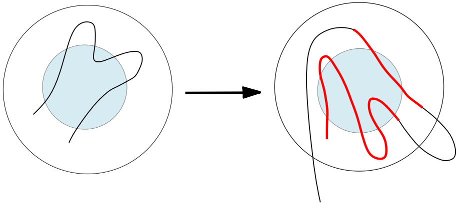
Given $\gamma\colon [0,1] \xrightarrow{C^r} M$, how to bound Length$(f^n(\gamma \cap B_f(x,n,\varepsilon)))$?

Consider $I_1, \ldots, I_{\ell(n)} \subset [0,1]$ and parametrizations $\psi_i\colon [0,1] \to I_i$ st:

(a) $\|f^n \circ \gamma \circ \psi_i\|_{C^r} \leq 1$

(b) $Im(\gamma) \cap B_f(x,n,\varepsilon) \subset \cup_i Im(\gamma \circ \psi_i)$.

The growth of $\ell(n)$ is estimated by induction: $\ell(n) \lesssim Lip(f)^{n/r}$.



$Lip(f) = \|f\|_{C^r}$ and $\|\gamma\|_{C^r} \leq 1 \Rightarrow \|D^r f \circ \gamma(L\,.\quad)\|_0 \leq 1$, where $L \sim Lip(f)^{-1/r}$.

Algebraic lemma. *One can subdivide and reparametrize $f(\gamma) \cap B(x,\varepsilon)$ into at most cte.$Lip(f)^{1/r}$ arcs γ' st $\|\gamma'\|_{C^r} \leq 1$.*