Creating and Using Entanglement

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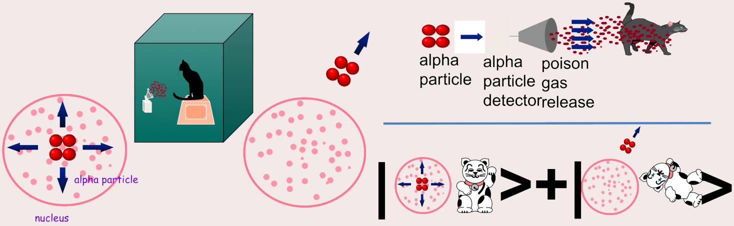
The nature of entanglement

Essence	Tensor-product (multipartite) structure + superposition.
Ubiquity	Separable pure states have zero measure.
Fragility	Rare for mixed states.
Mystery	Measurement paradox and many worlds.
Realism	Test Bell inequalities.
Resource	Bipartite entanglement can be useful for $\mathbb{Q}$ info processing.
Sharing	Limitations to sharing useful entanglement.

Einstein-Podolsky-Rosen 1935

- Measure 1$^{\text{st}}$ of two entangled particles.
- 2$^{\text{nd}}$ particle's state becomes indeterminate.
- Instantaneous interaction or pre-existing hidden information?
- Former relativistically impossible $\implies$ incompleteness.

Verschränkung



Bell Inequalities

Bell's Inequality

Correlation: $C(\mathbf{a},\mathbf{b}) = \int \mathrm{d}\lambda A(\mathbf{a},\lambda)B(\mathbf{b},\lambda)\rho(\lambda)$ with $A, B \in \{\pm 1\}$.
For some $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$, $1 + C(\mathbf{b},\mathbf{c}) \geq \|C(\mathbf{a},\mathbf{b}) - C(\mathbf{a},\mathbf{c})\|$.

Clauser-Horne-Shimony-Holt

$$S = C(\mathbf{a},\mathbf{b}) + C(\mathbf{a},\mathbf{b}') + C(\mathbf{a}',\mathbf{b}) - C(\mathbf{a}',\mathbf{b}').$$

Bounds: $|S_\mathrm{C}| \leq 2$ & $|S_\mathrm{Q}| \leq 2\sqrt{2}$.

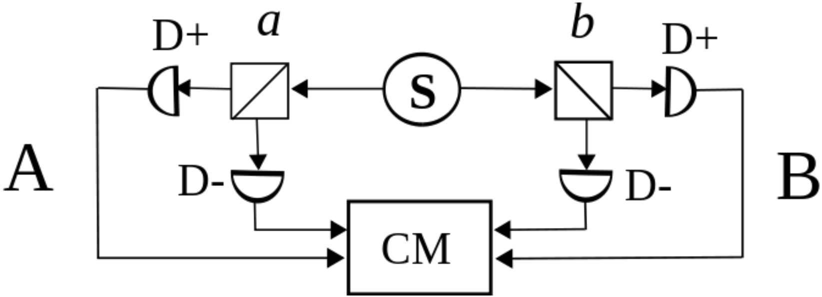
Clauser-Horne

For $\infty \implies$ removed polarizer,

$$\begin{aligned} -C(\infty,\infty) \leq C(\mathbf{a},\mathbf{b}) - C(\mathbf{a},\mathbf{b}') + C(\mathbf{a}',\mathbf{b}) \\ + C(\mathbf{a}',\mathbf{b}') - C(\mathbf{a}',\infty) - C(\infty,\mathbf{b}) \leq 0. \end{aligned}$$

Need fast 82.5%-efficient detectors with low dark-count rates.

Experimental Bell Inequality Test



http://upload.wikimedia.org/wikipedia/commons/3/39/Two_channel_bell_test.svg

Operators and States

Operator	Trace-class Θ in $L^2(\mathbb{R})$ acting on Hilbert space $\mathscr{H}$.
Norm	$\mathrm{tr}\sqrt{\Theta^\dagger\Theta} = \sqrt{\sum_i \lvert\theta_i\rvert^2}$, $\{\theta_i\}$ singular values of Θ.
Bases	basis$(\mathscr{H}) = \{\lvert i\rangle\}$; tensor-product basis$(\bigotimes \mathscr{H}) = \{\lvert \mathbf{i}\rangle\}$.
Repres$^{\mathrm{n}}$	$\rho = \sum_{i_1 i_1' \cdots i_n i_n'} \rho_{\mathbf{i}\mathbf{i}'} \lvert\mathbf{i}\rangle\langle\mathbf{i}'\rvert$
Eigen	Diagonalize as $\rho = \sum_{\mathbf{i}} p_{\mathbf{i}} \lvert\psi_{\mathbf{i}}\rangle\langle\psi_{\mathbf{i}}\rvert$.
Pure	$\rho^2 = \rho \implies p_{\mathbf{i}} = \delta_{\mathbf{i}\mathbf{i}_0}$ and $\rho = \lvert\psi\rangle\langle\psi\rvert$ for some $\lvert\psi\rangle \in \mathscr{H}$.
Purify	$\exists \lvert\Psi\rangle \in \mathscr{H}_{\mathrm{A}} \otimes \mathscr{H}_{\mathrm{B}}$ s.t. $\rho_{\mathrm{A}} = \mathrm{tr}_B \lvert\Psi\rangle\langle\Psi\rvert$.
Entropy	von Neumann: $S(\rho) = -\mathrm{tr}(\rho \ln \rho) = -\sum_{\mathbf{i}} p_{\mathbf{i}} \log p_{\mathbf{i}}$.

Bipartite Entanglement

Separability

- Partition tensor-product space to two spaces: A & B.
- Separable (convex) $\mathcal{S}_{\text{sep}}$: $\approx \sum_{i=1}^{d} p_i \rho_i^{\text{A}} \otimes \rho_i^{\text{B}}$.
- $S(\rho\|\sigma) := \operatorname{tr}\left[\rho\left(\ln\rho - \ln\sigma\right)\right]$. $\mathbb{C} \to$ Kullback-Leibler div.
- Relative entropy: $E_{\text{R}}(\rho) = \inf_{\sigma\in\mathcal{S}_{\text{sep}}} S(\rho\|\sigma)$.

Local Operations & $\mathbb{C}$ Communication (LOCC)

- Entanglement non-increasing under LOCC (subset of SEP).
- Considered to be 'easy' operation and germane to $\mathbb{Q}$ tasks:

  Form make entangled ρ from copies of bipartite maximally entangled state (e.g., singlet $(|01\rangle - |10\rangle)/\sqrt{2}$);

  Distill make $nS(\rho)$ bipartite maximally entangled states from $\rho^{\otimes n}$.
- Establishes equivalence class for entangled states.
- Generalized to stochastic LOCC for e.g. three-qubit states.

Entanglement Monotone and Entanglement Measures

Monotone

$E(\rho) \geq 0$ with $= 0$ only if $\rho \in \mathcal{S}_{\text{sep}}$; E nonincreasing under LOCC.

Some measures

- Entanglement of distillation.
- Entanglement cost.
- Entanglement of formation
- Relative entropy of entanglement E_R.
- Distillable secret key rate.
- Squashed entanglement.

Plenio and Virmani *QIC* **7** 1–51 (2007).

Positive partial transpose (PPT)

- $\rho^{\mathbb{1}\otimes\mathrm{T}} := (\mathbb{1}\otimes\mathrm{T})\rho = \sum_{ijkl} p^{ij}_{kl} |i\rangle\langle j| \otimes |l\rangle\langle k|$.
- Separability implies nonnegative eigenvalues of $\rho^{\mathbb{1}\otimes\mathrm{T}}$.
- Negativity: sum of negative eigenvalues of $\rho^{\mathbb{1}\otimes\mathrm{T}}$.
- Separable iff PPT for $2\otimes 2$ and $2\otimes 3$; else NP-Hard[a].
- Log negativity $\log_2 \|\rho^{\mathbb{1}\otimes\mathrm{T}}\|$.
- Nondistillable: PPT with entanglement.

[a] Non-deterministic Polynomial-time-Hard: all NP problems poly-time reducible to this one.

Operational Relative Entropy of Entanglement

Regularized relative entropy of entanglement

$$E_r = \lim_{n\to\infty} \frac{1}{n} E_R(\rho^{\otimes n}).$$

Convert copies of ρ to copies of ρ'

$\rho^{\otimes n} \mapsto \sigma^{\otimes m_n}$ for $m_n \leq n$ at rate

$$\lim_{n\to\infty} \frac{m_n}{n} = \frac{E_r(\rho)}{E_r(\sigma)}.$$

Pure states & von Neumann entropy

$$E_R(|\Psi\rangle^{AB}\langle\Psi|) = -\mathrm{tr}(\rho_r \log \rho_r).$$

Squashed Entanglement

Extended state

$$\mathrm{ext}\rho_{\mathrm{AB}} := \{\rho_{\mathrm{ABE}}; \mathrm{tr}_{\mathrm{E}}\rho_{\mathrm{ABE}} = \rho_{\mathrm{AB}}\}$$

Quantum conditional mutual information

$$I(\mathrm{A};\mathrm{B}|\mathrm{E}) := S(\rho_{\mathrm{AE}}) + S(\rho_{\mathrm{BE}}) - S(\rho_{\mathrm{ABE}}) - S(\rho_{\mathrm{E}})$$

Squashed entanglement

$$E_{\mathrm{sq}}(\rho_{\mathrm{AB}}) := \frac{1}{2} \inf_{\rho_{\mathrm{ABE}} \in \mathrm{ext}\rho_{\mathrm{ABE}}} \{I(\mathrm{A};\mathrm{B}|\mathrm{E})\}$$

Additive under tensor product

$$E_{\mathrm{sq}}(\rho_{\mathrm{AB}} \otimes \rho_{\mathrm{A'B'}}) = E_{\mathrm{sq}}(\rho_{\mathrm{AB}}) + E_{\mathrm{sq}}(\rho_{\mathrm{A'B'}})$$

Christandl & Winter *J. Math. Phys.* **45** 829–840 (2004).

Entanglement Witness

Witness Existence Theorem

$\forall$ entangled ρ $\exists$ Hermitian W s.t. $\mathrm{tr}(W\rho) < 0$, $\mathrm{tr}(W\sigma) \geq 0$.

CHSH Bell Inequality

Construct CHSH operator as a witness.

Computational complexity and QSEP

The quantum separability problem (QSEP) asking whether given ρ is entangled vs separable is NP-hard if ρ is an inverse exponential distance with respect to dimension from the border of $\mathcal{S}_{\text{sep}}$.

Horodecki Horodecki Horodecki *Phys. Lett. A* **223** 1–8 (1996).

Monogamy & polygamy of entanglement

Sharing entanglement & monogamy

Given tripartite mixed state ρ_{ABC}, $E_{A|B}$ denotes entanglement shared between A & B & $E_{A|BC}$ denotes entanglement shared between A and jointly B & C. $\tau_{A|B|C} := E_{\mathrm{A|BC}} - E_{\mathrm{A|B}} - E_{\mathrm{A|C}} \geq 0$.

Distributing entanglement & polygamy

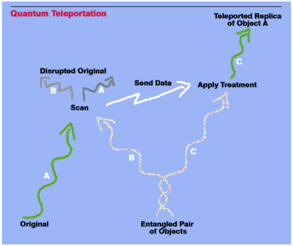
Mixed state ρ_{AB} can be purified to pure state $|\Psi\rangle_{\mathrm{ABC}}\langle\Psi|$, and C can assist A & B to share entanglement. The upper bound to sharing is the lower bound $\tau_{A|B|C}$ for distributing $\implies$ duality.

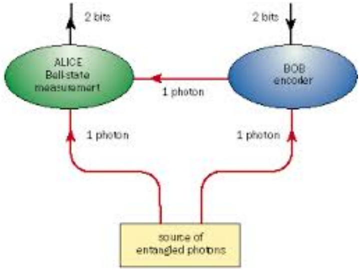
Squashed entanglement

Additive and monogamous but intractable.

Gour Kim Sanders *Contemp. Phys.* **53** 417–432 (2012).

Tasks


Quantum Teleportation
Teleported Replica of Object A
C
Disrupted Original
Send Data
Apply Treatment
B
A
Scan
B
C
A
Original
Entangled Pair of Objects


2 bits
2 bits
ALICE Bell-state measurement
BOB encoder
1 photon
1 photon
1 photon
source of entangled photons

Conclusion

- Entanglement is the property of nonseparability.
- Special types of entanglement are resourceful.
- Determining entanglement of a state is generically hard.