# Slow equilibration and Fluctuation Relations

# Quasi-integrable systems:

# four famous examples

- **Planetary systems**
- **Weakly nonlinear waves**
- **The Fermi-Pasta-Ulam chain**
- **Quantum chains**

# 1. Planetary systems

$\longrightarrow$ Kepler (Laplace-Lagrange) (integrable) + perturbation see Laskar

## Very short Lyapunov times. Unstable? Moser

| | |
|---|---|
| Mercury | 1.4M |
| Venus | 7.2 M |
| Earth | 4.8M |
| Mars | 4.5M |
| Jupiter | 8.4 M |
| Saturn | 6.4M |
| Uranus | 7.5M |
| Neptune | 6.7M |

**with some grains of salt (what is an individual Lyapunov?)**

# Diffusion of the eccentricity of Mercury, slightly different runs

Laskar

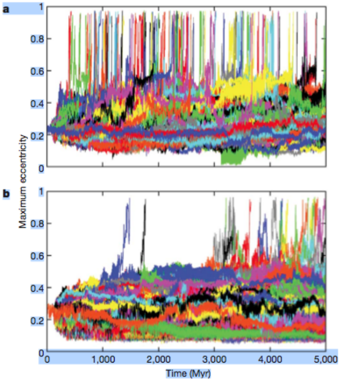
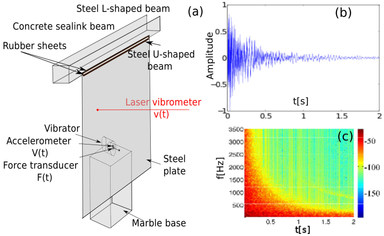



**Figure 1 | Mercury's eccentricity over 5 Gyr.** Evolution of the maximum

n.b. a thousand Lyapunov times

# 2. Weakly interacting waves

Zakharov, Nazarenko...




Düring, Rica, Josserand,...

$\rightarrow$ Plane waves (integrable) + perturbation

# 3. The Fermi-Pasta-Ulam chain

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{N} V(q_{i+1} - q_i)$$

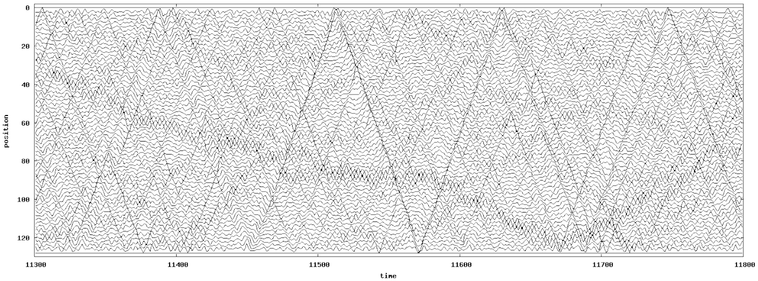
$$V(r) = \frac{r^2}{2} + \alpha\frac{r^3}{3} + \beta\frac{r^4}{4}$$

$$V(r) = \frac{r^2}{2} + \alpha\frac{r^3}{3} + \beta\frac{r^4}{4} = \underbrace{V_o(e^{\lambda r} - 1 - \lambda r)} + \text{`}small'$$

$\longrightarrow$ Toda (integrable) + perturbation (Benettin and Ponno)

# The Fermi-Pasta-Ulam chain

## Trajectories



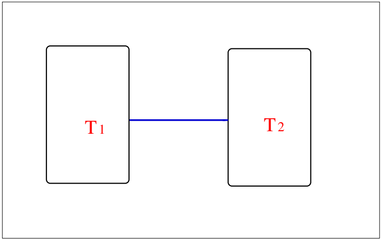
# 4. Quantum chains

$$H(J_2,\delta,U) = -J_1 \sum_{l=1}^{L} [1+(-1)^l \delta](c_l^\dagger c_{l+1} + \text{H.c.}) - J_2 \sum_{l=1}^{L} [c_l^\dagger c_{l+2} + \text{H.c.}] + U \sum_{l=1}^{L} n_l n_{l+1}.$$

Bertini, Essler, Groha, Robinson; DeLuca and Mussardo; ...

$\longrightarrow$ Various integrable limits

# Plus the simplest example:




$$P \propto e^{-\beta_1 E_1 - \beta_2 E_2} = e^{-\beta_+ E_+ - \beta_- E_-}$$

$E_+$ is conserved, $E_-$ is slowly evolving

$2\beta_+ = 1/T_1 + 1/T_2$ **and** $2\beta_- = 1/T_1 - 1/T_2$

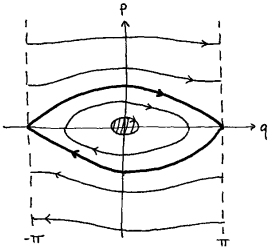
**clearly, temperatures associated with nonconserved quantities go to infinity**

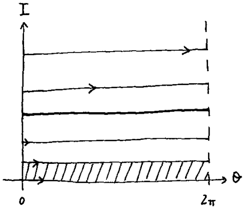
# Integrable systems

**$N$ independent constants of motion**

**Action $(I_1, ...I_N)$ and Angle $(\theta_1, ..., \theta_N)$ variables**

**Flow is *laminar*, restricted to tori $I_i = const$, $\theta_i = \omega_i t$**






**Action $(I_1, ...I_N)$ and Angle $(\theta_1, ..., \theta_N)$ variables**

# Action-angle representation

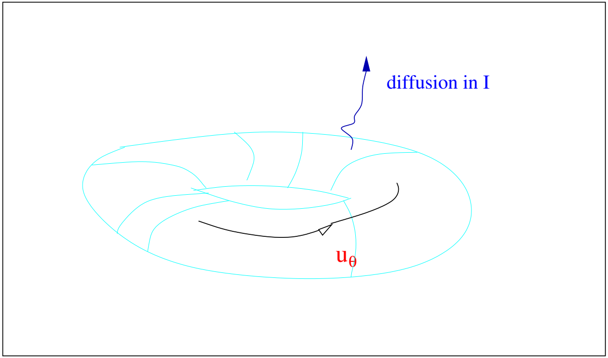
$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}
\end{aligned}
$$

---

$$
\dot{I}_i = -\frac{\partial H}{\partial \theta_i}
$$

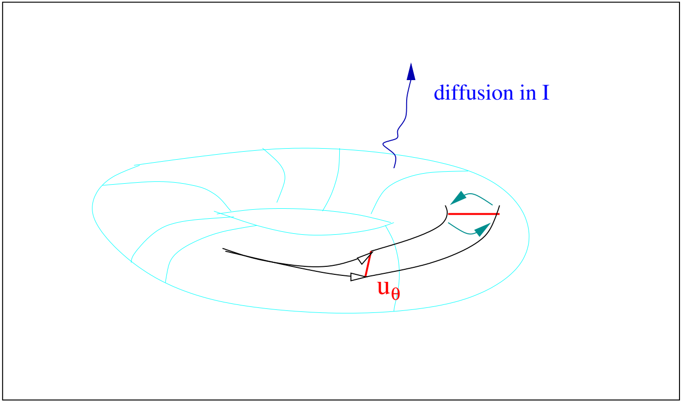
$$
\dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i(\mathbf{I})
$$

# Small perturbation:




**‘stability’ = lack of ergodicity**

# Lyapunov time is puzzlingly short




mostly chaos on the torus

Nguyen Thu Lam and JK

## We perturb with weak, additive noise

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}\\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + \varepsilon^{\frac{1}{2}}\xi_i(t)\end{aligned}$$

**consider the simple case in which the $\xi(t)$ are white noises:**

$$\langle \xi(t) \rangle = 0, \qquad \text{and} \qquad \langle \xi(t)\xi(t') \rangle = 2\delta(t - t').$$

**In the action-angle variables, the noise is no longer additive, and reads:**

$$\dot{I}_i = \varepsilon^{\frac{1}{2}} \sum_k \{I_i, q_k\} \xi_k(t)$$

$$\dot{\theta}_i = \omega_i + \varepsilon^{\frac{1}{2}} \sum_k \{\theta_i, q_k\} \xi_k(t)$$

# We concentrate for a while on FPU

# Constants of motion of FPU: differences of eigenvalues of

$$L = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdot & \cdot & \cdot & a_N \\ a_1 & b_2 & a_2 & 0 & & & & \\ 0 & a_2 & b_3 & a_3 & & & & \\ 0 & 0 & a_3 & b_4 & & & & \\ \cdot & & & & \cdot & & & \\ \cdot & & & & & \cdot & & \\ \cdot & & & & & & \cdot & \\ a_N & & & & & & & b_N \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdot & \cdot & \cdot & -a_N \\ -a_1 & 0 & a_2 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -a_2 & 0 & a_3 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ a_N & & & & & & & 0 \end{pmatrix}.$$

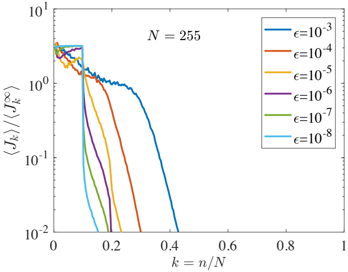
A simple computation shows that

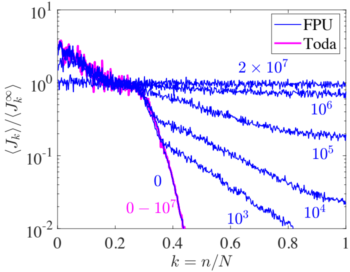
$$\dot{L} = [B, L] \equiv BL - LB .$$

$$a_n = \tfrac{1}{2} e^{-(Q_n - Q_{n-1})/2},$$

$$b_n = -\tfrac{1}{2} P_{n-1} .$$

$$H = \sum_{n=1}^{n} \left( \tfrac{1}{2} P_n^2 + e^{-(Q_n - Q_{n-1})} \right),$$



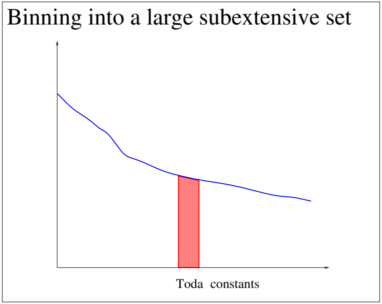

**Thermalization of FPU** analogous to Benettin and Ponno

**Evolution of Toda constants of motion** $J_1, \ldots, J_N$

# Generalized Gibbs Ensemble

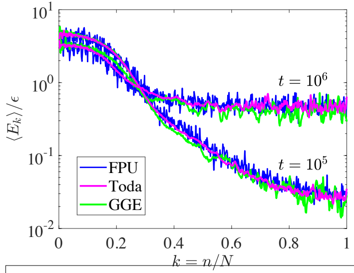
Rigol, Dunjko, Olshanii, Yurovsky

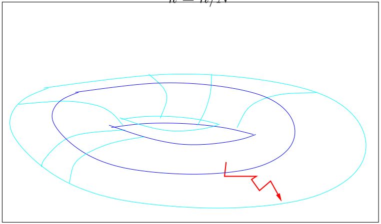
$$P \propto e^{-\sum \beta_q J_q}$$




**analogous to hydrodynamic limit**

Coarse-graining is an optional, but the only way to get non-fluctuating quantities, self-averaging and equivalence of ensembles



## From Toda torus to Toda torus

**Evolution of Toda constants of motion** $J_1, ..., J_N$

# An algorithm:

- **Compute time derivative of binned quasi-constants of motion ‘within a torus’**
- **Evolve the quasi-constants linearly**
- **Find a point with new constants (GGE or ‘generalized microcanonical’) and iterate**

**Works with FPU.**
**What about planets? And quantum?**

**A numerical version of wave turbulence...**

# Diffusion of the eccentricity of Mercury, slightly different runs

Laskar

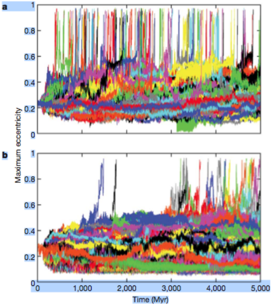


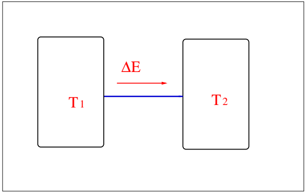

**Figure 1 | Mercury's eccentricity over 5 Gyr.** Evolution of the maximum

Is it possible to apply the numerical strategy for this case, computing at each level rate of change and diffusion constants of orbital parameters?

**Here we do not have determinism!**

# A fluctuation theorem

# A conduction problem, we expect a fluctuation theorem




of the type $P(heat)/P(-\ heat) = e^{\beta\ heat}$

Indeed: see Jarzynski and Wojcik

$$\frac{P(\Delta E_-)}{P(-\Delta E_-)} = e^{\beta_- \Delta E_-}$$

# With the analogy with quasi-constants of motion, we are led to the quantum and classical more general result

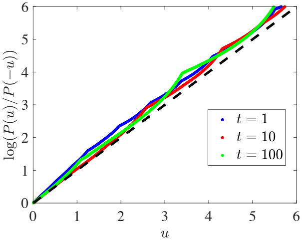
T Goldfriend and JK, see Hickey and Genway also J. Mur-Petit, A. Relao, R. A. Molina, D. Jaksch

$$\frac{P(\Delta Q)}{P(-\Delta Q)} = e^{Q}$$

$$Q = \sum_r \beta_r J_r$$

**which describes change of charges**

**and violations of irreversibility as the system thermalizes (the $\beta_r$ going to infinity).**




# FPU chain, Toda constants, $N = 15$

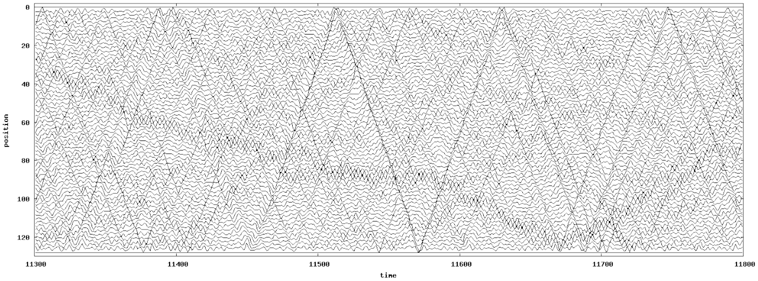
**Usually, fluctuation theorems are limited to very small systems and/or very short times.**

**Here the situation is different: the 'size' parameter is the integrability violation, so that even a macroscopic systems may exhibit 'reversals'**

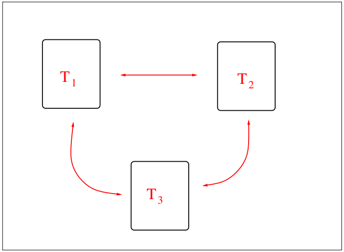
For example, it applies to a large FPU chain with a small number of decaying solitons...

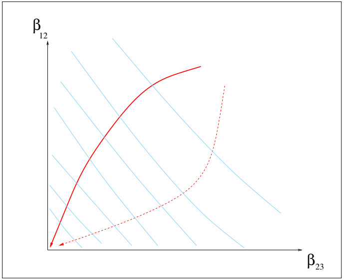
***What is the connection with the dynamical effective temperatures?*** Cugliandolo, Gambassi, Foini and Konik

# Trajectories



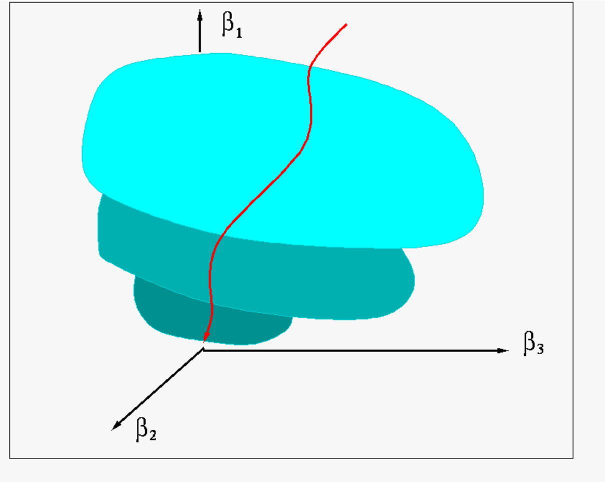
# The next-to-simplest GGE:



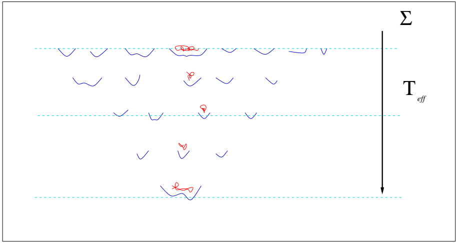

**downhill in constrained entropy (free energy) but (unfortunately) not necessarily along a gradient**

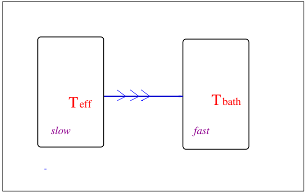
## Skiing in $\beta$ space downhill in $-S$:




**(Prob downhill) / (Prob uphill) $\rightarrow$ fluctuation theorem**

# Glass annealing may be schematically seen as:



**Expressed this way, the effective temperatures in glasses and in weakly integrable models seem very close**

**Fluctuation theorems for slowly evolving glasses?**