Viscosity divergence and dynamical slowing down at the jamming transition

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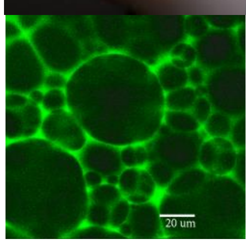
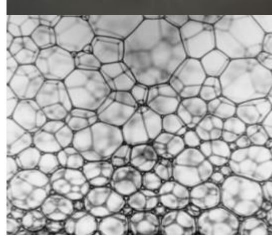
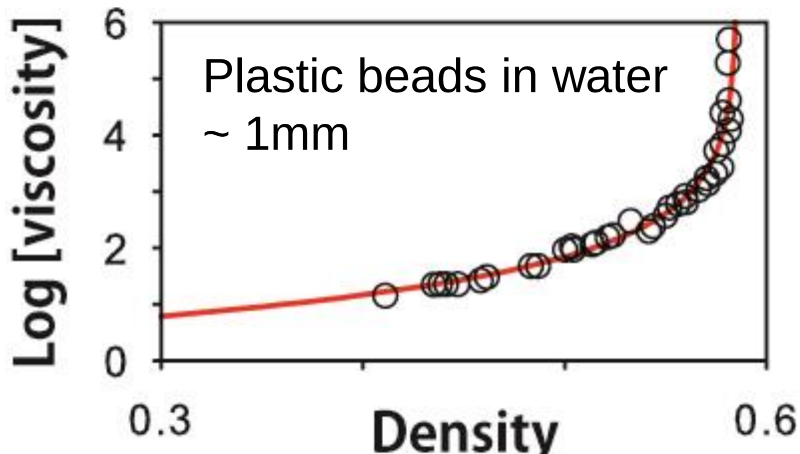
Jamming transition

- **Athermal particles**: particles that are large enough to be free from Brownian motions
  - Large colloidal particles, emulsions, foams, granular materials etc
- **Jamming transition = Viscosity divergence of athermal particles at the critical densitiy ~ random close packing density.**





$$\eta \propto (\varphi_{\mathrm{J}} - \varphi)^{-\gamma}, \quad \gamma \approx 2$$

[Boyer et al. 2011]

Glass vs Jamming

- **Glass transition = Viscosity divergence of thermal particles**
  - Small colloidal particles, atoms, molecules etc
  - Viscosity increases following **Vogel-Fulcher law**
  - **Relaxation time increases** as the viscosity increases (Green-Kubo formula)

- **Jamming transition = Viscosity divergence of athermal particles**
  - Large colloidal particles, foams, grains etc
  - Viscosity diverges following **the power-law**
  - **Relaxation time do NOT increase** near the transition

→ I will explain these points.

[Ikeda et al. 2012]

Jamming: Simple model

Athermal frictionless soft particles

Inter-particle interaction

$$v(r) = \epsilon \delta^2$$

$$\delta = \begin{cases} 1 - r/a & (r \le a) \\ 0 & (r > a) \end{cases}$$

Overlap length between particles

* Energy penalty when overlapping
* Finite range, repulsive
contacts between particles are well-defined

Overdamp equation of motion

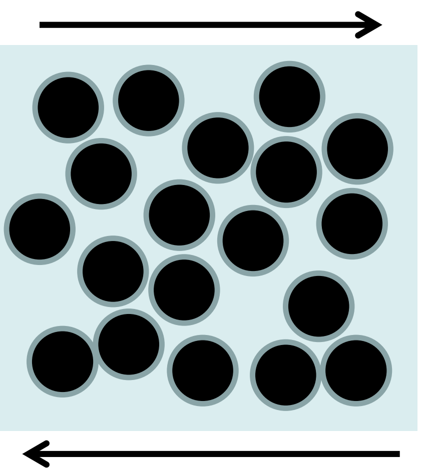
$$\xi \left(\frac{\partial \vec{r}_i}{\partial t} - \dot{\gamma} y_i \vec{e}_x \right) = - \sum_{j \neq i} \frac{\partial v(|\vec{r}_i - \vec{r}_j|)}{\partial \vec{r}_i}$$

Viscous dissipation

ξ = damping coefficient

$\dot{\gamma}$ = shear rate

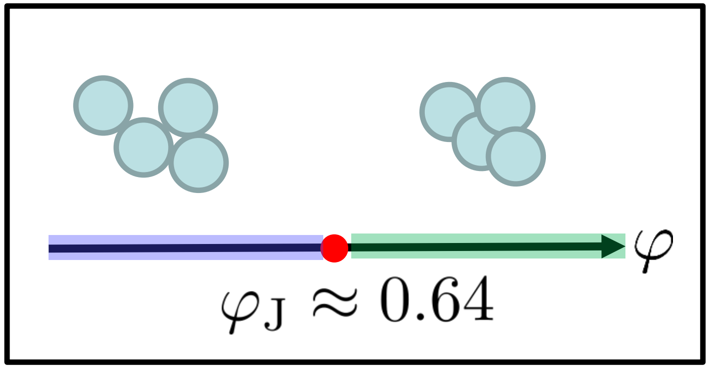
Inter-particle interaction



[Olsson-Teitel 2007, etc.]

Jamming: Phase diagram

- **Control parameter**
  - Packing density φ (or pressure)
  - Shear rate $\dot{\gamma}$ (or shear stress)



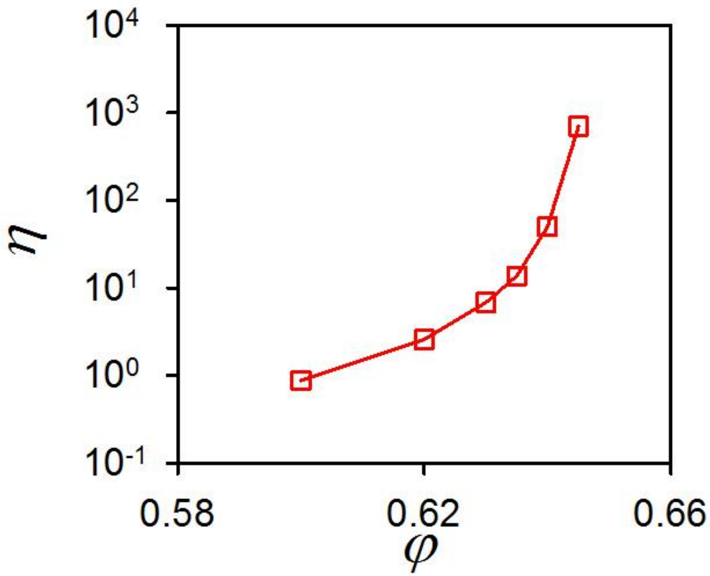
- **Phase diagram at low shear rate limit**
  - Low density: $\varphi < \varphi_{\mathrm{J}}$
    - Newtonian flow
    - Particles are just touching each other
  - High density: $\varphi > \varphi_{\mathrm{J}}$
    - Yielding of solid
    - Particles are overlapping
  - Critical density: $\varphi = \varphi_{\mathrm{J}}$
    - Marginally stable solid
    - Number of contacts per particle becomes isostatic

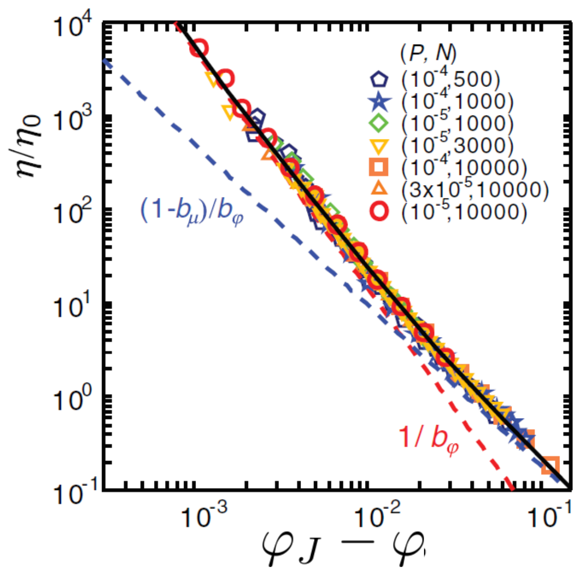
$$z = 2d \ (= 6)$$

[O'Hern 2003, Olsson-Teitel 2007]

Jamming: Viscosity

- **Newtonian viscosity** (at low shear rate)





- **Power-law divergence**

$$\eta \sim (\varphi_J - \varphi)^{-\gamma}, \qquad \gamma \in [2.6, 2.8]$$

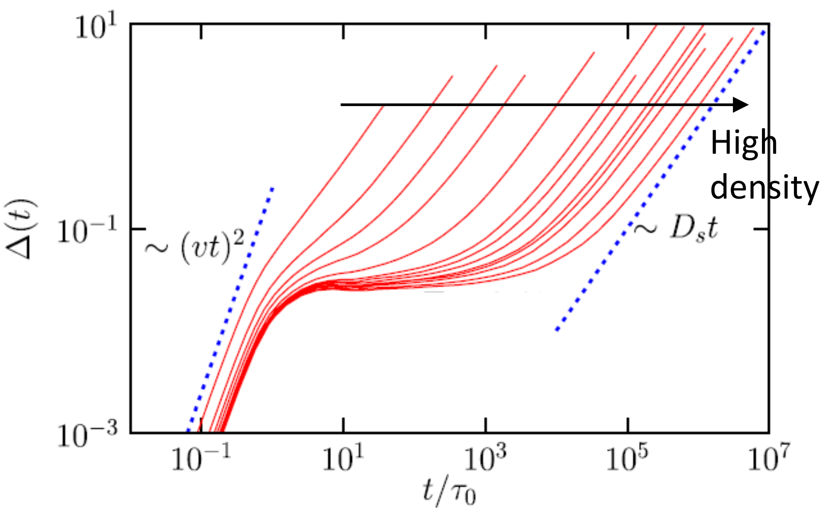
- Note: Exponent changes near the transition (1.7 → [2.6,2.8])

[Lerner et al. 2012, Kawasaki et al. 2015]

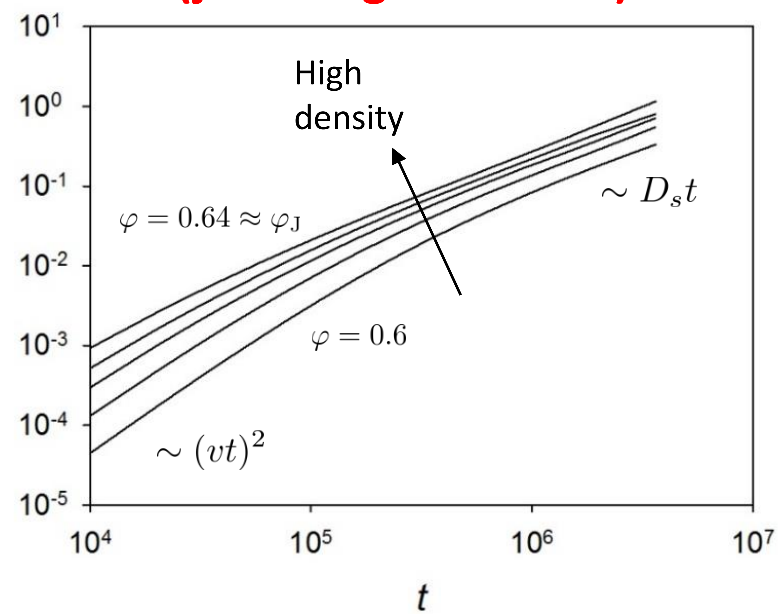
Jamming: Dynamics

- **Mean-square displacement** (at low shear rate, Newtonian regime)

Thermal particles (glass transition)



Athermal particles (jamming transition)



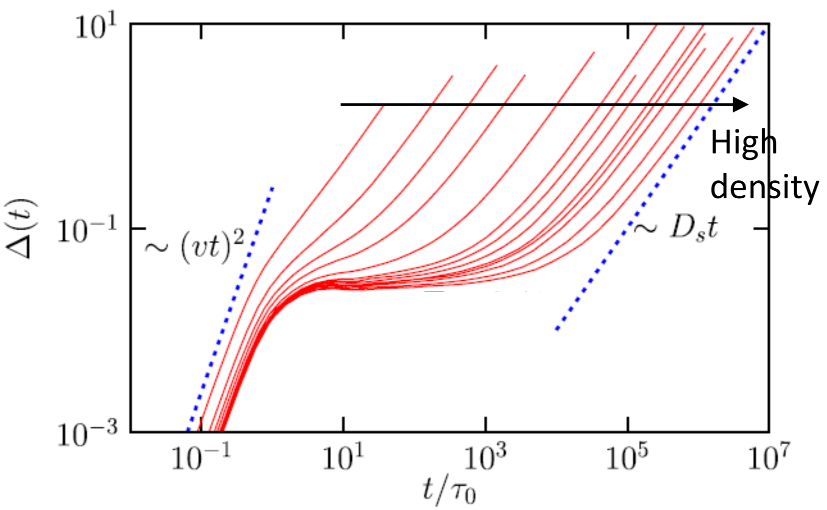
- **Jamming transition do not slow down the dynamics.**

[Heussinger et al. 2010]

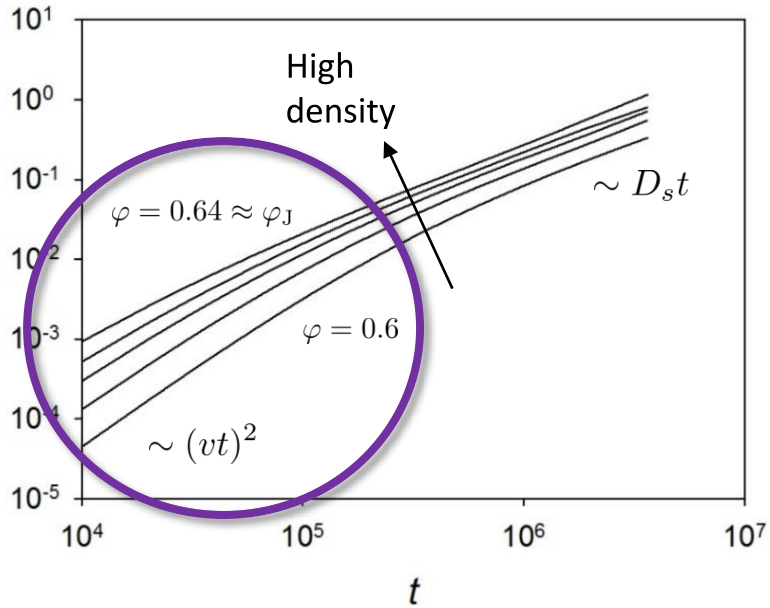
Jamming: Dynamics

- **Mean-square displacement** (at low shear rate, Newtonian regime)

Thermal particles (glass transition)



Athermal particles (jamming transition)

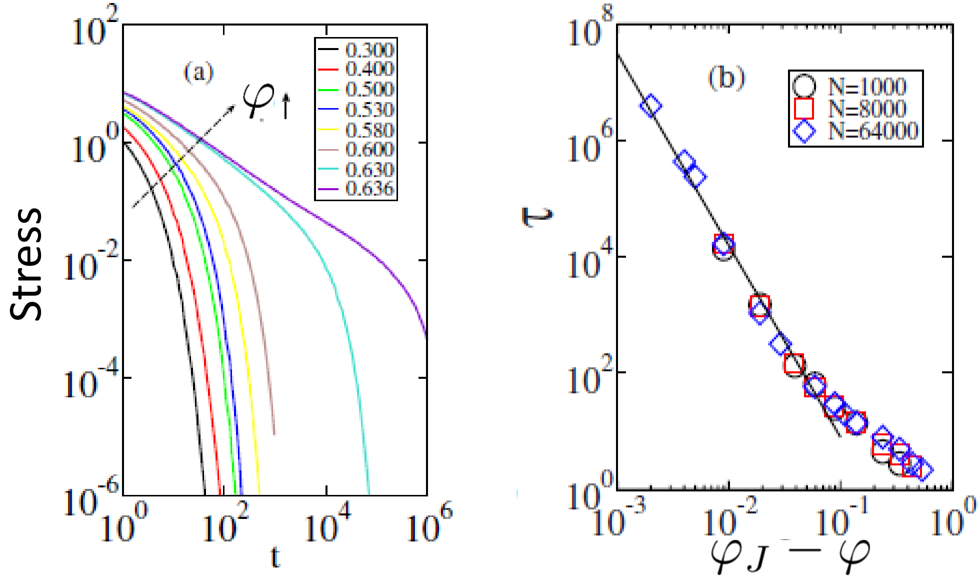


- **Jamming transition do not slow down the dynamics.**
- **Jamming even “speeds up” the short-time ballistic dynamics!**

[Heussinger et al. 2010]

Slowing down (1)

- **However in several settings, slowing down near the jamming have been observed**
  - Prepare particles configuration by the steepest descent without shear
  - Apply an infinitesimally small step strain.
  - Then, the system is relaxed by overdamped dynamics without shear and the relaxation of the stress is studied



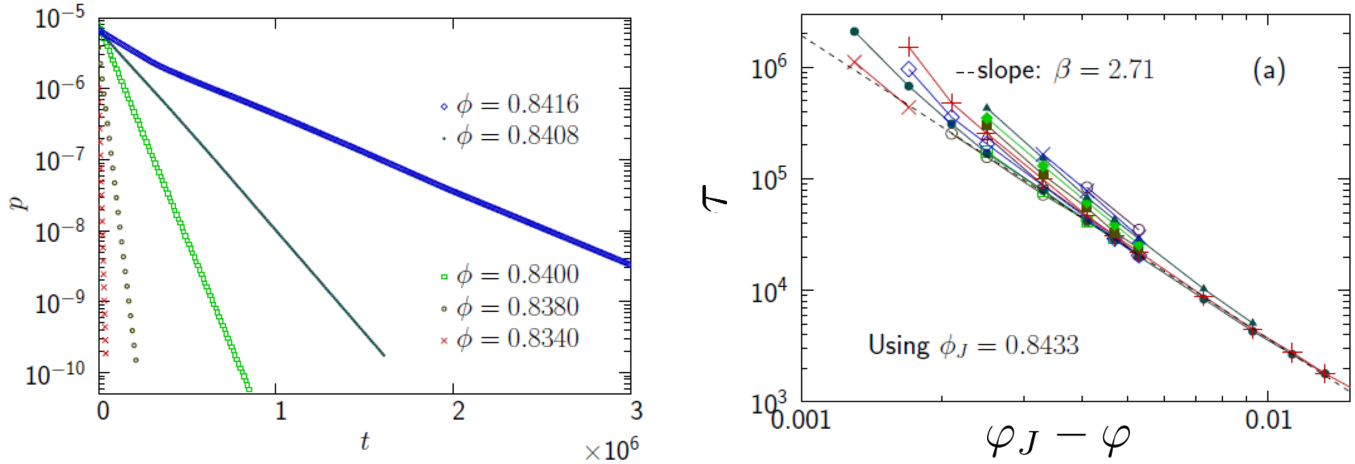
Relaxation time diverges at the jamming transition

$$\tau \propto (\varphi_J - \varphi)^{-3.3}$$

[Hatano 2010]

Slowing down (2)

- **However in several settings, slowing down near the jamming have been observed**
  - Perform overdamped dynamics with shear
  - Stop the shear. Then, the system is relaxed by overdamped dynamics without shear, and the relaxation of the pressure is studied



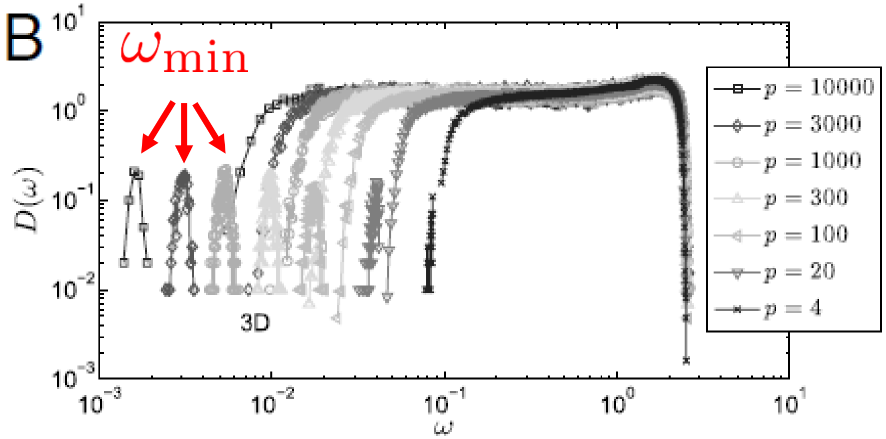
- **Viscosity and relaxation time diverges with the same power-law**

$$\eta \propto \tau \propto (\varphi_{\rm J} - \varphi)^{-2.7}$$

[Olsson 2014]

Slowing down (3)

- **However in several settings, slowing down (?) near the jamming have been observed**
  - Perform event-driven dynamics of athermal hardparticles with shear
  - Then, for the steady-state configurations, connect particles with spring and study the vibrational density of states



$$\omega_{\rm min} \sim (\Delta z)^{1.4} \sim (\varphi_{\rm J} - \varphi)^{-1.4}$$

- For hardspheres, one can show that $\eta \propto \omega_{\rm min}^{-2} \propto (\varphi_{\rm J} - \varphi)^{-2.8}$

[Lerner et al. 2012]

This work

? Speed up near the jamming in steady-state, while slowing down in some cases. Why ?

? Are all the observations of the slowing down related to the viscosity divergence ?

This work:

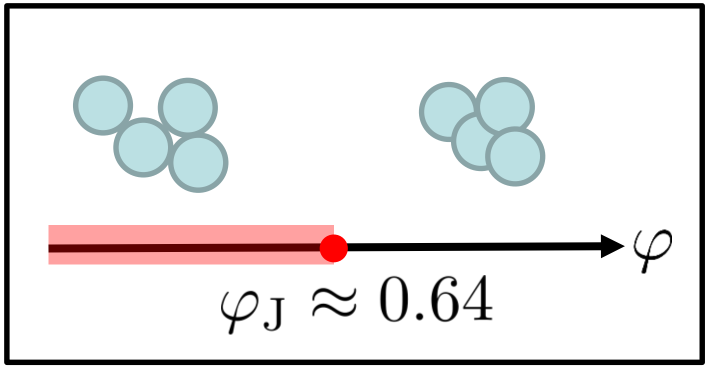
Comprehensive study of the dynamics of the system near the jamming transition.

Setting

- **Put particles randomly in a box.**

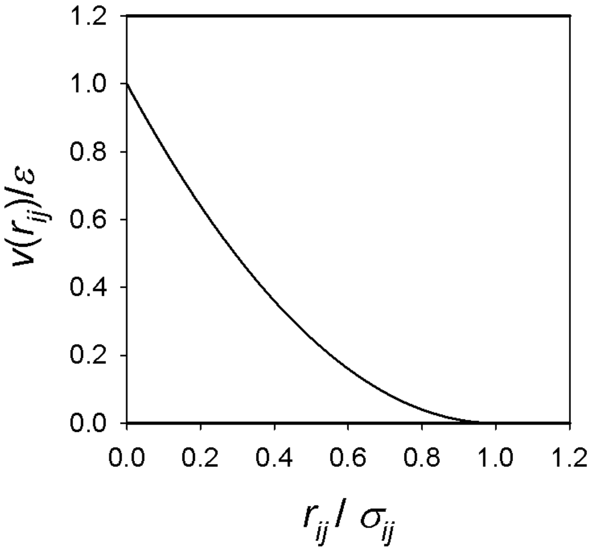
Focus only on the unjammed phase:

$$\varphi < \varphi_{\mathrm{J}}$$



- **Inter-particles interaction is**

$$v(r) = \begin{cases} \epsilon(1 - r/a)^2 & (r \leq a) \\ 0 & (r > a) \end{cases}$$

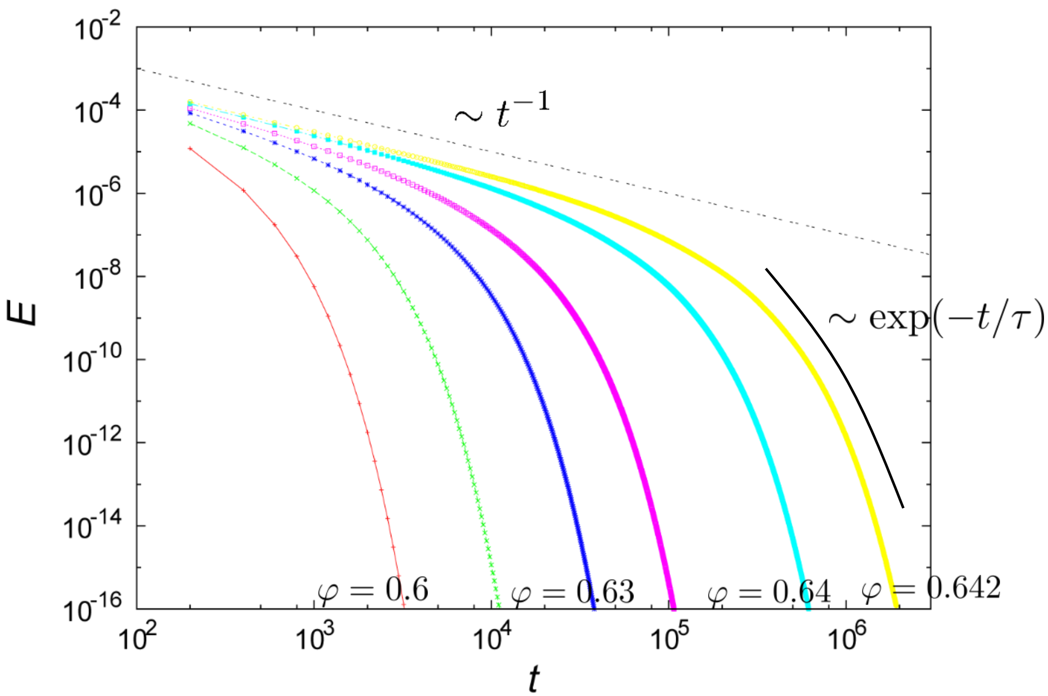


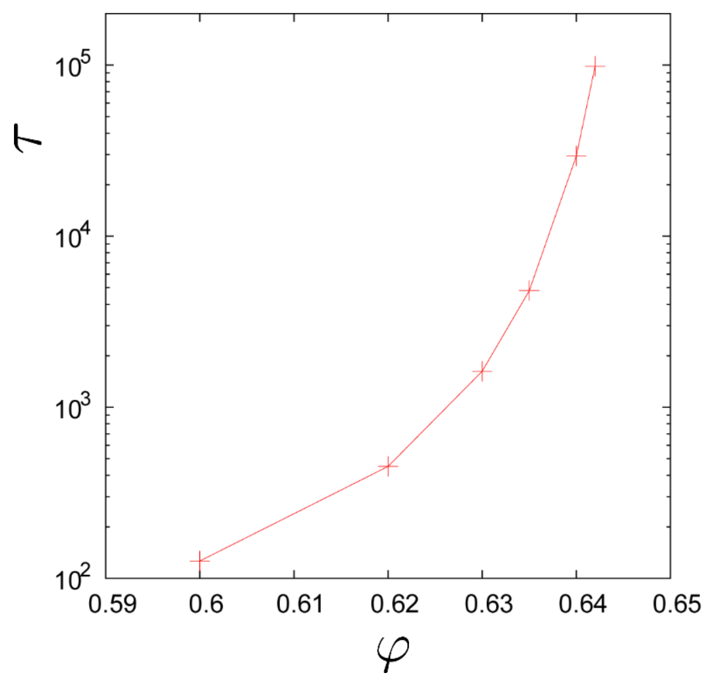
- **Then study the relaxation dynamics (No shear)**

$$\xi \frac{\partial \vec{r}_i}{\partial t} = -\sum_{j \neq i} \frac{\partial v(|\vec{r}_i - \vec{r}_j|)}{\partial \vec{r}_i}$$

Relaxation dymamics

- **Relaxation dynamics of the potential energy**

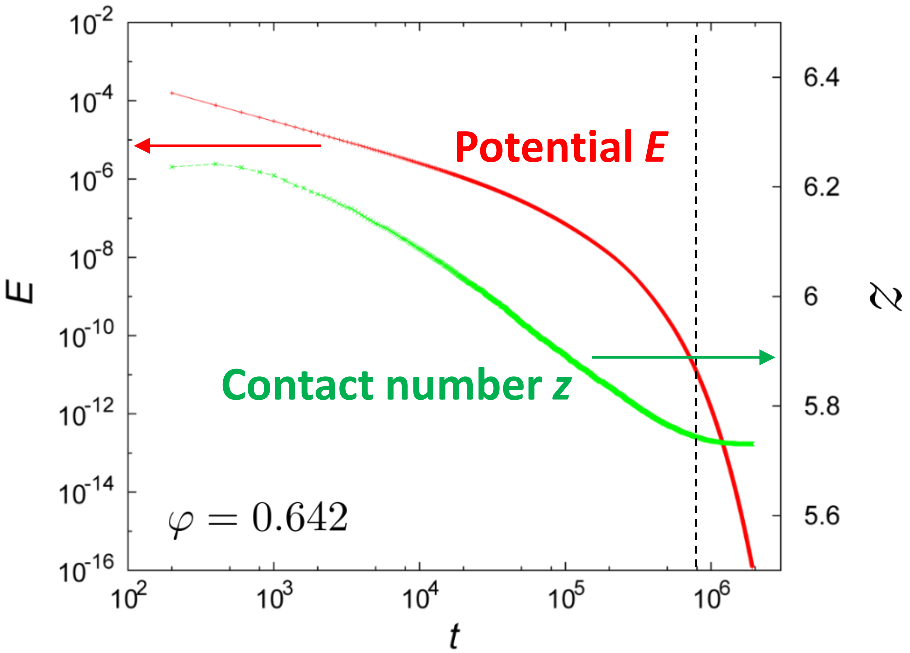




- Power-law*exponential: $E(t) \sim t^{-1} \exp(-t/\tau)$
- Relaxation time τ diverges at $\varphi \to \varphi_{\mathrm{J}}$

Contact number

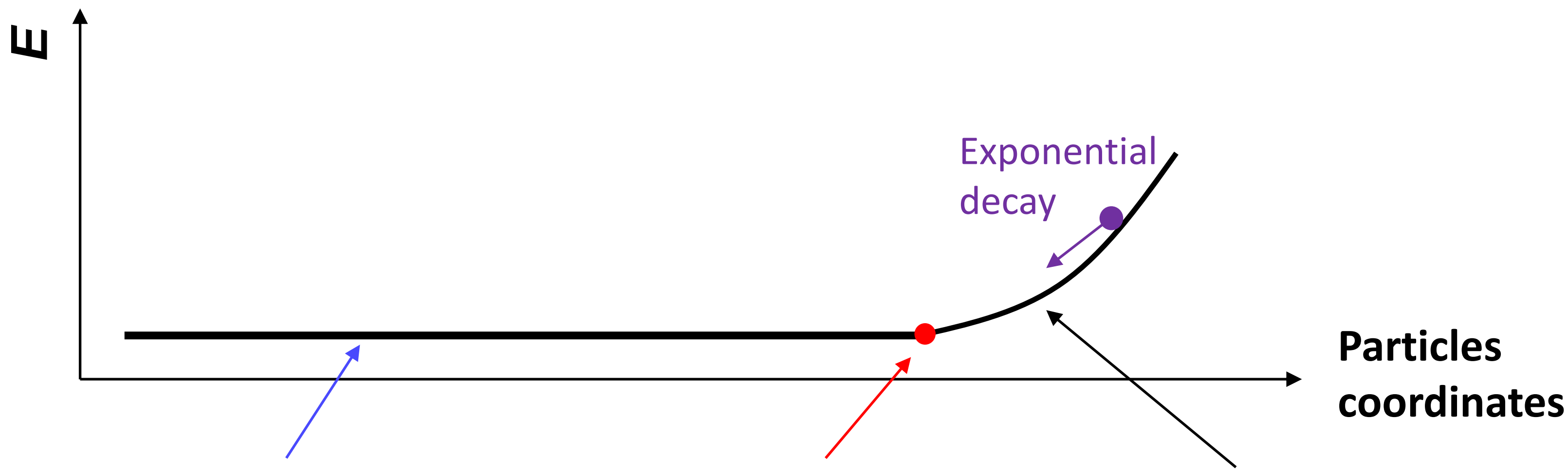
- Relaxation dynamics of contact number z
  - Contact number = average number of overlapping particles per particle



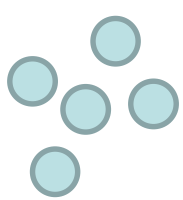
- Power-law region： z decreases (contacts are broken)
- **Exponential region： z converges into a constant**
  - **Relaxation without changing the contact network of particles**

What’s going on?

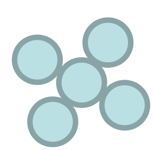
- Potential energy landscape at $\varphi < \varphi_J$



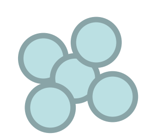
* Zero-energy configurations
* Particles are separated by finite distance, so the contact number is zero



* Zero energy configurations
* Particles are just touching, so the contact number is finite



* Finite energy configurations
* Particles are overlapping
* The contact network is the same as ● state



What’s going on?

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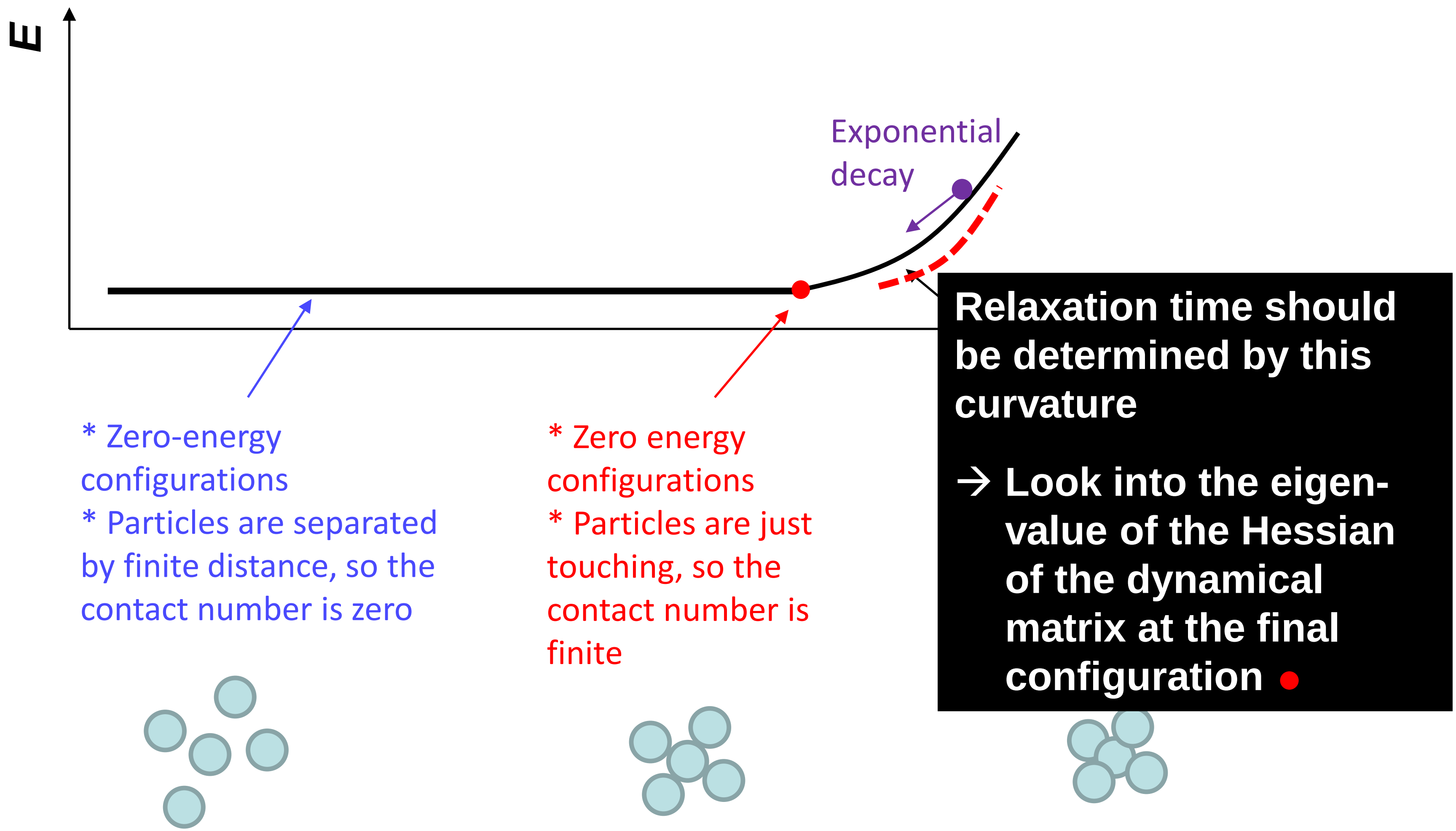


Eigenvalues

- For each φ , many final configurations are obtained from many initial configurations
- Obtain the Hessian of each final configuration

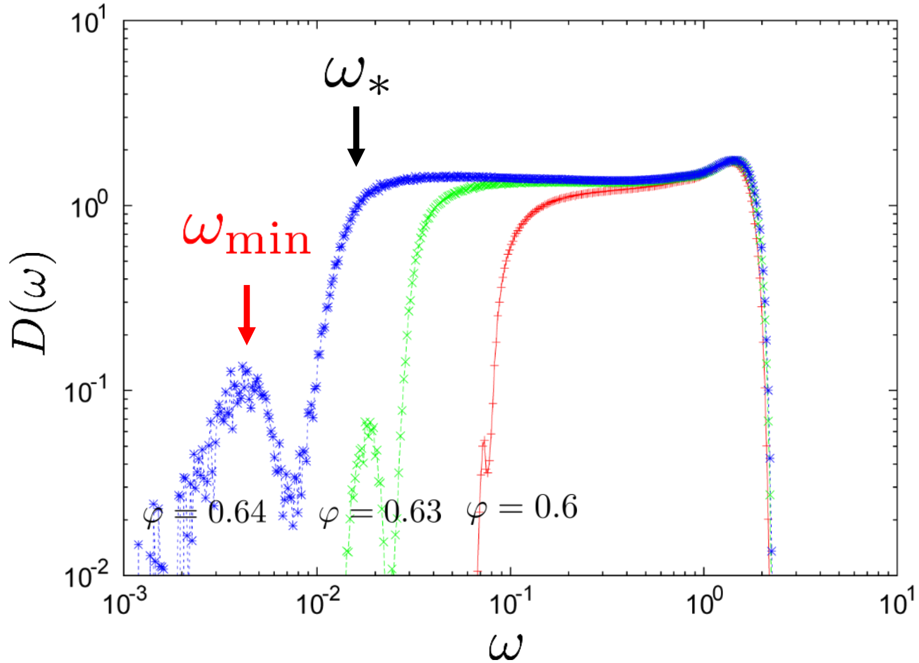
$$\mathbf{M} \equiv \frac{\partial^2 V}{\partial \vec{r}_i \partial \vec{r}_j} \qquad V = \sum_{i,j} v(|\vec{r}_i - \vec{r}_j|)$$

- Diagonalize the Hessian, obtain the eigenvalues $\{\lambda_\alpha\}$, calculate the vibrational density of states:

$$D(\omega) \equiv \frac{1}{N} \sum_\alpha \delta(\omega - \omega_\alpha) \qquad \omega_\alpha \equiv \sqrt{\lambda_\alpha}$$

  - We ignore the 3N*(6-z) zero modes, because we consider the relaxation from finite energy configurations
- Average $D(\omega)$ over obtained final configurations

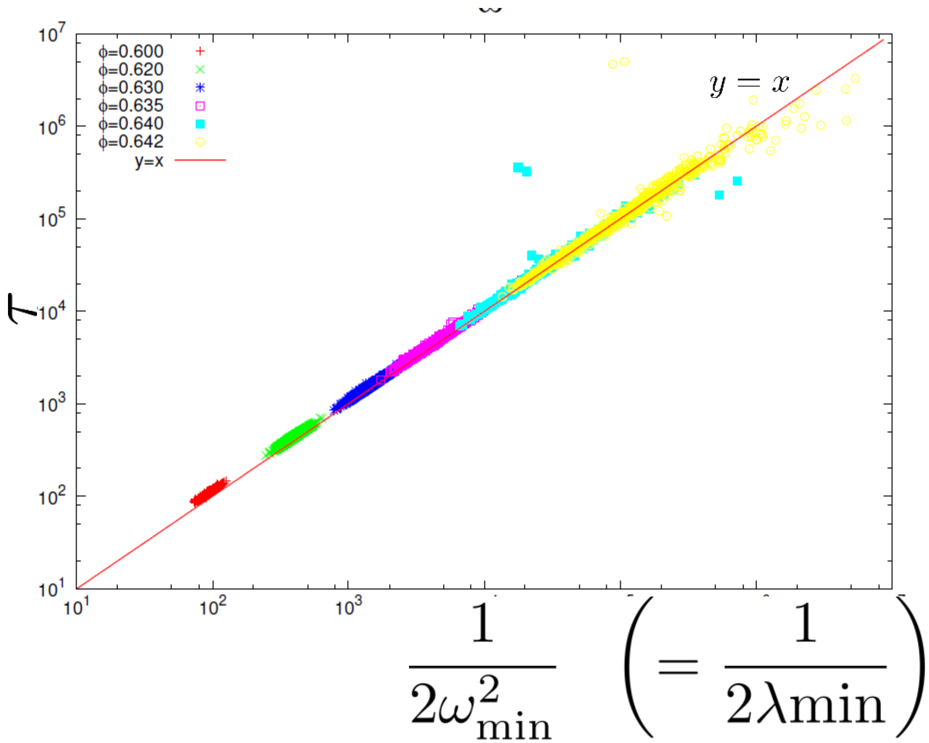
Eigenvalues

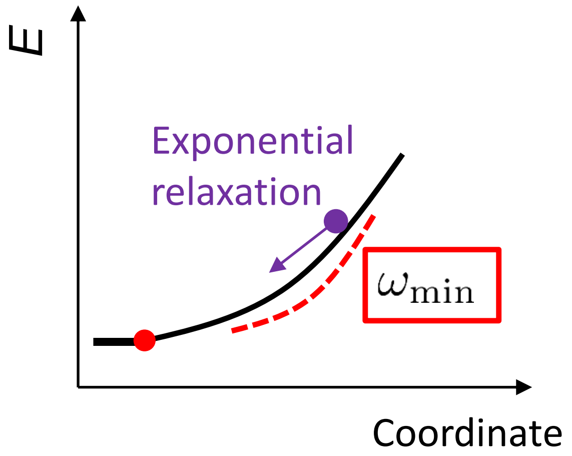


- $\omega > \omega_*$ ：Flat density of states
- $\omega = \omega_{\rm min}$ ：**There is one anomalously soft mode**
  - This mode is one isolated mode for one configuration

Relaxation time vs $\omega_{\min}$

- Plot the relaxation time against $\omega_{\min}$ for each configuration



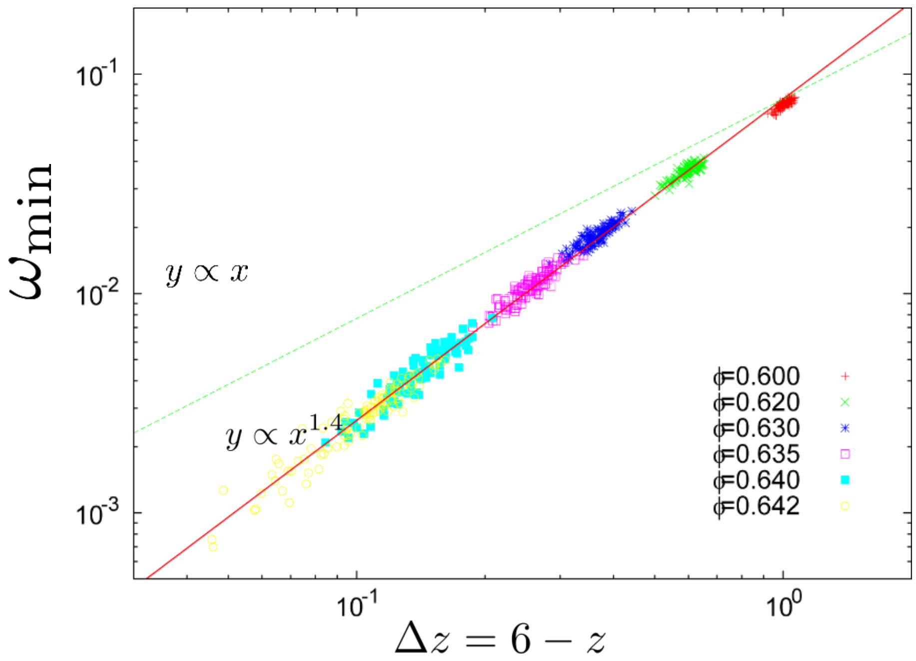


One configuration has only one extremely soft mode.

Exponential relaxation is along this mode.

Critical behavior

- The terminal contact number vs $\omega_{\min}$



- Lowest frequency: $\omega_{\min} \propto \Delta z^{1.4} \propto (\varphi_{\mathrm{J}} - \varphi)^{1.4}$

- Relaxation time： $\tau \sim (\varphi_{\mathrm{J}} - \varphi)^{-2.8}$

Summary of the first part

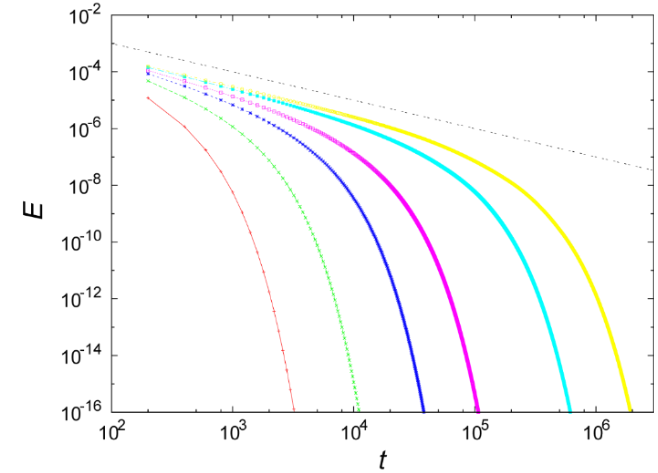
◆ **The terminal relaxation is exponential and without changing the contact network.**

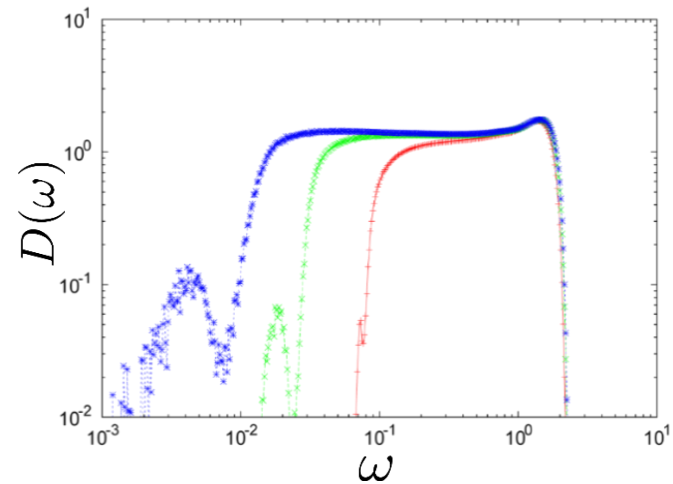
◆ **Each unjammed configuration has one anomalously soft mode. The terminal relaxation is along this mode.**

◆ **The relaxation time (= inverse of the eigenvalue of the anomalously soft mode) diverges as**

$$\tau \sim (\varphi_{\mathrm{J}} - \varphi)^{-2.8}$$

→ **Seemingly, the exponent is the same as that of the viscosity!**

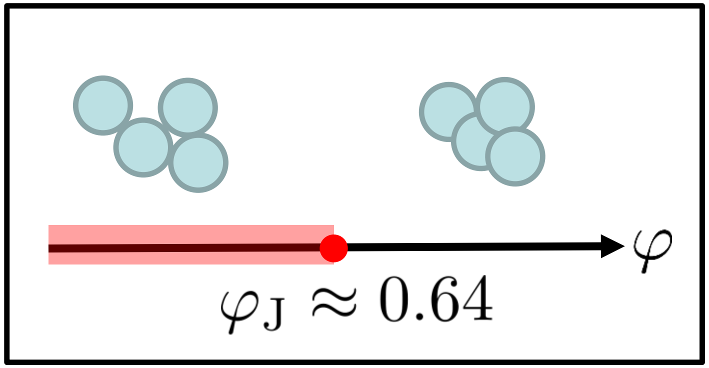




Setting

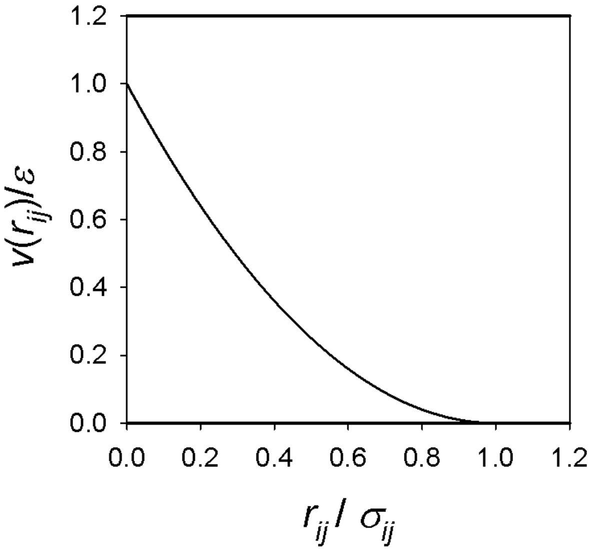
◆ **The same system as the first part, but consider the dynamics under shear.**

$$\xi\left(\frac{\partial \vec{r}_i}{\partial t} - \dot{\gamma} y_i \vec{e}_x\right) = -\sum_{j\neq i} \frac{\partial v(|\vec{r}_i - \vec{r}_j|)}{\partial \vec{r}_i}$$



◆ Shear rate is low enough to be in the Newtonian regime

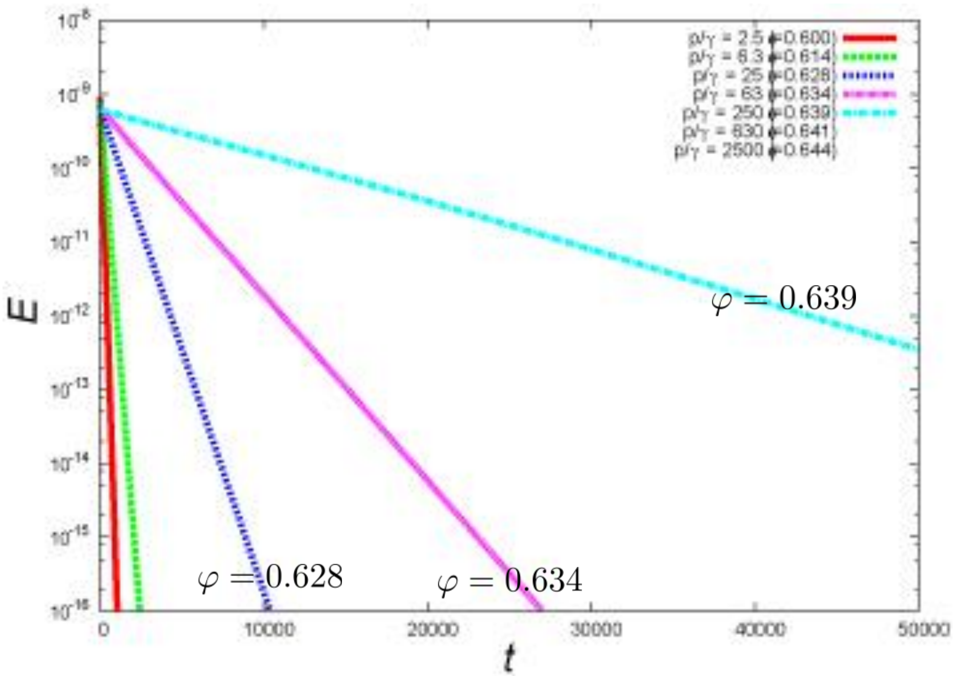
◆ **After achieving the steady state, stop the shear suddenly** (set $\dot{\gamma} = 0$) **and study the relaxation dynamics without shear.**



[2D case: Olsson 2014]

Relaxation dymamics

- **Relaxation dynamics of the potential energy**



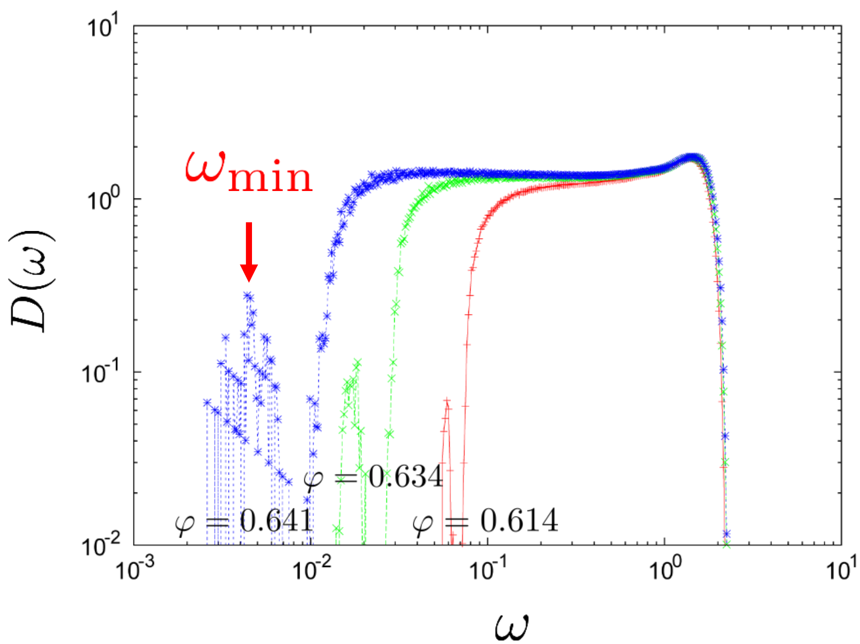
- Purely exponential relaxation

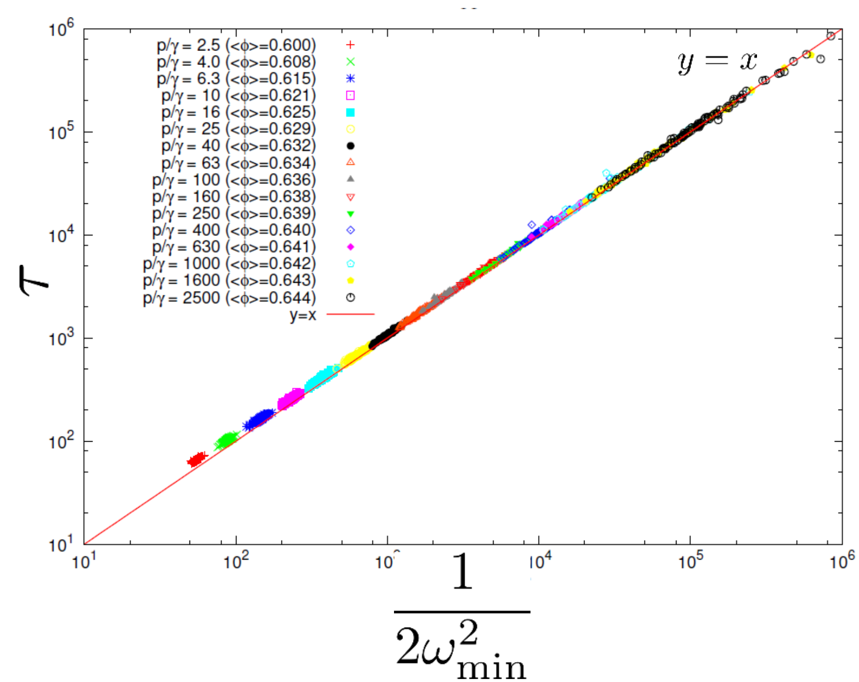
$$\sim \exp(-t/\tau)$$

[2D case: Olsson 2014]

Eigenvalues

- Vibrational density of state at the terminal configurations

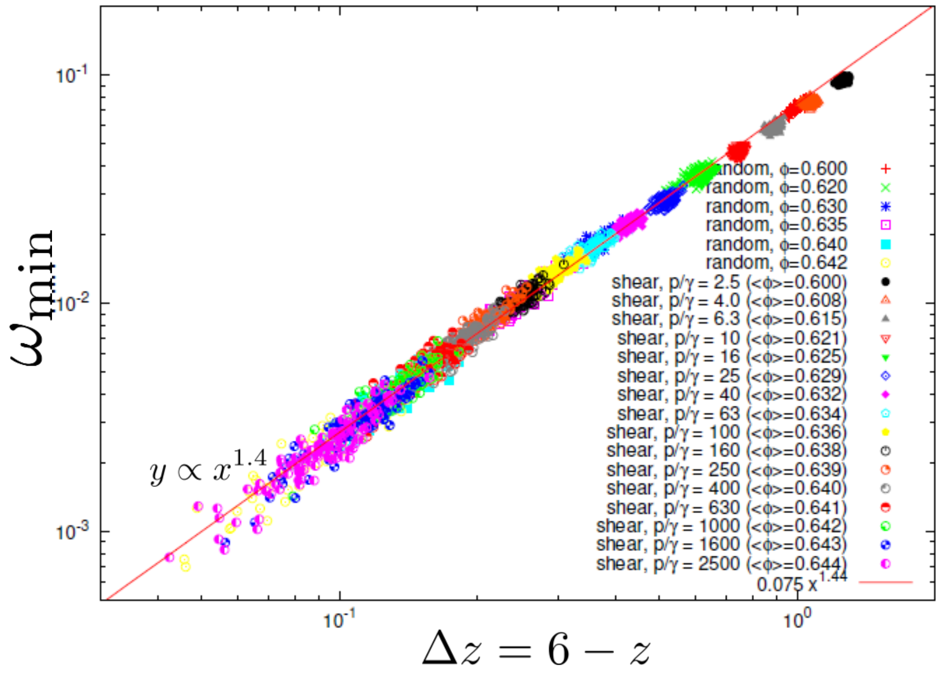




- **As in the case of random case, the configurations has anomalously soft mode.**
- **Single exponential relaxation = Only the softest mode is excited under the shear.**

Critical behavior

- $\omega_{\min}$ vs contact number, with/without shear
  - **Raw data are plotted as they are**



- **The law of the softest mode are exactly the same in both cases**

$$\omega_{\min} \propto \Delta z^{1.4} \propto (\varphi_{\mathrm{J}} - \varphi)^{1.4}$$

Summary

- Relaxation dynamics both from random and sheared configurations becomes sluggish near the jamming.

$$\tau \sim (\varphi_J - \varphi)^{-2.8}$$

- This relaxation is along the anomalously softest mode. The laws of the anomalous modes are the same in both random and sheared cases

$$\omega_{\min} \propto \Delta z^{1.4} \propto (\varphi_J - \varphi)^{1.4}$$

The exponent of the viscosity divergence is more universal than expected.

- Anomalously soft mode expresses that the contact forces are nearly balanced near the jamming transition.
- The speed up of the MSD is caused by the fact that increase of the contact force smears the slowing down.