Generating a second-order topological insulator with multiple corner states by periodic driving

Diptiman Sen

Indian Institute of Science, Bengaluru

Email: diptiman@iisc.ac.in

Outline

- Introduction to topology and topological systems
- Model of a second-order topological insulator
- Chern number, diagonal winding number and corner states
- Anisotropic model
- Generating multiple corner states by periodic driving
- Contributions of different momenta to topological transitions

Seshadri, Dutta and Sen, arXiv:1901.10495

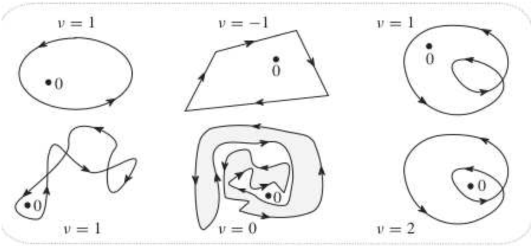
What is topology ?

Topology is a branch of mathematics where we study those properties of a system which remain the same if small changes are made in the system

If we can define an integer which remains the same under small changes, it is called a topological invariant

Closed curve in two dimensions

An example of a topological invariant is the number of times a closed curve in a plane winds around the origin in the anticlockwise direction. This integer is called the winding number



http://usf.usfca.edu/vca/PDF/vca-winding.pdf

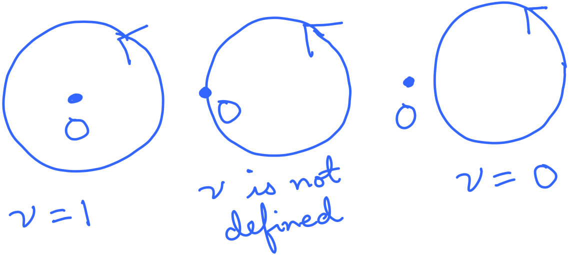
The winding number cannot change unless the curve passes through the origin

Robustness of topological invariants

Since a topological invariant is an integer, it cannot change slowly if some small changes are made in the system

The only way for a topological invariant to change is to become ill-defined at some point

For example, the winding number of a closed curve in a plane can only change if, at some point, the curve goes through the origin. Exactly at that point, the winding number is ill-defined



Topological systems

A topological condensed matter system has the following properties

- The bulk of the system is gapped, namely, there is a finite energy gap between the ground state and the first excited state. Hence the bulk is an insulator at low temperatures

- The bulk band structure is characterized by a topological invariant which is a non-zero integer

- There are gapless states at the boundaries of the system; these contribute to electronic transport

- Bulk-boundary correspondence: The number of boundary states is equal to the topological invariant; it does not change if the parameters in the Hamiltonian are changed a bit or if a small amount of disorder is present

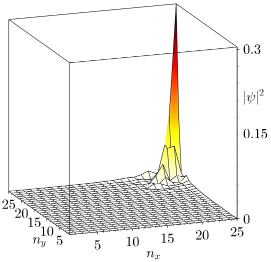
Hasan and Kane, Rev. Mod. Phys. 82, 3045 (2010)

Qi and Zhang, Rev. Mod. Phys. 83, 1057 (2011)

Higher-order topological insulators

The usual topological insulators (TIs) can be called first-order.
The $d-$dimensional bulk states are gapped, and the $(d-1)-$dimensional boundaries have gapless states

This has now been generalized to the idea of higher-order TIs.
For instance, a second-order TI has a $d-$dimensional bulk and a $(d-1)$-dimensional boundary which are both gapped, but the $(d-2)$-dimensional "boundary of the boundary" has gapless states.
Thus in a second-order two-dimensional TI, both the bulk and edge states are gapped, but there are gapless states at the corners



A model for a second-order 2D TI

Consider a four-band model with the Hamiltonian

$$\begin{aligned} H(\vec{k}) = & \left[M + t_0 (\cos k_x + \cos k_y)\right] \tau^z \otimes \sigma^0 \\ & + \Delta_1 (\sin k_x \, \tau^x \otimes \sigma^x + \sin k_y \, \tau^x \otimes \sigma^y) \\ & + \Delta_2 (\cos k_x - \cos k_y) \, \tau^y \otimes \sigma^0 \end{aligned}$$

$\vec{\tau}$ and $\vec{\sigma}$ are Pauli matrices acting on the orbital and spin degrees of freedom respectively, and τ^0 and σ^0 denote identity matrices acting in the two spaces

t_0 denotes a spin-independent but orbital-dependent hopping amplitude between nearest-neighbor sites, and Δ_1 denotes a spin-orbit coupling. For $\Delta_2 = 0$, there are gapless edge states and no corner states. Turning on Δ_2 gaps out the edge states but produces gapless corner states

Schindler et al., Science Advances 4, 0346 (2018)

Symmetries of the model

$$
\begin{aligned}
H(\vec{k}) &= \left[M + t_0 \left(\cos k_x + \cos k_y\right)\right] \tau^z \otimes \sigma^0 \\
&+ \Delta_1 \left(\sin k_x \, \tau^x \otimes \sigma^x + \sin k_y \, \tau^x \otimes \sigma^y\right) \\
&+ \Delta_2 \left(\cos k_x - \cos k_y\right) \tau^y \otimes \sigma^0
\end{aligned}
$$

The corresponding model on a square lattice is symmetric under time-reversal and C_4 (rotations by $\pi/2$) if $\Delta_2 = 0$. If $\Delta_2 \neq 0$, time-reversal and C_4 are separately broken, but their product remains a symmetry

The spectrum is given by

$$
\begin{aligned}
E(\vec{k}) &= \pm \left[(M + t_0 \left(\cos k_x + \cos k_y\right))^2 + \Delta_1^2 \left(\sin^2 k_x + \sin^2 k_y\right) \right.\\
&\left. + \Delta_2^2 \left(\cos k_x - \cos k_y\right)^2\right]^{1/2}
\end{aligned}
$$

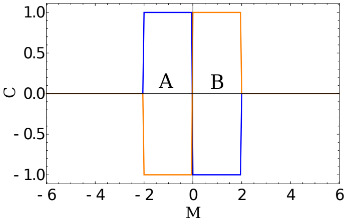
and each energy level has a double degeneracy

Chern number

For $\Delta_2 = 0$, the degenerate bands can be distinguished from each other by eigenvalues of $\tau^z \sigma^z$. In each such band, we can use the wave function $\psi_{\vec{k}}$ to calculate the Chern number

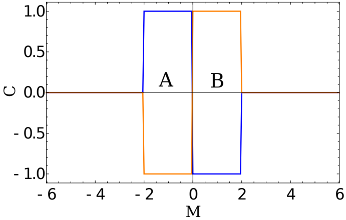
$$C = \frac{1}{\pi} \int d^2\vec{k} \; \mathrm{Im} \left[\frac{\partial \psi_{\vec{k}}^\dagger}{\partial k_x} \frac{\partial \psi_{\vec{k}}}{\partial k_y} \right]$$

For $t_0 = 1$, $\Delta_1 = 1$ and $\Delta_2 = 0$, we find that C varies with M as follows (the orange and blue lines are for negative energy bands with $\tau^z \sigma^z = \pm 1$)



Chern number $\cdots$

$t_0 = 1,\ \Delta_1 = 1$ and $\Delta_2 = 0$



There are topological phases in the ranges $-2 < M < 0$ and $0 < M < 2$ with gapless edge states, and non-topological phases when $M < -2$ and $2 < M$ where there are no edge states

The jumps in the Chern number occur when the band gaps close. The gaps close at the momenta $(k_x, k_y) = (0,0)$ and (π, π) producing jumps of 1 at $M = -2$ and 2 respectively, and at $(0, \pi)$ and $(\pi, 0)$ simultaneously producing a jump of 2 at $M = 0$

Chern number $\cdots$

$$
\begin{aligned}
H(\vec{k}) \;=\; & \left[M + t_0 \left(\cos k_x + \cos k_y\right)\right] \tau^z \otimes \sigma^0 \\
& + \Delta_1 \left(\sin k_x \, \tau^x \otimes \sigma^x + \sin k_y \, \tau^x \otimes \sigma^y\right) \\
& + \Delta_2 \left(\cos k_x - \cos k_y\right) \tau^y \otimes \sigma^0
\end{aligned}
$$

For $\Delta_2 \neq 0$, $\tau^z \sigma^z$ is not a good quantum number and cannot be used to distinguish between the degenerate bands

As a result, the Chern number cannot be calculated

Diagonal winding number

There is a different topological invariant, a diagonal winding number, which can be calculated for all values of Δ_2

$$\begin{aligned} H(\vec{k}) \;=\; & \left[M + t_0\,(\cos k_x + \cos k_y)\right] \tau^z \otimes \sigma^0 \\ & + \Delta_1\,(\sin k_x\, \tau^x \otimes \sigma^x + \sin k_y\, \tau^x \otimes \sigma^y) \\ & + \Delta_2\,(\cos k_x - \cos k_y)\, \tau^y \otimes \sigma^0 \end{aligned}$$

In general, the Hamiltonian is the sum of four anticommuting matrices and it is not possible to define a winding number since that requires a sum of two anticommuting matrices (whose coefficients will parametrize a point in a two-dimensional plane)

However, on the diagonals $k_x = \pm k_y$, the Hamiltonian becomes

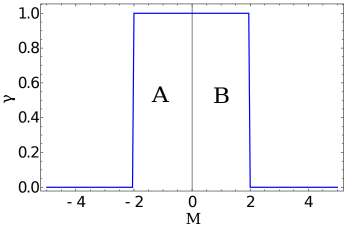
$$H(\vec{k}) = \left[M + 2t_0 \cos k_x\right] \tau^z \otimes \sigma^0 + \sqrt{2}\Delta_1 \sin k_x\, \tau^x \otimes \frac{\sigma^x \pm \sigma^y}{\sqrt{2}}$$

which is the sum of two anticommuting matrices

Diagonal winding number $\cdots$

We therefore have to calculate the winding number of a closed curve in a plane whose points have the coordinates $(M + 2t_0 \cos k_x, \sqrt{2}\Delta_1 \sin k_x)$, where k_x goes from $-\pi$ to π. Note that the curve does not depend on Δ_2 at all

For $t_0 = 1$ and $\Delta_1 = 1$, the winding number varies with M as follows

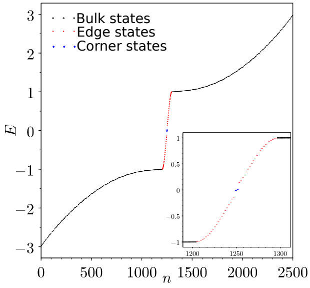


The figure looks similar to the variation of the Chern number

Corner states

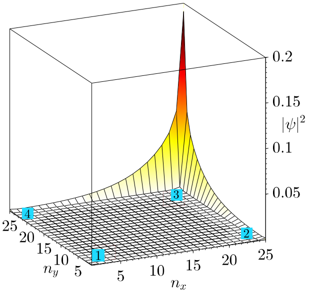
When $\Delta_2 \neq 0,$ the Chern number cannot be calculated any more, but the winding number remains the same

In the region $-2 < M < 2$ where the winding number is equal to $1,$ a gap of the order of Δ_2 opens in the spectrum of edge states and a gapless state appears at each corner of a square lattice. This is shown below for $M = 0,\ t_0 = \Delta_1 = 1$ and $\Delta_2 = 0.1$ for a lattice with 25×25 sites



Probability plot of a corner state

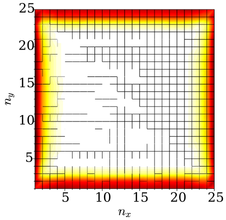
$M = 0$, $t_0 = \Delta_1 = 1$ and $\Delta_2 = 0.1$ for a lattice with 25×25 sites



The decay length into the bulk is much smaller than the decay length along the edges since the latter is proportional to $1/\Delta_2$

Explanation for corner states

$$
\begin{aligned}
H(\vec{k}) &= \left[M + t_0 (\cos k_x + \cos k_y)\right] \tau^z \otimes \sigma^0 \\
&+ \Delta_1 (\sin k_x \, \tau^x \otimes \sigma^x + \sin k_y \, \tau^x \otimes \sigma^y) \\
&+ \Delta_2 (\cos k_x - \cos k_y) \, \tau^y \otimes \sigma^0
\end{aligned}
$$



For $\Delta_2 = 0,$ the edge states satisfy massless Dirac equations. The Δ_2 term acts as a mass term and gaps out the edge states. The Hamiltonian shows that the mass term changes sign from one edge to the next. For the Dirac equation in one dimension, it is known that a change in the sign of the mass term at some point produces a zero energy state there

Anisotropic model

It is interesting to look at an anisotropic model where the Chern number and diagonal winding number do not behave identically

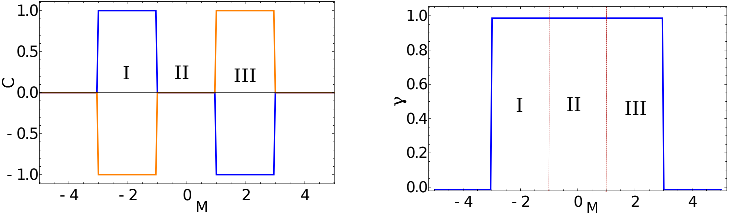
$$\begin{aligned} H(\vec{k}) &= (M + t_x \cos k_x + t_y \cos k_y)\, \tau^z \otimes \sigma^0 \\ &+ \Delta_1 (\sin k_x\, \tau^x \otimes \sigma^x + \sin k_y\, \tau^x \otimes \sigma^y) \\ &+ \Delta_2 (\cos k_x - \cos k_y)\, \tau^y \otimes \sigma^0 \end{aligned}$$

We allow the hoppings t_x and t_y to be different, but take the same coefficient for $\cos k_x$ and $\cos k_y$ in the last term (this allows us to define a diagonal winding number as before)

Now the Chern and diagonal winding number plots look different from each other

Topological invariants for anisotropic model

$t_x = 1, \quad t_y = 2, \quad \Delta_1 = 1$ and $\Delta_2 = 0$



In regions I and III, the Chern number is non-zero and there are gapless edge states on all the four edges

In region II, the Chern number is zero and there are gapless edge states only on the edges parallel to the $x-$direction when $|t_x| < |t_y|$. So the system is a weak topological insulator

When $\Delta_2 \neq 0$, corner states appear in regions I, II and III

Periodic driving of the parameter M

Given the Hamiltonian

$$H(\vec{k}) = \left[M + t_0 (\cos k_x + \cos k_y)\right] \tau^z \otimes \sigma^0 + \Delta_1 (\sin k_x \, \tau^x \otimes \sigma^x + \sin k_y \, \tau^x \otimes \sigma^y) + \Delta_2 (\cos k_x - \cos k_y) \tau^y \otimes \sigma^0$$

we will consider the effect of varying M periodically in time as a sequence of pulses

$$M(t) = \begin{cases} M_1 & \text{if } 0 < t < T/4 \\ M_2 & \text{if } T/4 < t < 3T/4 \\ M_1 & \text{if } 3T/4 < t < T \end{cases}$$

and $M(t+T) = M(t)$ for all t

Floquet operator U_F

We will calculate the time evolution operator for one time period T

$$U_F = \mathcal{T}\, e^{-i\int_0^T H(t')dt'}$$
$$= e^{-iH_1T/4}\, e^{-iH_2T/2}\, e^{-iH_1T/4}$$

where $\mathcal{T}$ denotes the time-ordered product, and H_1 and H_2 are the Hamiltonians with $M = M_1$ and M_2 respectively

The reason for choosing this ordering, rather than the simpler form $e^{-iH_1T/2}\, e^{-iH_2T/2}$, is to have an additional symmetry which allows us to define a diagonal winding number

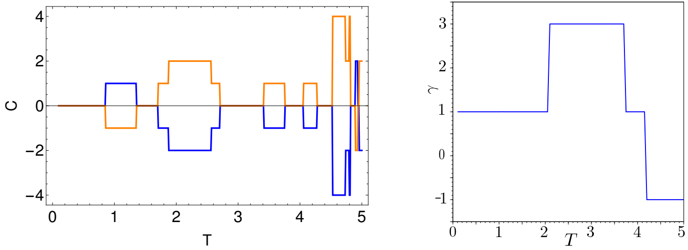
Topological invariants for driven system

For $\Delta_2 = 0,$ we can again use $\tau^z \sigma^z$ to distinguish between the different bands of eigenstates of U_F and then calculate the Chern number for each band. For $\Delta_2 \neq 0,$ the Chern number cannot be calculated

For any value of $\Delta_2,$ the driving protocol we are using implies that $U_F(\vec{k})$ is a symmetric matrix and is therefore the exponential of the sum of only two anticommuting matrices $(\tau^z \otimes \sigma^0$ and $\tau^x \otimes \sigma^x)$ on the diagonals $k_x = \pm k_y$

Hence a diagonal winding number can again be calculated

Topological invariants versus T

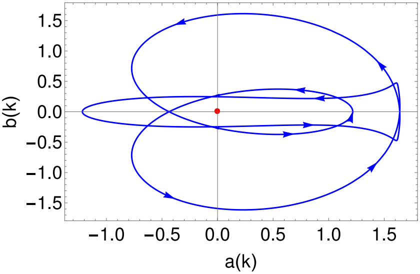


Topological invariants versus T calculated for a driven system with $M_1 = -0.9$, $M_2 = -0.45$, $\Delta_1 = 1$, $t_x = 1$ and $t_y = 2$

The fluctuations in the Chern number at $T \simeq 4.8 - 4.9$ are numerical artefacts. The gaps between some of the quasienergy bands become very small at certain values of $\vec{k}$ which leads to large fluctuations in the numerically calculated value of the Berry curvature and therefore of the Chern number

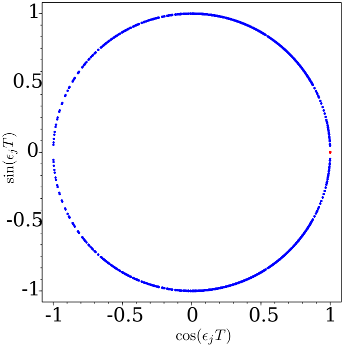
Multiple corner states

For a periodically driven system, the diagonal winding number can sometimes be larger than 1. This means that more than one gapless state can appear at each corner



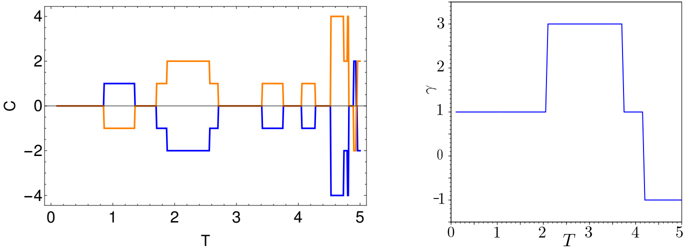
Closed curve with winding number equal to $3,$ for a driven system with $M_1 = -0.9,$ $M_2 = 0.9,$ $\Delta_1 = 1,$ $t_x = 1,$ $t_y = 2,$ and $T = 3$

Eigenvalues of Floquet operator



Floquet eigenvalues for a driven system on a square of size 30×30 with $M_1 = -0.9$, $M_2 = 0.9$, $\Delta_1 = 1$, $\Delta_2 = 0.3$, $t_x = 1$, $t_y = 2$, and $T = 3$. We find 12 states with eigenvalue $+1$ which are separated from the bulk states by a gap. These correspond to 3 states at each corner

Jumps in the topological invariants



Is there a simple way to find the values of T where the topological invariants jump from one integer value to another ?

The jumps occur when the gaps between some of the quasienergy bands close at certain values of the momentum $\vec{k}$. This happens when the Floquet operator $U_F(\vec{k}) = \pm I_4$ (where I_4 is the 4×4 identity matrix), since $U_F(\vec{k})$ then has degenerate eigenvalues

Condition for $U_F(\vec{k}) = \pm I_4$

There are four time-reversal invariant momenta $\vec{k} = (0,0)$, $(0,\pi)$, $(\pi,0)$ and (π,π) where the degeneracy condition can be found analytically. For $\Delta_2 = 0$, the Hamiltonian is

$$H_{1,2}(\vec{k}) = [M_{1,2} + t_x \cos k_x + t_y \cos k_y] \tau^z \otimes \sigma^0 + \Delta_1 (\sin k_x \tau^x \otimes \sigma^x + \sin k_y \tau^x \otimes \sigma^y)$$

At the four momenta above, H_1 and H_2 commute. Hence $U_F(\vec{k}) = e^{-iH_1T/4} e^{-iH_2T/2} e^{-iH_1T/4}$ will be equal to $\pm I_4$ when

$$[M_1 + M_2 \pm 2(t_x + t_y)] \frac{T}{2} = n\pi \quad \text{for} \quad \vec{k} = (0,0) \text{ and } (\pi,\pi)$$

or

$$[M_1 + M_2 \pm 2(t_x - t_y)] \frac{T}{2} = n\pi \quad \text{for} \quad \vec{k} = (0,\pi) \text{ and } (\pi,0)$$

where n is an integer. This explains the locations of many of the jumps that we see in the Chern number

Degeneracy condition for eigenvalues of $U_F(\vec{k})$

We also find that jumps in the Chern number can appear due to other values of the momentum $\vec{k}$. For $\Delta_2 = 0$, we have

$$H_{1,2}(\vec{k}) = \left[M_{1,2} + t_x \cos k_x + t_y \cos k_y\right] \tau^z \otimes \sigma^0 + \Delta_1 \left(\sin k_x \, \tau^x \otimes \sigma^x + \sin k_y \, \tau^x \otimes \sigma^y\right)$$

Now H_1 and H_2 do not commute generally. Hence $U_F(\vec{k}) = e^{-iH_1T/4} \, e^{-iH_2T/2} \, e^{-iH_1T/4}$ will be equal to $\pm I_4$ only if $e^{-iH_1T/2}$ and $e^{-iH_2T/2}$ are separately equal to $\pm I_4$. This implies that $\vec{k}$ must satisfy two conditions simultaneously,

$$[(M_1 + t_x \cos k_x + t_y \cos k_y))^2 + \Delta_1^2(\sin^2 k_x + \sin^2 k_y)]^{1/2} \frac{T}{2} = n_1 \pi$$

$$[(M_2 + t_x \cos k_x + t_y \cos k_y))^2 + \Delta_1^2(\sin^2 k_x + \sin^2 k_y)]^{1/2} \frac{T}{2} = n_2 \pi$$

where n_1 and n_2 are positive integers

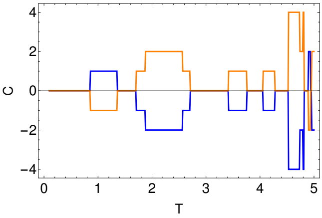
Degeneracy condition for eigenvalues of $U_F(\vec{k})$

The simplest solution of the above equations is given by $n_1 = n_2 = 1.$ This leads to the conditions

$$\cos k_x = -\frac{(M_1 + M_2)\, t_x}{2\,(t_x^2 + t_y^2)}$$

$$\cos k_y = -\frac{(M_1 + M_2)\, t_y}{2\,(t_x^2 + t_y^2)}$$

Since this has 4 solutions for $(k_x, k_y),$ this explains the jump of 4 in the Chern number that we see at $T \simeq 4.49$



Summary

- We have presented isotropic and anisotropic models of second-order topological insulators in two dimensions
- There are two topological invariants, a Chern number and a diagonal winding number. The Chern number is defined only if a parameter Δ_2 is zero and it gives the number of gapless edge states. The winding number gives the number of gapless corner states which appear when $\Delta_2 \neq 0$
- For the anisotropic model, the regions of non-zero Chern number and non-zero diagonal winding number do not coincide
- The same topological invariants can be used for the periodically driven system. The winding number, and therefore the number of states at each corner, can be larger than 1
- The jumps in the topological invariants (indicating topological phase transitions) can occur at either the time-reversal invariant momenta or at momenta which have no special symmetries

Seshadri, Dutta and Sen, arXiv:1901.10495